Putnam study meeting notes
Tuesday November 1, 2022

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November 18, 2022

We worked on (and somebody was able to solve) two A1 problems. Some of us also tried part A of the 2004 exam on the previous Saturday October 29. I will try to put some notes on what I understood about those problems as well.

1 A1 (2004)

$S(N)$ is the number of successful free throws (in basketball for a particular player) out of the first N attempted free throws. If for some N_0 it is known that $S(N_0)$ is less than 80% of N_0 and for some $N_1 > N_0$ one has $S(N_1)$ is greater than 80% of N_1, determine if there must be some N for which $S(N)$ is precisely 80% of N.
Note: This was the practice exam we worked on for three hours (timed). Several of us got pretty much full credit solutions for this problem under simulated testing conditions.
Solution: For each $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ either $S(n + 1) = S(n)$ (if the $(n + 1)$-st free throw is missed) or $S(n + 1) = S(n) + 1$ (if the $(n + 1)$-st free throw is made). From this it follows that for $n \in \{2, 3, 4, \ldots\}$

$$S(n-1) = S(n) \qquad \textbf{or} \qquad S(n-1) = S(n) - 1.$$

Let n_0 be the minimum integer greater than N_0 for which

$$S(n_0) > \frac{4n_0}{5}.$$

Notice $N_1 \in \{N \geq N_0 : S(N) > 4N/5\}$, so n_0 is the minimum integer in this **nonempty** set. (Technically, we're using the greatest lower bound property of the natural numbers here.)

Note first that $n_0 \neq N_0$ because $S(N_0) < 4N_0/5$. Therefore,

$$n_0 > N_0.$$

In particular, $n_0 \in \{2, 3, 4, \ldots\}$. Also, since $S(n_0) > 4n_0/5 \geq 8/5$, we know $S(n_0) > 1$. (Incidentally, $n_0 > 2$ because if $n_0 = 2$, then we must have $N_0 = 1$ which means $S(N_0) = S(1) < 4/5$ so that $S(1) = 0$, and $S(2) \leq S(1) + 1 = 1 < 4n_0/5 = 8/5$. This contradicts the fact that $S(n_0) = S(2) > 4n_0/5$. Thus, $n_0 \in \{3, 4, 5, \ldots\}$ and $S(n_0) > 4n_0/5 \geq 12/5 > 2$. But we don't need any of this to solve the problem.)

Exercise 1 *What is the smallest possible value of n_0?*

I do need $n_0 > 1$ because I'm about to consider $S(n_0 - 1)$.

Second, it must be the case that

$$S(n_0 - 1) = S(n_0) - 1.$$

Otherwise,

$$S(n_0 - 1) = S(n_0) > \frac{4n_0}{5} > \frac{4(n_0 - 1)}{5}$$

and $n_0 - 1 \in \{N \geq N_0 : S(N) > 4N/5\}$ is smaller than the smallest element in the set, which is a contradiction (of the definition of n_0).

Finally, since $n_0 - 1 \geq N_0$, it must be the case that

$$S(n_0 - 1) \leq \frac{4(n_0 - 1)}{5}.$$

Again, the alternative gives $n_0 - 1 \in \{N \geq N_0 : S(N) > 4N/5\}$.

The inequality

$$S(n_0) > \frac{4n_0}{5}$$

implies

$$4n_0 < 5S(n_0).$$

The inequality

$$S(n_0 - 1) = S(n_0) - 1 \leq \frac{4(n_0 - 1)}{5}$$

implies

$$5S(n_0) \leq 4n_0 + 1.$$

Thus $5S(n_0)$ is a natural number for which

$$4n_0 < 5S(n_0) \leq 4n_0 + 1.$$

The only natural number between the natural numbers $4n_0$ and $4n_0+1$ (not including $4n_0$) is $4n_0 + 1$. Therefore,

$$5S(n_0) = 4n_0 + 1 \qquad \text{and} \qquad S(n_0 - 1) = S(n_0) - 1 = \frac{4n_0 + 1}{5} - 1 = \frac{4(n_0 - 1)}{5}.$$

Indeed for $N = n_0 - 1$ the value of $S(N)$ is exactly 80% of N.

