Putnam study meeting notes
Tuesday November 1, 2022

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A1 (2007)

(verbatim) Find all values of α for which the curves $y = \alpha x^2 + \alpha x + 1/24$ and $x = \alpha y^2 + \alpha y + 1/24$ are tangent to each other.

Preliminary comments/notes:

Though the statement of this problem is pretty clear, there are aspects with which one could nitpick. One is that it should be said that α should be a real number. If α is not a real number, e.g., if α is allowed to be a complex number, then it's not entirely clear what it means for an expression like

$$y = \alpha x^2 + \alpha x + \frac{1}{24}$$

to be a "curve." Of course, this expression itself is **not** a curve, so that is another slightly problematic aspect of the statement. The proper way to say it might be

$$\left\{ (x, y) \in \mathbb{R}^2 : \alpha x^2 + \alpha x + \frac{1}{24} \right\}$$

is a set which is (or turns out to be) a curve. Generally, the technical definition for such a thing to be true is something like this:

> A set $C \subset \mathbb{R}^2$ is a **curve** if for each point $p \in C$, there exist positive numbers r and ϵ and a continuous function $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^2$ such that
>
> $$C \cap \{(x, y) \in \mathbb{R}^2 : |(x, y) - p| < r\} = \{\gamma(t) : |t| < \epsilon\}.$$

In some contexts people think about curves in pretty much this form. For example, you can look up (and study) something called a **Peano** (space filling) **curve**, which is an object pretty much of this sort. Suffice it to say, the image of such a "curve," is not one that necessarily fits most people's intuition about what a curve should be. For that one usually considers a more restricted class of functions γ. One improvement—getting us closer to what we might think of as a curve—is to require γ to be differentiable or continuously differentiable, i.e., $\gamma(t) = (x(t), y(t))$ where $x, y : (-\epsilon, \epsilon) \to \mathbb{R}^2$ are differentiable functions or differentiable functions with continuous derivatives.

This leads into thinking about curves from a different point of view. Let's take the time to explain this a little more and make it precise:

One point of view (and that implicitly taken by the problem) is that curves are **particular kinds of sets**. In particular, they are sets which can be **parameterized locally** by functions $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^2$ more or less as described above.

The second point of view is that curves are **the parameterizations themselves**. More precisely, a (parameterized) **curve** is a continuous function $\gamma : I \to \mathbb{R}^2$ where I is some interval in the real line. Then from there one can restrict the kinds of continuous functions under consideration by imposing various requirements on the function γ like differentiability. From this point of view the set

$$\{\gamma(t) : t \in I\} \qquad \text{is the } \textbf{image} \text{ of the curve}$$

rather than the curve itself.

On the face of it, these two points of view concerning curves lead to different kinds of things, and it surely helps to consider some examples. Starting with the parameterization point of view is perhaps the easiest: As the parameter t moves along the interval I, one can imagine the point $\gamma(t) \in \mathbb{R}^2$ moves around the plane. For example, if $\gamma : [0, 2\pi] \to \mathbb{R}^2$ by $\gamma(t) = (\cos t, \sin(2t))$, the image is indicated in Figure 1. You can plot some points along this curve starting with $t = 0$ at $\gamma(0) = (1, 0)$ and imagine how $\gamma(t)$ moves along the image (curve) as the parameter changes.

When we think of a curve as a set, on the other hand, we are thinking about something like the circle shown in Figure 2. Here we can start with the point $p = (1, 0)$ and take the radius $r = 1/2$ and consider

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cap \left\{(x, y) \in \mathbb{R}^2 : |(x, y) - p| < \frac{1}{2}\right\}.$$

Incidentally, a set like $\{(x, y) \in \mathbb{R}^2 : |(x, y) - p| < r\}$ is called a **ball** or **disk** of radius

