Putnam study meeting notes
Tuesday October 25, 2022

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We decided to try some A1 problems from various years with possibly the goal for many to get (at least) one problem correct—let's be ambitious and say "nail one problem"—on the upcoming practice exam this Saturday October 29 at 10AM in Skiles room 006. It was further suggested that on Saturday we only do/work on part A of the exam from 10AM to 1PM and then discuss the exam (perhaps over lunch—actually that detail wasn't discussed, but maybe I'll come up with something and let you know).

As usual, let me know what I'm missing or have gotten wrong in these notes.

1 A1 (2001) solution by Dennis

Statement: Given a set S and a binary operation $* : S \times S \to S$ satisfying

$$(a * b) * a = b \qquad \text{for every } a, b \in S, \tag{1}$$

prove that $a * (b * a) = b$ for every $a, b \in S$.

Preliminary comments: You may have noticed how Dennis and I had some difficulty communicating the statement of the problem with him reading it and me trying to write it on the board. In the old days, we had a kind of standard way to express such things. For statement (1) we would say "The quantity a star b stared with a is b for all a and b." So those parentheses were expressed with the words "the quantity." Sometimes we might also say "The quantity a star b **all** stared with $a\ldots$" with the word "all" ending the parentheses. Of course, another possibility is "Open parentheses, a star b, closed parentheses, star a."

As a second preliminary comment it may be noted that when one is done solving this problem, one has shown, in particular, that the operation in question is **associative**.
Solution: For clarity, let us write the fundamental relation (1) as

$$(A * B) * A = B \tag{2}$$

which is assumed to hold for all $A, B \in S$. Now, let $a, b \in S$. Taking $A = b$ and $B = a$ in the fundamental relation (2) we know

$$(b * a) * b = a.$$

Or turning this around we can replace the first a in $a * (b * a)$ with $(b * a) * b$ to get

$$a * (b * a) = [(b * a) * b] * (b * a).$$

Now applying the fundamental relation (2) to the last expression with $A = b * a$ and $B = b$ we get

$$a * (b * a) = [(b * a) * b] * (b * a) = b.$$

2 A1 (2018) solutions by Anant and Drake

Statement: Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}. \tag{3}$$

Preliminary comments: I usually write the set of positive integers (also known as the **natural numbers**) as $\mathbb{N} = \{1, 2, 3, \ldots\}$. The set of all ordered pairs of natural numbers is then denoted by $\mathbb{N} \times \mathbb{N}$ or $\mathbb{N}^2$.

A first step in both solutions was to factor 2018 into primes. Clearly, $2018 = 2(1009)$. The prime numbers $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$, and 31 do not divide 1009

and some of their quotients with remainders are as follows:

$$\begin{aligned}1009 &= 7(144) + 1\\ &= 11(91) + 8\\ &= 13(77) + 8\\ &= 17(59) + 6\\ &= 19(53) + 2\\ &= 23(43) + 20\\ &= 29(34) + 23\\ &= 31(32) + 17.\end{aligned}$$

It will be noted that the next prime 37 (and all larger primes) must have quotient less than 32, in fact $1009 = 37(27) + 10$, so no greater prime can divide 1009, or else we would have found a prime smaller than 32 that divided 1009. The conclusion of course, is that 1009 is prime, and the prime factorization of 2018 is just

$$2018 = 2(1009).$$

Anant's solution: Rearranging the equation (3) we obtain

$$2018(a+b) = 3ab \qquad \text{or} \qquad 2018a + 2018b = 3ab.$$

It follows that a divides $2018b$, that is $2018b = k_1 a$ for some $k_1 \in \mathbb{N}$. Similarly, b divides $2018a$, so there is some $k_2 \in \mathbb{N}$ with $2018a = k_2 b$. This means

