Notes from Tuesday September 13, 2022

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We discussed two (main) problems (unless I missed something), both of which had a good deal to offer I think. Here are some notes.

1 William's square integer problem

Let a, b, c, x, y, z be natural numbers. Show the following relations cannot all hold:

$$\begin{aligned} a^2 + b + c &= x^2 \\ a + b^2 + c &= y^2 \\ a + b + c^2 &= z^2. \end{aligned} \tag{1}$$

(one of) William's elegant solution(s):

Note that x, y, and z are (strictly) larger integers then the integers a, b, and c respectively. This follows from the first, second, and third equations respectively. For example, the first equation says $x^2 = a^2 + b + c$, and since $b, c > 0$, you know $x^2 > a^2$ so that $x > a$.

Consequently, there are three other natural numbers k, ℓ, and m for which

$$\begin{aligned} x &= a + k, \\ y &= b + \ell, \text{ and} \\ z &= c + m. \end{aligned}$$

Making these substitutions on the right in the original relations gives

$$\begin{aligned}
a^2 + b + c &= a^2 + 2ak + k^2 \\
a + b^2 + c &= b^2 + 2b\ell + \ell^2 \\
a + b + c^2 &= c^2 + 2cm + m^2,
\end{aligned}$$

or

$$\begin{aligned}
b + c &= 2ak + k^2 \\
a + c &= 2b\ell + \ell^2 \\
a + b &= 2cm + m^2.
\end{aligned}$$

Adding these three relations we get

$$2(a + b + c) = 2(ak + b\ell + cm) + k^2 + \ell^2 + m^2$$

or

$$2[a(1 - k) + b(1 - \ell) + c(1 - m)] = k^2 + \ell^2 + m^2.$$

The left side is non-positive and the right side is positive, so this is a contradiction. In summary, $k, \ell, m \geq 1$ implies $1 - k, 1 - \ell, 1 - m \leq 0$, so if we assume the relations (1) hold, then

$$0 < k^2 + \ell^2 + m^2 = 2[a(1 - k) + b(1 - \ell) + c(1 - m)] \leq 0. \qquad \square$$

An alternative solution (or at least approach) was offered by Drake some version of which was eventually presented (at least in part to everyone and in its entirety to me) by Joseph (Campbell). That solution is the following:

One of the integers a, b, c must be greater than or equal to the other two. Say $c \geq a, b$. Then if the third relation $a + b + c^2 = z^2$ holds one notes

$$c^2 < c^2 + a + b = z^2 < (c + 1)^2.$$

The last inequality holds because $a+b < 2c+1$, and of course $2 \leq a+b \leq c+c < 2c+1$. This implies

$$c < z < c + 1,$$

but this is a contradiction because there are no integers between c and $c + 1$.

The same argument applies using the second equation of (1) if $b \geq a, c$ and the first equation if $a \geq b, c$. $\square$

Note: Several of you expressed an interest in "improving your problem solving skills." I think our discussion of this problem had something significant to offer in this regard. Drake certainly had the basic idea of a solution—and likewise if you can cast about and somehow hit upon the basic idea, then you can proceed to try to polish the details and/or work out the kinks. Of course, how one hits on the basic idea in a problem like this is (still perhaps) a bit of a mystery. A second thing to note is that your presentation (and probably your ability to see a solution clearly and/or avoid getting confused) can possibly be enhanced by improved notation. Remember Drake's $c = a + \epsilon_1 = b + \epsilon_2$, and compare to William's $x = a + k$ or Joseph's solution where no new notation was introduced.

Taiki's problem

The other problem we discussed in Problem 5 from *Putnam and Beyond* which was brought to my attention by Taiki who discussed it with me in an email. Here is the way I phrased it:

> *Given nine planar regions of unit area whose union has area* 5*, show that some pair of the regions intersect in an area at least* 1/9*.*

I believe it was (possibly, correct me if I'm wrong) Evan who suggested a solution along the following lines:

Take the first region P_1. It contributes unit area (or area 1) to the union.

Adding in, that is to say taking the union of the first region P_1 with P_2, the area of the union must go up by at least 8/9.

Let's think about this for a second. If $|P_1 \cup P_2|$ denotes the area of $P_1 \cup P_2$, then (because P_2 is assumed to intersect P_1 in area less then 1/9) there must be 8/9 "new" area added to $|P_1|$ in $|P_1 \cup P_2|$.