2 A1 (2009)

Let f be a real-valued function defined on the plane such that for every square (with vertices) A, B, C, and D in the plane, there holds

$$f(A) + f(B) + f(C) + f(D) = 0.$$

Does it follow that $f(P) = 0$ for every point P in the plane?
Notes: I think Jonathan and Evan gave solutions for this. If I remember correctly, Jonathan's solution (the first I try to "recreate" below) was based on the observation that if a square with (conscutive) vertices A_1, A_2, A_3, and A_4 is divided evenly into four sub-squares so that the midpoint of side A_jA_{j+1} is P_j, for $j = 1, 2, 3$ then the sums of the values of two vertices on one of the rectangles must be the same as the sum of the values of the two oppsite vertices. For example,

$$f(A_1) + f(P_1) = f(P_3) + f(A_4).$$

From there (as I understood it) Jonathan wrote down various algebraic equations and determined that the opposite vertices of every square had to have opposite values: $f(A_1) = -f(A_3)$. But then also the center of the square P_0 must have value $-f(A_1)$ as well as $-f(A_3) = f(A_1)$. Thus, $f(P_0) = -f(P_0)$ for $f(P_0) = 0$, and since every point in the plane is the center of some square, one has $f \equiv 0$. We'll see if we can figure it out.

Evan (as I recall) did something like cycle the vertices and used that the midpoints of the sides P_1, P_2, P_3, and P_4 are also the vertices of a square.
First solution: Given any point P_0 in the plane construct around P_0 a square with vertices A_j, $j = 1, 2, 3, 4$ with center P_0 and side midpints P_j, $j = 1, 2, 3, 4$ as

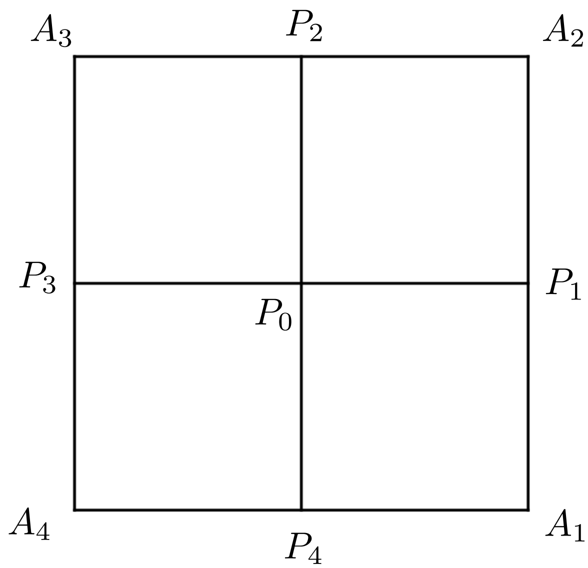

Figure 1: A square composed of four subsquares.

indicated in Figure 1. Since $f(P_4) + f(A_1) = -[f(P_0) + f(P_1)]$ and $f(P_2) + f(A_2) = -[f(P_0) + f(P_1)]$, we get

$$f(P_4) + f(A_1) = f(P_2) + f(A_2),$$

and a similar assertion holds for the opposite sides of each rectangle in the figure, at least with respect to the shorter sides of the rectangles with two short sides and two long sides. In particular,

$$f(P_2) + f(A_3) = f(P_4) + f(A_4).$$

It follows from these two equations that

$$f(A_2) - f(A_3) = f(A_1) - f(A_4) \qquad \text{or} \qquad f(A_1) - f(A_2) + f(A_3) - f(A_4) = 0.$$

Combining this last equation with the basic equation $f(A_1)+f(A_2)+f(A_3)+f(A_4) = 0$ gives

$$2[f(A_1) + f(A_3)] = 0.$$

So this indeed tells us the value at opposite vertices of a square are the negatives of each other. Applying this consecutively to A_1 and P_0, and then to P_0 and A_3 gives

$f(P_0) = -f(A_1)$ and $f(P_0) = -f(A_3)$. But also $f(A_1) = -f(A_3)$, so

$$f(P_0) = -f(P_0) \qquad \text{or} \qquad f(P_0) = 0.$$

Second solution: Proceeding as above, about any point P_0 in the plane construct around P_0 the same square. Consider the four subsquares consecutively along with the fact that P_1, P_2, P_3, and P_4 are the vertices of a square:

$$\begin{aligned} f(A_1) + f(P_1) + f(P_0) + f(P_4) = 0 \\ f(A_2) + f(P_2) + f(P_0) + f(P_1) = 0 \\ f(A_3) + f(P_3) + f(P_0) + f(P_2) = 0 \\ f(A_4) + f(P_4) + f(P_0) + f(P_3) = 0. \end{aligned}$$

Adding these four equations gives

$$\sum_{j=1}^{4} f(A_j) + 4f(P_0) + \sum_{j=1}^{4} f(P_j) = 0.$$

Since

$$\sum_{j=1}^{4} f(A_j) = \sum_{j=1}^{4} f(P_j) = 0,$$

we get $4f(P_0) = 0$ and $f(P - 0) = 0$.