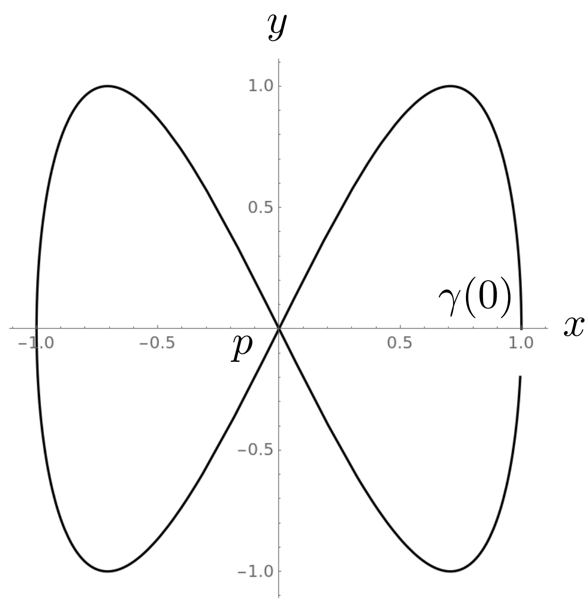

Figure 1: The image of a parameterized curve $\gamma(t) = (\cos t, \sin(2t))$.

r with center p. It has a notation:

$$B_r(p) = \{(x, y) \in \mathbb{R}^2 : |(x, y) - p| < r\}.$$

In any case, we can parameterize the subset

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cap B_{1/2}(0, 0)$$

by $\gamma : (-\pi/6, \pi/6) \to \mathbb{R}^2$ with $\gamma(t) = (\cos t, \sin t)$. This gives a local parameterization identifying the set $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ locally as a curve. Of course, a similar local parameterization can be found at any point $p \in C$.

Exercise 1 *Find a local parameterization in an open ball centered at every other point $p \in C$.*

In fact, according to our (provisional) definition of a curve as a set given above, we can find a **global parameterization** of C for each point $p \in C$ as follows: Let $r > 2$, then $C \subset B_r(p)$ and $\gamma : (-2\pi, 2\pi) \to \mathbb{R}^2$ by $\gamma(t) = (\cos t, \sin t)$ satisfies the requirements we've put on a local parameterization.

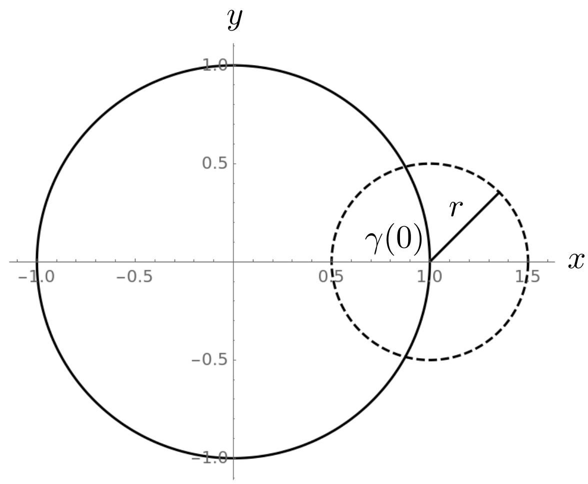

Figure 2: The circle determined by the relation $x^2 + y^2 = 1$.

It may occur to you that the last global parameterization has an undesirable characteristic, namely, it is not one-to-one on the entire interval $(-\epsilon, \epsilon) = (-2\pi, 2\pi)$. As mentioned above, we can impose other requirements on our local parameterizations, and we could definitely impose the requirement that γ be one-to-one onto its image. Usually an even stricter requirement is imposed. To understand this requirement we need to make sense of the idea that the inverse function $\gamma^{-1} : C \cap B_r(p) \to (-\epsilon, \epsilon)$ is continuous. The usual "ϵ-δ definition of continuity will work in this instance:

> For each $q_0 \in C \cap B_r(p)$ and each $\tau > 0$, there is some $\rho > 0$ for which $|\gamma^{-1}(q) = \gamma^{-1}(q_0)| < \tau$ whenever $q \in C \cap B_\rho(q_0)$.

