Putnam study meeting notes
Tuesday November 22, 2022

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We didn't have too many people at this meeting, but we solved a couple pretty nice A2 problems. I went over my solution of A2 (2007) which is in the November 15 notes.

1 A2 (2009)

Assume f, g, h are differentiable functions defined in some open interval containing $x = 0$ and satisfy the (ordinary differential) equations and intitial conditions

$$\begin{cases} f' = 2f^2gh + \dfrac{1}{gh}, & f(0) = 1 \\[2ex] g' = fg^2h + \dfrac{4}{fh}, & g(0) = 1 \\[2ex] h' = 3fgh^2 + \dfrac{1}{fg}, & h(0) = 1. \end{cases} \tag{1}$$

Find an explicit formula for $f = f(x)$ in some open interval containing $x = 0$.

Preliminary comment: This is an autonomous system of ordinary differential equations (ODEs) of precisely the type one sees and considers in any standard course on ODEs. Though one should know many things about such a system, I'm not sure those things help one very much with this problem. In the end, the solution below is just sort of seeing a kind of "trick" and then doing some relatively easy integration. In that sense, the problem is a little dissappointing.

Solution: I think I made the initial observation that the equations can be rewritten as

$$\begin{aligned} f'gh &= 2f^2g^2h^2 + 1 \\ fg'h &= f^2g^2h^2 + 4 \\ fgh' &= 3f^2g^2h^2 + 1. \end{aligned}$$

Thus, addng the three equations, gives an equation for the single (product) function $p = fgh$:

$$p' = (fgh)' = 6(p^2 + 1).$$

This equation is separable:

$$\frac{p'}{p^2 + 1} = 6$$

so integrating from 0 to x gives

$$\tan^{-1} p - \tan^{-1}(1) = 6x \qquad \text{or} \qquad fgh = \tan(6x + \pi/4).$$

So this determines a formula for the product. The next observation was due to Anant: Substitute for the product gh in the first equation to get a non-autonomous ODE for f:

$$f'\frac{\tan(6x + \pi/4)}{f} = 2\tan^2(6x+\pi/4)+1 \qquad \text{or} \qquad (\ln f)' = 2\tan(6x+\pi/4)+\cot(6x+\pi/4).$$

Again, integrating from 0 to x, we get

$$\begin{aligned} \ln f &= 2\int_0^x \frac{\sin(6t + \pi/4)}{\cos(6t + \pi/4)}\,dt + \int_0^x \frac{\cos(6t + \pi/4)}{\sin(6t + \pi/4)}\,dt \\ &= -\frac{1}{3}\left(\ln|\cos(6x + \pi/4)| - \ln\frac{\sqrt{2}}{2}\right) + \frac{1}{6}\left(\ln|\sin(6x + \pi/4)| - \ln\frac{\sqrt{2}}{2}\right) \\ &= \ln\frac{\sqrt[6]{\sin(6x + \pi/4)}}{\sqrt[3]{\cos(6x + \pi/4)}} + \ln\sqrt[6]{\frac{\sqrt{2}}{2}}. \end{aligned}$$

Notice the absolute values can be removed in an open set containing zero where both

$$\sin(6x+\pi/4) > \frac{1}{2}\sin(\pi/4) = \frac{\sqrt{2}}{4} > 0 \qquad \text{and} \qquad \cos(6x+\pi/4) > \frac{1}{2}\cos(\pi/4) = \frac{\sqrt{2}}{4} > 0.$$

So, it's not necessarily so pretty, and maybe it can be dolled up in some way, but it does seem like this gives an explicit solution for $|x| < \pi/24$:

$$f(x) = \sqrt[6]{\frac{\sqrt{2}}{2}}\, \frac{\sqrt[6]{\sin(6x + \pi/4)}}{\sqrt[3]{\cos(6x + \pi/4)}} = \frac{1}{2^{1/12}} \left[\tan\left(6x + \frac{\pi}{4}\right) \sec\left(6x + \frac{\pi}{4}\right)\right]^{1/6}.$$

2 A2 (2010)

Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

I think Jonathan mostly solved this one, though I think Anant was at least pretty close to a solution and may have also had it. Jonathan's solution (which I remember) was this:

First notice that for all $x \in \mathbb{R}$,

$$f'(x+n) = \frac{f(x+2n) - f(x+n)}{n}.$$

Adding the expression for $f'(x)$ above gives

$$f'(x+n) + f'(x) = \frac{f(x+2n) - f(x)}{n} = 2f'(x).$$

Thus, $f'(x+n) = f'(x)$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Next, differentiate the expression for $f'(x)$, noting that the right side is definitely differentiable with respect to x for $n \in \mathbb{N}$ fixed:

$$f''(x) = \frac{f'(x+n) - f'(x)}{n} \equiv 0.$$

Thus, $f(x) = ax+b$ for some $a, b \in \mathbb{R}$. Any function of this form satisfies the condition because in this case

$$a \equiv f'(x) \qquad \text{and} \qquad \frac{f(x+n) - f(x)}{n} = \frac{a(x+n) + b - (ax+b)}{n} = \frac{an}{n}.$$

Follow up/note: Of course, it should be noted that the expression on the right in the given expression for the derivative is a **difference quotient**, and the mean value theorem gives that for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$ there exists some ξ with $x < \xi < x+n$ with

$$f'(x) = \frac{f(x+n) - f(x)}{n} = \frac{f'(\xi)(x+n\ -\ x)}{n} = f'(\xi).$$

I'm not sure how to use this to solve the problem, but the following may also be noted: The solution above does not (as least as far as I can see) lend itself to drawing pictures to, shall we say, "illustrate what is going on,", i.e., what drives the proof. There are good pictures associated with the mean value theorem and difference quotients in general. One guesses there might be a clever proof admitting illustration, but I do not know it.