3 A1 (2012)

Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.
Notes and preliminary comments: Lawrence gave the solution for this one. The solution was based on a couple important facts about triangles that are probably worth keeping in mind.

First of all, three positive numbers a, b, and c are the sides of some triangle if and only if they satisfy the "triangle inequalities"

$$\begin{aligned} a &< b + c \\ b &< a + c \\ c &< a + b. \end{aligned} \tag{1}$$

Second the angle C opposite the side of length c is acute, i.e., of angle measure strictly less than $\pi/2$, if and only if

$$c^2 < a^2 + b^2.$$

This is an implication of the law of cosines which says

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Thus, one can always solve for $\cos C$ to get

$$\cos C = \frac{1}{2ab}[c^2 - (a^2 + b^2)].$$

Since $0 < C < \pi$ and $\cos C$ decreases from 1 to -1 on this interval with $\cos C > 0$ if and only if $0 < C < \pi/2$, the implication follows. Thus, and acute triangle, in which all the angles are acute, has side lengths satisfying

$$\begin{aligned} a^2 &< b^2 + c^2 \\ b^2 &< a^2 + c^2 \\ c^2 &< a^2 + b^2, \end{aligned} \tag{2}$$

and conversely, three positive numbers a, b, and c satisfying (1) and (2) are the sides of an acute triangle.

Lawrence made a couple other basic and important observations, but I'll save those for the solution.

Solution: By reordering one may assume $d_1 \leq d_2 \leq \cdots \leq d_{12}$. I believe Joseph suggested these inequalities can be taken as strict. I think that is not the case, however. The statement says to find "distinct" indices, but does not say the numbers are distinct. In any case, Lawrence asserts that the desired side lengths a, b, and c must be three consecutive numbers from this list:

$$a = d_j, \quad b = d_{j+1}, \quad \text{and} \quad c = d_{j+2} \qquad \text{for some } j \in \{1, 2, \ldots, 12\}.$$

Thus, he starts with the first three $d_1 \leq d_2 \leq d_3$ and assumes these three lengths are **not** the side lengths of an acute triangle. A first crucial observation at this point is that if one of the triangle inequalies (1) is violated then the "corresponding" acuteness[1] inequality must also be violated. For example, if $a \geq b + c$, then

$$a^2 \geq (b + c)^2 = b^2 + 2bc + c^2 > b^2 + c^2.$$

[1]Isn't this a-cute word? Although. . . it may not be a word.

As a consequence, if one can show all the acuteness inequalities hold for some a, b, and c, then the triangle inequalities will necessarily also hold. Next, if the three numbers a, b, and c are ordered $a \leq b \leq c$, and there is a violation of one of the acuteness inequalities, then one can assume it is the inequality

$$c^2 < a^2 + b^2$$

that is violated. This is because the other two $a^2 < b^2+c^2$ and $b^2 < a^2+c^2$ necessarily follow from the assumption $a^2 \leq c^2$ and $b^2 \leq c^2$.

With this in mind, if no three consecutive numbers

$$a = d_j, \quad b = d_{j+1}, \quad \text{and} \quad c = d_{j+2}$$

from the list are the sides of an acute triangle, then one must have

$$\begin{aligned}
d_3^2 &\geq d_1^2 + d_2^2 > 1^2 + 1^2 = 2,\\
d_4^2 &\geq d_2^2 + d_3^2 > 1^2 + 2 = 3,\\
d_5^2 &\geq d_3^2 + d_4^2 > 2 + 3 = 5,\\
d_6^2 &\geq d_4^2 + d_5^2 > 3 + 5 = 8,\\
d_7^2 &\geq d_5^2 + d_6^2 > 13,\\
d_8^2 &\geq d_6^2 + d_7^2 > 21,\\
d_9^2 &\geq d_7^2 + d_8^2 > 34,\\
d_{10}^2 &\geq d_8^2 + d_9^2 > 55,\\
d_{11}^2 &\geq d_9^2 + d_{10}^2 > 89,\\
d_{12}^2 &\geq d_{10}^2 + d_{11}^2 > 144.
\end{aligned}$$

Thus, $d_{12} > 12$ which is a contradiction. Notice the estimates for the sequence of squares are Fibonacci numbers.