$$k_1 k_2 ab = (2018)^2 ab \qquad \text{or} \qquad k_1 k_2 = (2018)^2.$$

There are only finitely many choices of integers k_1 and k_2 in $\mathbb{N}$ whose product is $(2018)^2$. Each pair $(k_1, k_2) \in \mathbb{N}$ with $k_1 k_2 = (2018)^2$ determines a pair $(a, b) \in \mathbb{N}^2$ with

$$2018\left(a + \frac{2018a}{k_2}\right) = 3a\frac{2018a}{k_2} \qquad \text{and} \qquad 2018\left(\frac{2018b}{k_1} + b\right) = 3\frac{2018b}{k_1}b,$$

or

$$a = \frac{1}{3}(2018 + k_2) \qquad \text{and} \qquad b = \frac{1}{3}(2018 + k_1) \tag{4}$$

which may be checked to see if that pair satisfies equation (3). Any pair (a, b) satisfying equation (3) determines a pair (k_1, k_2), so we will find all solutions by checking the finitely many possibilities. The possibilities for the pairs $(k_1, k_2) \in \mathbb{N}$ with

$$k_1 k_2 = (2018)^2 = 2^2(1009)^2$$

are as follows:

$$\begin{aligned}(k_1, k_2) &= (1, 2^2(1009)^2)\\ &= (2, 2(1009)^2)\\ &= (2^2, (1009)^2)\\ &= (1009, 2^2(1009))\\ &= (2(1009), 2(1009))\end{aligned}$$

and the pairs (k_2, k_1) determined by these with the coordinates reversed and corresponding to the reversed pairs (b, a). There are essentially five possibilities to check. Here are the corresponding pairs (a, b):

$$\begin{aligned}
(k_1, k_2) = (1, 2^2(1009)^2) \qquad & (a, b) = \tfrac{1}{3}(2018 + 2018^2, 2018 + 1) = (2018(673), 673)\\
(2, 2(1009)^2) \qquad & \tfrac{1}{3}(2018 + 2018(1009), 2018 + 2)\\
(2^2, (1009)^2) \qquad & \tfrac{1}{3}(2018 + (1009)^2, 2018 + 4) = (1009(337), 674)\\
(1009, 2^2(1009)) \qquad & \tfrac{1}{3}(2018 + 4(1009), 2018 + 1009) = (1009(2), 1009)\\
(2(1009), 2(1009)) \qquad & \tfrac{2}{3}(2018, 2018).
\end{aligned}$$

The second and last possibilities do not lead to integers for a and b. Thus, we have three possibilities:

$$(a, b) = (2018(673), 673) \qquad \text{for which} \qquad \frac{1}{2018(673)} + \frac{1}{673} = \frac{2019}{2018(673)} = \frac{3}{2018},$$

$$(a, b) = (1009(337), 674) \qquad \text{for which} \qquad \frac{1}{1009(337)} + \frac{1}{674} = \frac{2 + 1009}{2018(337)} = \frac{3}{2018},$$

and

$$(a, b) = (1009(2), 1009) \qquad \text{for which} \qquad \frac{1}{1009(2)} + \frac{1}{1009} = \frac{3}{1009(2)} = \frac{3}{2018}.$$

In summary, there are six pairs of positive integers (a, b) satisfying equation (3).

These are

$$\begin{aligned}(a, b) &= (2018(673), 673) = (1358114, 673)\\ &= (1009(337), 674) = (340033, 674)\\ &= (1009(2), 1009) = (2018, 1009)\\ &= (673, 1358114)\\ &= (674, 340033) \text{ and}\\ &= (1009, 2018).\end{aligned}$$

Drake's solution: Rearranging the equation (3) as before and multiplying through by 3 we get

$$9ab - 3(2018)a - 3(2018)b = (3a - 2018)(3b - 2018) - 2018^2 = 0$$

or

$$(3a - 2018)(3b - 2018) = 2018^2.$$

Since $3a$ and $3b$ are positive integers, the factors $3a-2018$ and $3b-2018$ are definitely integers, and they are integers with absolute values (both) less than 2018. This means, the factors cannot be both negative—their product would be too small. They cannot be of different signs either, since $2018^2 > 0$. Thus, we must have two positive integers $k_1 = 3a - 2018$ and $k_2 = 3b - 2018$ with $k_1 k_2 = 2018^2$. I think from this point on, the solutions goes pretty much like the one above, except that I have switched the roles of k_1 and k_2.

