Putnam study meeting notes
Tuesday October 11, 2022

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I'm not sure if I remember everything, and I'm not sure about the order. Let me know what I'm missing.

1 Sums of powers

Drake continued with various results related to (and relating)

$$\sum_{j=1}^{k} j^p \qquad \text{and} \qquad \sum_{j=1}^{k} j.$$

These were inspired by the observation that

$$\sum_{j=1}^{k} j^p = \left(\sum_{j=1}^{k} j \right)^q$$

for $p = 3$ and $q = 2$.

2 B5 (2020)

I gave an update on Problem B5 (2020) which asks one to show that if $z_1, z_2, z_3, z_4 \in \partial B_1(0) \backslash \{1\} \subset \mathbb{C}$, then

$$\sum_{j=1}^{4} z_j - \prod_{j=1}^{4} z_j \neq 3.$$

There is a series of conjectures associated with this problem as follows:

Conjecture 1 *For* $n \geq 2$ *and* $z_1, z_2, \ldots, z_n \in \partial B_1(0) \backslash \{1\}$,

$$\sum_{j=1}^{n} z_j - \prod_{j=1}^{n} z_j \neq n - 1.$$

Conjecture 2 *For* $n \geq 2$ *and* $z_1, z_2, \ldots, z_n \in \partial B_1(0)$,

$$\sum_{j=1}^{n} z_j - \prod_{j=1}^{n} z_j \neq n - 1$$

unless $\#\{j : z_j = 1\} \geq n - 1$.

Conjecture 3 *For* $n \geq 2$ *and* $z_1, z_2, \ldots, z_n \in \partial B_1(0)$,

$$\left| n - \sum_{j=1}^{n} z_j \right| \geq \left| 1 - \prod_{j=1}^{n} z_j \right|$$

with strict inequality unless $\#\{j : z_j = 1\} \geq n - 1$.

I mentioned that it is probably enough to show any one of these conjectures inductively, if one can show the following:

> If the inductive assertion (equality or inequality) is violated for some points $z_1, z_2, \ldots, z_n \in \partial B_1(0) \backslash \{1\}$, then it is also violated for some points $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n$ with
>
> $$\#\{j : \tilde{z}_j \neq 1\} > 0 \qquad \text{and} \qquad \#\{j : \tilde{z}_j = 1\} > 0.$$

It will also be noted that the conjectures above are successively stronger in the sense that each next one implies the previous one. I previously discussed the verification of most of them in the case $n = 2$. In this case, all of the assertions hold, and we know precisely what happens.

I have a somewhat different conjecture which looks promising, though I didn't mention it, and I'm not entirely sure where it leads, but I'll mention it anyway:

Conjecture 4 *For* $n \geq 2$ *and* $z_1, z_2, \ldots, z_n, \zeta \in \partial B_1(0) \backslash \{1\}$,

$$\left| n + 1 - \zeta - \sum_{j=1}^{n} z_j \right| > \left| \overline{\zeta} - \prod_{j=1}^{n} z_j \right|.$$

Counterexamples to any of these conjectures for any n would be interesting.

3 A3 (2018)

Joseph suggested this problem.

Putnam exam problem A3 (2018): Find the maximum value of

$$\sum_{j=1}^{10} \cos(3\theta_j)$$

if

$$\sum_{j=1}^{10} \cos(\theta_j) = 0. \tag{1}$$

Joseph argued roughly as follows: $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$. Therefore, given the constraint (1) which we can generalize for $n \geq 3$ to

$$\sum_{j=1}^{n} \cos(\theta_j) = 0,$$

the value of the objective function is

$$\sum_{j=1}^{n} \cos(3\theta_j) = 4\sum_{j=1}^{n} \cos^3(\theta_j).$$

Thus, writing $x_j = \cos\theta_j$ for $j = 1, 2, \ldots, n$, the problem is equivalent to maximizing

$$\sum_{j=1}^{n} x_j^3$$

on the set $U = [-1, 1]^n \subset \mathbb{R}^n$ subject to the constraint

$$\sum_{j=1}^{n} x_j = 0$$

(at least for $n = 10$).

We didn't get a solution of this problem, though Lawrence suggested a positive value obtained by some algorithm I don't think I completely understand. I thought it was something like assuming $x_j = 1$ for $j = 1, 2, \ldots, k$ and $x_\ell = x_{k+1}$ for $\ell \geq k + 1$.

For example, if one takes $x_1 = x_2 = 1$ and $x_3 = x_4 = \cdots = x_{10} = -1/4$, then the constraint is satisfied and the objective function takes the value

$$\sum_{j=1}^{2} 1 - \sum_{j=1}^{8} \frac{1}{64} = 2 - \frac{1}{8}.$$

I'm pretty sure Lawrence claimed to get a value just shy of 10, but I do not see how to get such a value this way. It seems this way I only get

$$k - (10-k)\left(\frac{k}{10-k}\right)^3$$

which seems to top out at around $k = 3$ or $k = 4$ at a value less than 2.5. I must be missing something.

4 One more

Finally, I posed the problem of minimizing

$$\sum_{j=1}^{n} |z_j|^2$$

for $z_1, z_2, \ldots, z_n \in \mathbb{C}$ subject to the constraint

$$\sum_{j=1}^{n} z_j = 1.$$

I believe it was Juntao (correct me if I'm wrong) who gave a solution showing sequentially that any mimimizer $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ must have

1. z_j real for all j,
2. z_j positive for all j, and
3. z_j equal for all j.

Lawrence gave a second proof based on an inequality. I don't think I can reproduce Lawrence's solution, but it struck me as correct at the time. Perhaps Lawrence can write it up for us.