A continuous function $\gamma : I \to C$ that is invertible and has a continuous inverse is called a **homeomorphism**. If the function is differentiable, and its inverse is well-defined and differentiable, then the function is called a **diffeomorphism**. In this context, one should always consider the following example:

Exercise 2 *Show $\gamma : [0, 2\pi) \to \mathbb{R}^2$ by $\gamma(t) = (\cos t, \sin t)$ is differentiable and one-to-one onto its image $C = \partial B_1(\mathbf{0})$, but γ is not a homeomorphism onto C.*

We have also mentioned that differentiability is also usually (or at least often) imposed on the parameterization γ in the definition of a curve as a set. Peano curves do not satisfy such a condition, but many more techniques are available if one does have differentiability. One can then consider the **tangent** or **velocity** vector $\gamma'(t) = (x'(t), y'(t))$. We will come back to another way to enhance this requirement.

Switching perspectives back again for a moment, and considering curves (globally) as continuous (or differentiable) functions $\gamma : I \to \mathbb{R}^2$, these kinds of curves too may have some characteristics we might not want to associated with our idea of what it means to be a curve. Consider, for example, the image in Figure 1. Applying our intuition about curves as sets, the set $\{(\cos t, \sin(2t)) : 0 \le t \le 2\pi\} \cap B_{1/2}(p)$ centered at the point $p = (0, 0)$ labeled in the figure does not look so very much like the image of an interval. At least it does not look in any way very similar to the piece of the circle in Figure 2. We would say this is "two crossing curves."

Exercise 3 *Find a differentiable local parameterization* $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^2$ *of the set* $\{(\cos t, \sin(2t)) : 0 \le t \le 2\pi\} \cap B_{1/2}(p)$ *where* $p = (0, 0)$ *is the origin as illustrated in Figure 2.*

If this exercise seems difficult, or impossible consider yet a third example: The differentiable parameterized curve $\gamma : (-1, 1) \to \mathbb{R}^2$ by $\gamma(t) = (t^2, t^3)$ has image as indicated in Figure 3. Notice this curve has a kind of "turn around" point at $\gamma(0)$. Of course, to

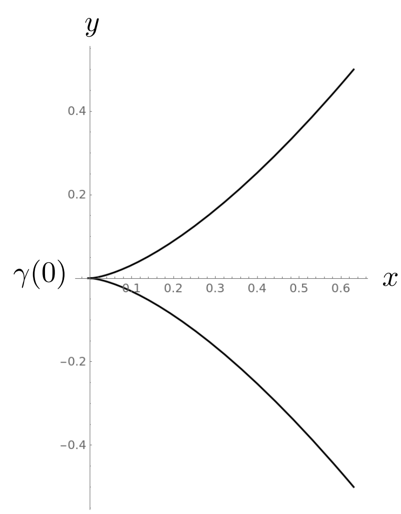

Figure 3: The image of a parameterized curve $\gamma(t) = (t^2, t^3)$ with a cusp.

parameterize the curve in Exercise 3 one needs to use "turn around" points and also "turn a corner points." The idea is the same: You need to slow down—or you need to have your parameterization slow down—and have a zero derivative: $\gamma'(t_0) = (0, 0)$ in order to abruptly change direction but remain differentiable.

In spite of the assertion of Exercise 3 there is still no comprehensive identification between curves as sets and curves as functions without further assumptions.