Final comment: Since you (or at least some of you) are going to take a 2022 Putnam exam on December 3, 2022, it does not seem unreasonable to prime factor the integer 2022 in advance... and maybe note anything else interesting about that integer you might notice. Obviously, $2022 = 2(1011)$. The sum of the digits of 1011 is 3, so I know also $2022 = 2(3)(337)$. The integer 337 is not divisible by $2, 3, 5, 7$ and

$$\begin{aligned}337 &= 11(30) + 7\\ &= 13(25) + 12\\ &= 17(19) + 14\\ &= 19(17) + 14.\end{aligned}$$

Therefore, 337 is prime, and the prime factorization of 2022 is

$$2022 = 2(3)(337).$$

3 A1 (2005) solution by Siddharth

Statement: Show every positive integer n is a sum

$$\sum_{j=1}^{k} x_j$$

of positive integers x_j each of which is a product $2^r 3^s$ of a power of 2 and a power of 3 with exponents r and s nonnegative integers and such that x_j does not divide x_i when $i \neq j$.
Preliminary comments: I like to denote the nonnegative integers by $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, though this notation for the **natural numbers with zero** has not quite caught on and become universal (yet).

To state the assertion here with a little more precision takes a lot of indices. It might be done as follows: Show every $n \in \mathbb{N}$ can be written as

$$n = \sum_{j=1}^{k_n} x_{nj}$$

where $k_n \in \mathbb{N}$ and

$$x_{nj} = 2^{r_{nj}} 3^{s_{nj}} \qquad \text{for some } r_{nj} \text{ and } s_{nj} \text{ in } \mathbb{N}_0$$

such that $x_{ni}/x_{nj} \notin \mathbb{N}$ for $i \neq j$. That is,

$$n = \sum_{j=1}^{k_n} 2^{r_{nj}} 3^{s_{nj}}$$

with

$$\frac{2^{r_{ni}} 3^{s_{ni}}}{2^{r_{nj}} 3^{s_{nj}}} \notin \mathbb{N} \qquad \text{for } i \neq j \text{ and } 1 \leq i, j \leq k_n.$$

One obvious question I asked myself before looking at this problem for very long was: If this is true, is the representation unique? I haven't thought much about that, and maybe it is not too difficult to answer.

Solution: Let's start with some preliminary observations. First of all, the first few integers can be handled:

$$\begin{aligned}
1 &= 2^0 3^0\\
2 &= 2^1 = 2^1 3^0\\
3 &= 3^1 = 2^0 3^1\\
4 &= 2^2 = 2^2 3^0\\
5 &= 2 + 3 = 2^1 3^0 + 2^0 3^1\\
6 &= 2 \cdot 3 = 2^1 3^1\\
7 &= 3 + 2^2 = 2^0 3^1 + 2^2 3^0.
\end{aligned}$$

So far, it looks like the representations are unique. For example, you can write $4 = 3 + 1$, but then 1 divides 3.

We are also given the example $23 = 9 + 8 + 6 = 3^2 + 2^3 + 2 \cdot 3$.

Beyond this, Siddharth seems to have made two or three crucial observations. One is that among the integers

$$n = \sum_{j=1}^{k} x_j$$

admitting such a representation **at most one of the natural numbers x_i can be odd**. This is because if $x_i = 2^r 3^s$ is odd, then $r = 0$. Hence, if there were two such terms 3^{s_1} and 3^{s_2}, one of them would have to divide the other.