Continuing this idea, the third region P_3 must add at least 7/9 "new" area to the union and so on, so we see

$$|P_1 \cup P_2 \cup \cdots \cup P_9| \geq 1 + \frac{8}{9} + \frac{7}{9} + \frac{6}{9} + \frac{5}{9} + \frac{4}{9} + \frac{3}{9} + \frac{2}{9} + \frac{1}{9} = 5.$$

Actually, the inequality is strict, so this is a contradiction. $\square$

I suggested reorganizing this solution and firming it up with an induction (or two) along the following lines. Before I try to type some of that up, let me mention what seems to me a good (or at least decent) idea underlying this approach—and I think every approach that was mentioned:

Think of the area of the union as being accumulated as you add one region after another. If you assume the area of the pairwise intersections are all smaller than 1/9, then you'll be forced to add "too much" area in this process.

Here are some details (as I see them) added somewhat successively: I will follow Taiki in denoting the regions by $A_1, A_2, \ldots, A_9$. Assume by way of contradiction that every pairwise intersection has area less than 1/9, that is,

$$|A_i \cap A_j| < 1/9 \qquad \text{for } 1 \le i < j \le 9.$$

Then we consider "accumulating" the area in the union one region at a time:

$$\left|\bigcup_{j=1}^{9} A_j\right| \ge |A_1| = 1.$$

$$\left|\bigcup_{j=1}^{9} A_j\right| \ge \left|\bigcup_{j=1}^{2} A_j\right| = |A_1| + |A_2 \backslash A_1| = |A_1| + |A_2| - |A_1 \cap A_2| > 1 + 1 - 1/9$$

because $|A_1 \cap A_2| < 1/9$. Let's try one more step:

$$\begin{aligned}
\left|\bigcup_{j=1}^{9} A_j\right| &\ge \left|\bigcup_{j=1}^{3} A_j\right| \\
&= \left|\bigcup_{j=1}^{2} A_j\right| - \left|A_3 \backslash \bigcup_{j=1}^{2} A_j\right| \\
&= \left|\bigcup_{j=1}^{2} A_j\right| + |A_3| - \left|A_3 \cap \left(\bigcup_{j=1}^{2} A_j\right)\right| \\
&\ge \left|\bigcup_{j=1}^{2} A_j\right| + |A_3| - |A_3 \cap A_1| - |A_3 \cap A_2| \\
&> 1 + (1 - 1/9) + 1 - 2/9
\end{aligned}$$

because $|A_3 \cap A_1|, |A_3 \cap A_2| < 1/9$.

For me, this was enough of a pattern to formulate an inductive hypothesis/conjecture:

$$\left|\bigcup_{j=1}^{k} A_j\right| > k - \sum_{j=1}^{k-1} \frac{j}{9} = k - \frac{k(k-1)}{18}.$$

In the last expression I've used the formula for the sum of the first $k-1$ natural numbers. I'll leave it to you to prove this by induction. (Remember the base case is $k = 2$.) Here's sort of a possibly interesting question: Induction (when successful) usually implies the inductive hypothesis holds for all natural (large enough) numbers; what happens in this case when k reaches $k = 9$, and you try to apply the inductive argument with $k = 10$?

In any case, if you can establish this inductive hypothesis for $k = 9$, then you get

$$5 = \left| \bigcup_{j=1}^{9} A_j \right| > 9 - \frac{9(8)}{18} = 5$$

which is a contradiction. $\square$

Now you might also find the following inductive hypothesis useful (and "interesting"):

$$\left| A_\ell \cap \left(\bigcup_{j=1}^{\ell-1} A_j \right) \right| = \left| \bigcup_{j=1}^{\ell-1} (A_\ell \cap A_j) \right| \leq \sum_{j=1}^{\ell-1} |A_\ell \cap A_j| < \frac{\ell - 1}{9}.$$

I guess you can show the direct set equality

$$A_\ell \cap \left(\bigcup_{j=1}^{\ell-1} A_j \right) = \bigcup_{j=1}^{\ell-1} (A_\ell \cap A_j)$$

and then the last inequlalities follow. Maybe you don't even need induction here.

Incidentally, this problem was in the section on proof by contradiction rather than induction. Perhaps using induction is a bit of overkill. The solution given in the back of the book *Putnam and Beyond* was just essentially Evan's solution. They called the regions S_j instead of P_j or A_j, and used no special notation for area.

As a final note, Lawrence showed me a solution of this problem using something (I think he) called the *principle of inclusion and exclusion*. I guess it's a combinatorics thing. It might be nice to learn about. I'm sure it makes (some) sense.

And as a final final note: Can you prove that given k planar regions of unit area whose intersection has area $A < k$ there is a specific optimal bound B on the minimal area of intersection of some pair of the regions? Incidentally, can you demonstrate the optimality in the problem above? I.e., find nine regions intersecting pairwise in area exactly 1/9.