Exercise 4 *Give an example of a globally parameterized differentiable curve* $\gamma : I \to \mathbb{R}^2$ *having the property that for each* $p = \gamma(t_0)$*, there is some* $\tau > 0$ *such that* $\gamma : (t_0 - \tau, t_0 + \tau) \to \{\gamma(t) : t_0 - \tau < t < t_0 + \tau\}$ *is a diffeomorphism, but* $\{\gamma(t) : t \in I\}$ *does not satisfy the definition of a curve as a subset of* $\mathbb{R}^2$*.*

The usual approach to giving a unified treatment in which curves as sets and curves as functions are essentially the same objects, and that these objects do not have any of the peculiar behavior of some of the examples above is to combine several of the concepts we've considered and to impose an additional assumption called **regularity**. A curve (as a set or parameterized) is said to be **regular** if the (local or global) parameterization either has or can be taken to have γ' non-vanishing. A rather nice class of curves is defined as follows:

1. A curve as a set $C \subset \mathbb{R}^2$ is a set for which given $p \in C$, there are positive numbers r and ϵ and a diffeomorphism $\gamma : (-\epsilon, \epsilon) \to C \cap B_r(p)$ with nonvanishing derivative γ'.

2. A curve as a global parameterization is a function $\gamma : I \to \mathbb{R}^2$ where I is an (open) interval and for each $t_0 \in I$, there exists some positive numbers r and τ such that $\gamma : (t_0 - \tau, t_0 + \tau) \to \mathbb{R}^2$ is a diffeomorphism onto $\{\gamma(t) : t \in I\} \cap B_r(p)$.

There is one small aspect we have glossed over. Namely, the set C should be (required to be) **connected**. Otherwise, you might get C being several (or a whole bunch) of curves. In any case, aside from this issue of C being connected, these regular curves (as sets and and parameterizations) can be comprehensively identified; talking about either one, is talking about the same thing.

In this problem, no particular pathological behavior is in play. These are very simple curves, considered as sets $\{(x, y) \in \mathbb{R}^2 : y = \alpha x^2 + \alpha x + 1/24\}$, as regular parameterizations,

$$\gamma(t) = (t, \alpha t^2 + \alpha t + 1/24) \qquad \text{for } t \in \mathbb{R}$$

with

$$\gamma'(t) = (1, 2\alpha t + \alpha) \neq (0, 0),$$

or simply as relations $y = \alpha x^2 + \alpha x + 1/24$ as given in the problem. In fact, these curves are parabolas, and I expect there is a much simpler (or at least somewhat simpler) approach to solving this problem which does not involve the consideration of parameterization directly at all. Having said that, I was happy to see that Joseph was able to solve the problem using the general theory of curves, and I'll note that he seemed to be the only one who solved the problem completely and correctly.

Finally, I will add that that one aspect of parameterized curves probably is more or less necessary to understand to solve this problem. This is the nature of tangency. Say you have two (regular parameterized) curves $\gamma_1(t) = (x_1(t), y_1(t))$ and $\gamma_2(t) = (x_2(t), y_2(t))$. Then the **tangent vectors** (which are nonzero because the parameterizations are regular) are

$$\gamma_j'(t) = (x_j'(t), y_j'(t)) \qquad \text{for } j = 1, 2.$$

For the two curves to be tangent means there are parameter values t_1 and t_2 for which the curves intersect,

$$\gamma_1(t_1) = \gamma_2(t_2)$$

and the curves are **tangent** at the point of intersection meaning the tangent vectors are **parallel**, which we denote by writing $\gamma_1'(t_1) \parallel \gamma_2'(t_2)$. In terms of an equation, probably the easiest way to express the condition $\gamma_1'(t_1) \parallel \gamma_2'(t_2)$ is to say the vector orthogonal, i.e., perpendicual, to one of these vectors is orthogonal to the other, that is to say

$$\gamma_1'(t_1) \perp (\gamma_2'(t_2))^\perp \qquad \text{or} \qquad (\gamma_1'(t_1))^\perp \perp \gamma_2'(t_2).$$

Orthogonality of two vectors can be expressed in terms of the dot product in $\mathbb{R}^2$, and given any vector $\mathbf{v} = (v_1, v_2)$ either of the vectors $(-v_2, v_1)$ and $(v_2, -v_1)$ are orthogonal to $\mathbf{v}$. The counterclockwise rotation is usually denoted by $\mathbf{v}^\perp = (-v_2, v_1)$. In any case, the equation for orthogonality would be

$$(x_1'(t_1), y_1'(t_1)) \cdot (-y_2'(t_2), x_2'(t_2)) = 0 \qquad \text{or} \qquad -x_1'(t_1)y_2'(t_2) + y_1'(t_1)x_2'(t_2) = 0.$$

