Sums and Induction

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These are some notes from our Putnam prep meeting of Tuesday September 6, 2022. Some of the discussion was around my effort to break up the solution of Problem A3 on the 2021 exam into some kind of pieces.

1 Problem A3 (2021)

I'm going to throw in here a justification for the assertion that the four points (m, m, m), $(m, -m, -m)$, $(-m, m, -m)$ and $(-m, -m, m)$ are the vertices of a regular tetrahedron. You may recall that the written proof we were offered (I believe by Drake Goldman) was along the following lines:

$$\sqrt{(\pm 2m)^2 + (\pm 2m)^2 + 0} = 2\sqrt{2}\, m.$$

These problems I gave you have the quality that they are sort of so simple, it is a bit of a challenge to write down a solution. In particular, I think that in the spirit of writing a solution that is clear and receives points on the Putnam examination, while you won't really get a problem like this and you could just make the assertion itself, figuring out how to compose some added explanation is good practice for you. Here is what I would suggest:

Four points are the vertices of a regular tetrahedron if each is the same distance from the other three. In this case, each point has exactly one coordinate in common with each of the other three points, and the difference of the remaining two coordinates is $\pm 2m$. Therefore, the distance from any one of these four points to any one of the other three is always

$$\sqrt{(\pm 2m)^2 + (\pm 2m)^2 + 0} = 2\sqrt{2}\, m.$$

My point: The added words give the reader/grader an explanation of what the calculation means. Just an answer won't get much credit. Just a calculation may get a little more, but what is really expected, i.e., what the grader is looking for, is a full explanation.

Perhaps a "next" piece would be the following:

Exercise 1 *Find the integers m and N so that the four points above lie on the sphere $x^2 + y^2 + z^2 = N$ where N is also a positive integer.*

2 sums and induction

I also posed the problem of expressing the sum

$$1 + 2^2 + 3^2 + \cdots + k^2$$

as a cubic polynomial in k. I believe it was William Joyner who offered the following clever solution. The solution does not use induction but it uses the formula for the sum of the first k integers:

$$1 + 2 + 3 + \cdots + k = \sum_{j=1}^{k} j.$$

The formula for this can be found in several ways including various pairings and geometric interpretations. One that was mentioned is the following: If we set $S_1 = 1 + 2 + 3 + \cdots + k$, then writing the sum in the reverse order and pairing we find $S_1 = k + \cdots + 3 + 2 + 2 + 1$ and

$$2S_1 = \sum_{j=1}^{k} [j + (k + 1 - j)] = \sum_{j=1}^{k} (k + 1) = k(k + 1).$$

Thus,

$$S_1 = \frac{k(k+1)}{2}. \tag{1}$$

This same formula can be obtained by induction which, in principle does away with all ellipsis, i.e., $\cdots$. William's derivation does use ellipsis, which could also be eliminated using induction, however you need to know the formula to use induction, and this derivation gives you (at least heuristically) the formula.

William's solution

Consider $S_2 = 1 + 2^2 + 3^2 + \cdots + k^2$. We can write

$$S_2 = 1 + (2+2) + (3+3+3) + \cdots + (k+k+\cdots+k) = \sum_{j=1}^{k}\sum_{\ell=1}^{j} j.$$

Rearranging the terms we get

$$S_2 = (1+2+3+\cdots+k)+(2+3+\cdots+k)+(3+\cdots k)+\cdots+[(k-1)+k]+k = \sum_{j=1}^{k}\sum_{\ell=k+1-j}^{k}\ell$$

We can apply (1) to the $j=k$ term here and write

$$S_2 = \frac{k(k+1)}{2} + \sum_{j=1}^{k-1}\sum_{\ell=k+1-j}^{k}\ell,$$

and more generally

$$\sum_{\ell=k+1-j}^{k}\ell = \sum_{\ell=1}^{k}\ell - \sum_{\ell=1}^{k-j}\ell = \frac{k(k+1)}{2} - \frac{(k-j)(k-j+1)}{2}.$$

With this substitution

$$S_2 = \sum_{j=1}^{k}\left(\frac{k(k+1)}{2} - \frac{(k-j)(k-j+1)}{2}\right) = \frac{k^2(k+1)}{2} - \sum_{j=1}^{k}\frac{(k-j)(k-j+1)}{2}.$$

The second (summation) term here is

$$-\frac{1}{2}\sum_{j=1}^{k}[k^2+k-2kj+j^2-j] = -\frac{k^2(k+1)}{2}+\frac{2k+1}{2}\sum_{j=1}^{k}j-\frac{S_2}{2} = -\frac{k^2(k+1)}{2}+\frac{k(k+1)(2k+1)}{4}-\frac{S_2}{2}.$$

We conclude

$$S_2 = \frac{k(k+1)(2k+1)}{4} - \frac{S_2}{2},$$

or

$$3S_2 = \frac{k(k+1)(2k+1)}{2}. \qquad \square$$

Exercise 2 *Give a "picture proof" that the sum of the first k odd integers is k^2:*

$$1 + 3 + 5 + \cdots + (2k-1) = k^2.$$

Then give the proof by induction.

3 Tomorrow and beyond

Okay, I'm going to work on the double integral problem (Problem A4 (2021)). My suggestion (which is sort of obvious): You all should be trying to work as many problems from the book and from the old exams as you can. If you can't work problems, then at least try to understand the solutions completely and present them in our meetings. Incidentally, here is the archive of problems and solutions I've been referencing:

`https://kskedlaya.org/putnam-archive/`

I believe the (otherwise amazing) solution for Problem A3 (2021) has a calculation error in it, though the basic idea still works. Can you find the error?