4 A2 (2004)

If triangles T_1 and T_2 have side lengths a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then do the conditions

1. Triangle T_2 is an acute triangle, and
2. $a_1 \leq a_2$, $b_1 \leq b_2$, and $c_1 \leq c_2$,

imply the area of triangle T_2 is as least as great as the area of triangle T_1?

Note: This seemed to be a more difficult problem. I'm not sure any of us got a full credit solution during the actual practice exam. I think the solution below is complete and correct, but I thought about it for several days. I did have most of the basic ideas during the three hours of the practice test, but I didn't really get them written down properly...at all. And some of the ideas I had were not fully developed or incorrect.

I will use the basic facts discussed in the preliminary discussion of the previous problem.

Solution: The answer is "yes."

We can work with certain coordinate triangles having the same areas as triangles T_1 and T_2. Say T is any triangle with side lengths a, b and c. There is a unique triangle having the same area as that of T and satisfying the following properties:

1. The vertices of T are $B = (0, 0)$, $C = (a, 0)$, and $A = (x, y)$ where
2. $y > 0$, and
3. (x, y) is the **unique intersection** of the circle $x^2 + y^2 = c^2$ (with center $B = (0, 0)$ and radius c) and the circle $(x - a)^2 + y^2 = b^2$ (with center $C = (a, 0)$ and radius b) **in the first quadrant**.

Let us call such a triangle a "coordinatized representative."

Before we consider coordinatized representatives of T_1 and T_2 let us note that the numbers a_1/a_2, b_1/b_2, and c_1/c_2 satisfy

$$0 < \frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2} \leq 1.$$

Furthermore, we can put these numbers in some (weakly) ascending order. By renaming the corresponding sides if necessary, we may assume

$$0 < \frac{c_1}{c_2} \leq \frac{b_1}{b_2} \leq \frac{a_1}{a_2} \leq 1. \tag{3}$$

My plan is to execute a series of reductions to nominally simpler cases. The first reduction is obtained by scaling all sides of triangle T_2 by the ratio a_1/a_2. In this way we obtain a triangle T_3 with side lengths

$$a_3 = \frac{a_1}{a_2} a_2 = a_1, \qquad b_3 = \frac{a_1}{a_2} b_2, \qquad \text{and} \qquad c_3 = \frac{a_1}{a_2} c_2.$$

It is very easy to check that the lengths a_3, b_3 and c_3 satisfy the triangle inequalities and thus are indeed the sides of a triangle. In fact, both the triangle inequalities and the acuteness inequalities are invariant under uniform scaling. For example,

$$a_3 = \frac{a_1}{a_2} a_2 \le \frac{a_1}{a_2}(b_2 + c_2) = b_3 + c_3,$$

and

$$a_3^2 = \left(\frac{a_1}{a_2}\right)^2 a_2^2 \le \left(\frac{a_1}{a_2}\right)^2 (b_2^2 + c_2^2) = b_3^2 + c_3^2.$$

In point of fact, I don't think we will need the angle opposite the side of length a, once we have ordered the sides according to the ratios as in (3), but we'll see about that as we go along.

The traanlge T_3 also satisfies

$$\text{area}(T_3) \le \text{area}(T_2).$$

In fact, T_3 is similar to T_2; one can check that the angles are the same using the law of cosines, and the area is half the product of two sides and the sine of the included angle. Thus,

$$\text{area}(T_3) = \left(\frac{a_1}{a_2}\right)^2 \text{area}(T_2) \le \text{area}(T_2) \qquad \text{to be exact.}$$

Thus, if we are able to show $\text{area}(T_3) \ge \text{area}(T_1)$, then we are done. Notice now that triangle T_3 has a side length $a_3 = a_1$ in common with the side of length a_1 of triangle T_1. Furthermore, the remaining two sides satisfy

$$b_3 = \frac{a_1}{a_2} b_2 \le b_2 \qquad \text{and} \qquad c_3 = \frac{a_1}{a_2} c_2 \le c_2.$$

We also have information on the ordering of the new ratios c_1/c_3, b_1/b_3, and $a_1/a_3 = 1$. Namely,

$$0 < \frac{c_1}{c_3} \le \frac{b_1}{b_3} \le \frac{a_1}{a_3} = 1.$$

This is because

$$\frac{c_1}{c_3} = \frac{a_2}{a_1}\frac{c_1}{c_2} \le \frac{a_2}{a_1}\frac{b_1}{b_2} = \frac{b_1}{b_3} \le \frac{a_2}{a_1}\frac{a_1}{a_2} = \frac{a_1}{a_3} = 1.$$