Solution:

We can parameterize the first parabola by

$$\gamma_1(t) = \left(t, \alpha t^2 + \alpha t + \frac{1}{24}\right)$$

and the second one by

$$\gamma_2(t) = \left(\alpha t^2 + \alpha t + \frac{1}{24}, t\right).$$

The tangent vectors are

$$\gamma_1'(t) = (1, 2\alpha t + \alpha) = (1, \alpha(2t+1))$$

and

$$\gamma_2'(t) = (2\alpha t + \alpha, 1) = (\alpha(2t+1), 1).$$

Since neither of these vectors can vanish, this indicates that both parameterizations are regular. The equations for intersection become

$$\begin{aligned} t_1 &= \alpha t_2^2 + \alpha t_2 + \frac{1}{24} \\ t_2 &= \alpha t_1^2 + \alpha t_1 + \frac{1}{24} \end{aligned}$$

and the equation of tangency

$$\alpha^2(2t_1+1)(2t_2+1) = 1.$$

Noting that each quadratic relation

$$\eta = \alpha\xi^2 + \alpha\xi + \frac{1}{24} = \alpha\left(\xi + \frac{1}{2}\right)^2 - \frac{\alpha}{4} + \frac{1}{24}$$

can be written as

$$4\eta = \alpha(2\xi+1)^2 - \alpha + \frac{1}{6},$$

we can rewrite the intersection equations as

$$\begin{aligned} 4t_1 &= \alpha(2t_2+1)^2 - \alpha + \frac{1}{6} \\ 4t_2 &= \alpha(2t_1+1)^2 - \alpha + \frac{1}{6}, \end{aligned}$$

or

$$
\begin{aligned}
2(2t_1+1) &= \alpha(2t_2+1)^2 - \alpha + \frac{13}{6} \\
2(2t_2+1) &= \alpha(2t_1+1)^2 - \alpha + \frac{13}{6},
\end{aligned}
$$

Writing $A = 2t_1 + 1$ and $B = 2t_2 + 1$, we arrive at the nonlinear system

$$
\begin{aligned}
\alpha^2 AB &= 1 \\
2A &= \alpha B^2 - \alpha + 13/6 \\
2B &= \alpha A^2 - \alpha + 13/6.
\end{aligned}
$$

First of all notice that in general, according to the first equation, it is impossible to have any solution with $\alpha = 0$ (or $A = 0$ or $B = 0$). Thus, we have no concerns about dividing by α (or A or B).

Subtracting the third equation from the second we get

$$2(A - B) = -\alpha(A - B)(A + B).$$

If $A \neq B$ then $\alpha(A+B) = -2$, and dividing by α we have $B = -A - 2/\alpha$. But then the first equation becomes

$$\alpha^2 A \left(A + \frac{2}{\alpha}\right) = -1 \qquad \text{or} \qquad \alpha^2 A^2 + 2\alpha A + 1 = (\alpha A + 1)^2 = 0$$

so that $A = -1/\alpha$ and $B = 1/\alpha - 2/\alpha = -1/\alpha$ contradicting our assumption $A \neq B$.

Thus, we are reduced to the case $A = B$, so that $t_1 = t_2$, the point of intersection must be on the line $y = x$, and the system becomes

$$
\begin{aligned}
\alpha^2 A^2 &= 1 \\
\alpha A^2 - 2A - \alpha + 13/6 &= 0.
\end{aligned}
$$

Again, we know $\alpha \neq 0$, so $A^2 = 1/\alpha^2$. If $A = -1/\alpha$, then

$$\frac{1}{\alpha} + \frac{2}{\alpha} - \alpha + \frac{13}{6} = 0 \qquad \text{or} \qquad 6\alpha^2 - 13\alpha - 18 = 0.$$