Expanding on this observation, if an even integer n admits the desired representation, then **none of the terms x_j is odd**. For even integers n of the form

$$n = \sum_{j=1}^{k} x_j \qquad \text{with } x_j = 2^{r_j} 3^{s_j} \text{ for } j = 1, \ldots, k,$$

we must have $r_j \geq 1$. Also, if an odd integer n admits the desired representation, then **exactly one of the terms x_j is odd**, having the form $x_j = 3^{s_j}$.

Having made these preliminary observations, we proceed by induction. If $n \geq 3$, then there is a unique largest power m for which $3^m \leq n$. If $n = 3$, then $m = 1$ and we have $3^m = n = 3$ as above. In general, if we assume $n \geq 4$ is even and every integer $\ell < n$ admits the desired representation, then $n = 2\ell$ for some $\ell \in \mathbb{N}$ with $\ell < n$, and we know by induction that

$$\ell = \sum_{j=1}^{k} 2^{r_j} 3^{s_j} \qquad \text{with} \qquad \frac{2^{r_i} 3^{s_i}}{2^{r_j} 3^{s_j}} \notin \mathbb{N}, \quad \text{for } i \neq j.$$

Also, we know $r_i \geq 1$ for $i = 1, \ldots, k$, but I'm not sure we need that. In any case, we have

$$n = 2\ell = \sum_{j=1}^{k} 2^{r_j+1} 3^{s_j}.$$

Since

$$\frac{2^{r_i+1} 3^{s_i}}{2^{r_j+1} 3^{s_j}} = \frac{2^{r_i} 3^{s_i}}{2^{r_j} 3^{s_j}} \notin \mathbb{N} \quad \text{for } i \neq j,$$

we see n has the desired representation.

If, on the other hand, $n \geq 5$ is odd, then as mentioned above, there is a unique $m \in \mathbb{N}$ such that

$$3^m \leq n \qquad \text{but} \qquad 3^{m+1} > n. \tag{5}$$

In this case, consider $\ell = n - 3^m$. If $\ell = 0$, then $n = 3^m$ is the desired representation. Otherwise, $\ell \in \mathbb{N}$, and we have by induction that

$$\ell = \sum_{j=1}^{k} 2^{r_j} 3^{s_j}.$$

Note that ℓ in this case is even, so $r_j \geq 1$ for all $j = 1, \ldots, k$. Furthermore, $s_j < m$ for $j = 1, \ldots, k$ becuase if $m \leq s_{j_0}$, then since we also know $r_{j_0} \geq 1$

$$n = 3^m + \sum_{j=1}^{k} 2^{r_j} 3^{s_j} \geq 3^m + 2^{r_{j_0}} 3^m = 3^m (2^{r_{j_0}} + 1) \geq 3^{m+1}$$

contradicting (5). It follows that

$$\frac{2^{r_j} 3^{s_j}}{3^m} \notin \mathbb{N} \qquad \text{for } j = 1, \ldots, k, \tag{6}$$

and

$$n = 3^m + \sum_{j=1}^{k} 2^{r_j} 3^{s_j}$$

gives the desired representation for the odd integer $n \geq 5$. Just to make this assertion completely clear, note that in addition to (6) we have

$$\frac{2^{r_i} 3^{s_i}}{2^{r_j} 3^{s_j}} \notin \mathbb{N} \qquad \text{for } i \neq j$$

because

$$\ell = \sum_{j=1}^{k} 2^{r_j} 3^{s_j}$$

is a representation of the desired form for ℓ, and

$$\frac{3^m}{2^{r_j} 3^{s_j}} \notin \mathbb{N} \qquad \text{for } j = 1, \ldots, k$$

because $r_j \geq 1$ for $j = 1, \ldots, k$.

This completes the proof by induction. I've phrased the last case for odd $n \geq 5$ just because we handled the case $n = 3$ separately, but I think the reasoning goes through without any trouble if $n = 3$ is included as well.