From this quadratic equation we get two real values of α, namely,

$$\alpha = \frac{13 \pm \sqrt{169 + 432}}{12} = \frac{13 \pm \sqrt{601}}{12}$$

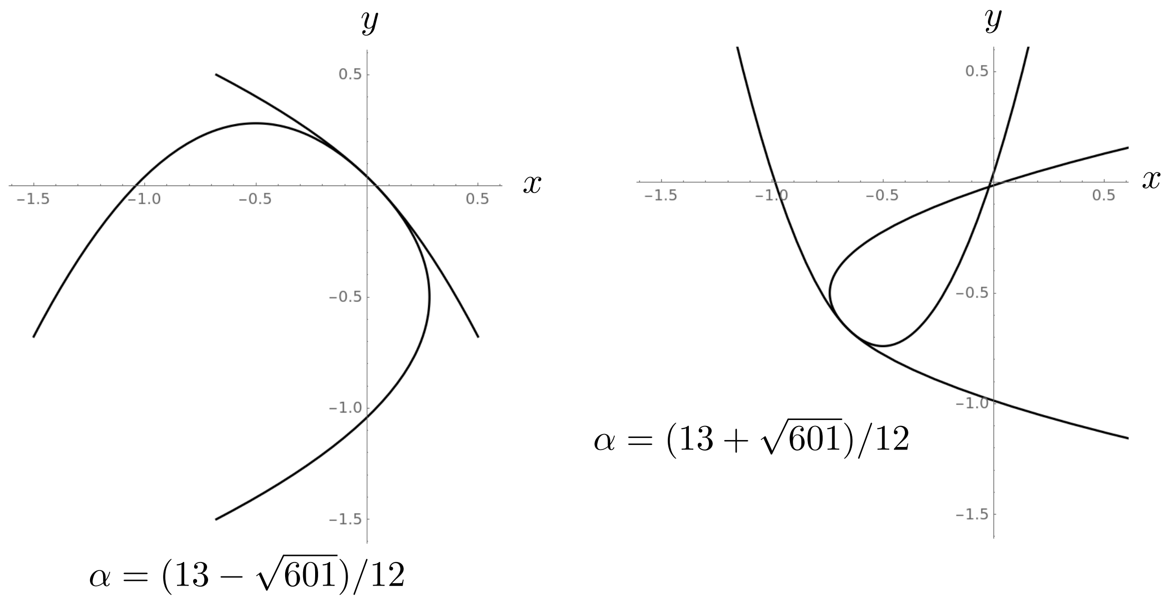

Figure 4: The parabolas corresponding to $\alpha = (13 - \sqrt{601})/12$ (left) and $\alpha = (13 + \sqrt{601})/12$ (right).

which presumably corresponds to a point of tangency of the curves. Incidentally 601 is prime. Also, substituting these values into a numerical plotter produces the the output indicated in Figure 4. Thus, apparently these values do indeed give tangency for the two parabolas, though I'm not precisely sure how one would check that conveniently without mathematical software. I guess there is a calculation that would do it, but I'll also guess that it would be rather unpleasant.

The other possibility is $A = 1/\alpha$. In this case, the second equation becomes

$$\frac{1}{\alpha} - \frac{2}{\alpha} - \alpha + \frac{13}{6} = 0 \qquad \text{or} \qquad 6\alpha^2 - 13\alpha + 6 = 0.$$

Thus we get two more real values of α, namely,

$$\alpha = \frac{13 \pm \sqrt{169 - 144}}{12} = \frac{13 \pm 5}{12}$$

The smaller value is $\alpha = 2/3$. This gives $A = 2t_1 + 1 = 3/2$ or $t_1 = t_2 = 1/4$. The parabola

$$y = \frac{2}{3}\left(x + \frac{1}{2}\right)^2 - \frac{1}{6} + \frac{1}{24} = \frac{2}{3}\left(x + \frac{1}{2}\right)^2 - \frac{1}{8}$$

passes through the point $(1/4, 1/4)$ with slope $y' = 1$. The parabola

$$x = \frac{2}{3}\left(y + \frac{1}{2}\right)^2 - \frac{1}{8} = \frac{1}{24}\left[4(2y+1)^2 - 3\right]$$

has upper half given by the graph of

$$y = -\frac{1}{2} + \frac{1}{4}\sqrt{24x + 3}.$$

It is easy to see that this graph also passes through $(1/4, 1/4)$ and also has slope $y' = 1$ at that point. The tangency corresponding to $\alpha = 2/3$ is indicated on the left in Figure 5, and this one can be plotted by hand without too much trouble.

The remaining root is $\alpha = 3/2$. This gives a second "external" tangency of the parabolas. Here we have $t_1 = t_2 = -1/6$. The parabola with vertical axis is

$$y = \frac{3}{2}\left(x + \frac{1}{2}\right)^2 - \frac{3}{8} + \frac{1}{24} = \frac{3}{2}\left(x + \frac{1}{2}\right)^2 - \frac{1}{3}.$$

passes through the point $(-1/6, -1/6)$. Again, the slope is $y' = 1$. It is also natural to simply consider the parabola

$$x = \frac{3}{2}\left(y + \frac{1}{2}\right)^2 - \frac{1}{3},$$

with respect to the reversed y, x-coordinates. (You just turn your head sideways to do this. In reversed y, x-coordinates this parabola passes through $(y, x) = (-1/6, -1/6)$ with slope $x' = 1$, and this is, of course, a coincident tangency with the first parabola. The result is shown on the right in Figure 5.

I also made an animation of the two parabolas plotted together as the parameter α moves.

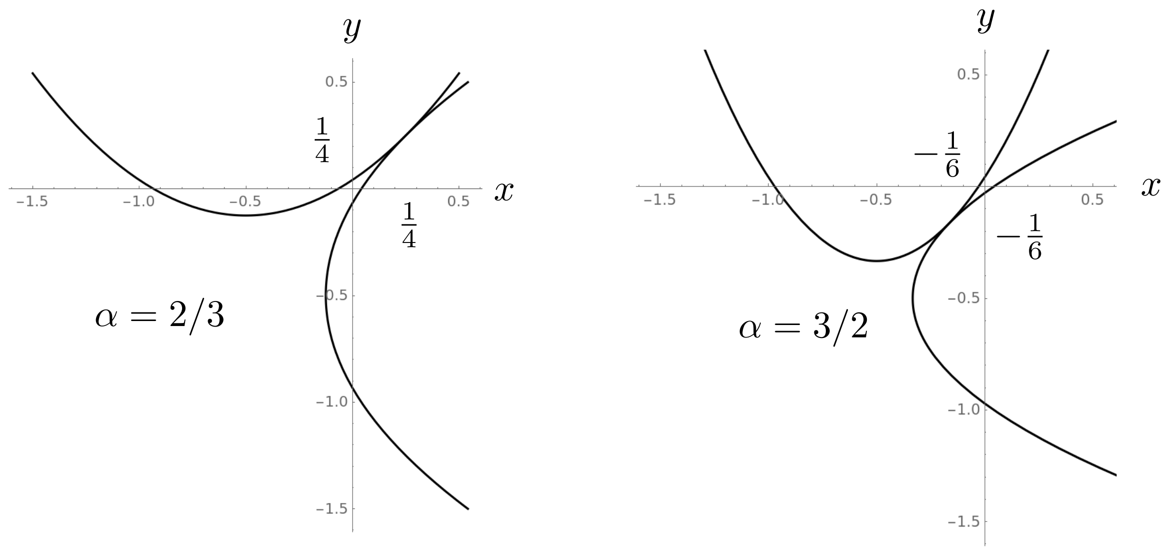

Figure 5: The parabolas corresponding to $\alpha = 2/3$ (left) and $\alpha = 3/2$ (right).