We are now in a position to execute our second reduction. We consider the three numbers

$$a_4 = a_3 = a_1, \qquad b_4 = \frac{b_1}{b_3}b_3 = b_1, \qquad \text{and} \qquad c_4 = \frac{b_1}{b_3}c_3.$$

First we should check to see if these three numbers are the side lengths of a triangle. In fact,

$$c_4 = \frac{b_1}{b_3}c_3 \le \frac{b_1}{b_3}(a_3 + b_3) = \frac{b_1}{b_3}a_3 + b_4 \le a_4 + b_4.$$

$$b_4 = \frac{b_1}{b_3}b_3 \le \frac{b_1}{b_3}(a_3 + c_3) = \frac{b_1}{b_3}a_3 + c_4 \le a_4 + c_4,$$

and

$$a_4 = a_1 \le b_1 + c_1 = b_4 + \frac{c_1}{c_3}c_3 \le b_4 + \frac{b_1}{b_3}c_3 = b_4 + c_4.$$

So we have a triangle T_4. We wish to show next that $\text{area}(T_4) \le \text{area}(T_3)$. Then we will have

$$\text{area}(T_4) \le \text{area}(T_3) \le \text{area}(T_2)$$

and our task will be reduced to showing $\text{area}(T_1) \le \text{area}(T_4)$ where T_4 is a triangle having side lengths $a_4 = a_1$, $b_4 = b_1$ (i.e., two side lengths in common with the corresponding side lengths of T_1) and

$$c_4 = \frac{b_1}{b_3}c_3 = \frac{b_1}{b_2}\frac{a_2}{a_1}\ \frac{a_1}{a_2}c_2 = \frac{b_1}{b_2}c_2 \ge \frac{c_1}{c_2}c_2 = c_1.$$

The area of T_4 is given by

$$\text{area}(T_4) = \frac{1}{2}a_4 b_4 \sin(C_4) = \frac{1}{2}a_3 b_4 \sin(C_4) \le \frac{1}{2}a_3 b_3 \sin(C_4)$$

where C_4 is the angle opposite the side of length c_4 in triangle T_4. Note that we have used

$$b_4 = \frac{b_1}{b_3}b_3 \le b_3 \qquad \text{since} \qquad \frac{b_1}{b_3} \le 1.$$

The law of cosines gives

$$\begin{aligned}\cos(C_4) &= \frac{1}{2a_4b_4}[a_4^2 + b_4^2 - c_4^2] \\ &= \frac{1}{2(b_1/b_3)a_3b_3}[(a_3^2 + (b_1/b_3)^2 b_3^2 - (b_1/b_3)^2 c_3^2] \\ &= \frac{1}{2a_3b_3}\left[\frac{a_3^2}{b_1/b_3} + (b_3^2 - c_3^2)(b_1/b_3)\right]. \end{aligned} \tag{4}$$

At this point we use that the angle opposite the side of length b is acute which tells us

$$b_3^2 < a_3^2 + c_3^2 \qquad \text{or} \qquad a_3^2 > b_3^2 - c_3^2.$$

Therefore,

$$a_3^2 > b_3^2 - c_3^2 \geq (b_1/b_3)(b_3^2 - c_3^2).$$

This implies the following string of inequalities:

$$\begin{aligned} a_3^2\left(1 - \frac{b_1}{b_3}\right) &\geq (b_3^2 - c_3^2)\frac{b_1}{b_3}\left(1 - \frac{b_1}{b_3}\right) \\ \left(\frac{b_1}{b_3}\right)^2 (b_3^2 - c_3^2) &\geq \frac{b_1}{b_3}a_3^2 + \frac{b_1}{b_3}(b_3^2 - c_3^2) \\ \frac{a_3^2}{b_1/b_3} + (b_1/b_3)(b_3^2 - c_3^2) &\geq a_3^2 + b_3^2 - c_3^2 \\ \frac{1}{2a_3b_3}\left[\frac{a_3^2}{b_1/b_3} + (b_1/b_3)(b_3^2 - c_3^2)\right] &\geq \frac{1}{2a_3b_3}[a_3^2 + b_3^2 - c_3^2] \\ \cos(C_4) \geq \cos(C_3) \geq 0. \end{aligned}$$

where $C_3 = C_2$ is the angle in similar triangles T_3 and T_2 opposite corresponding sides (of lengths) c_3 and c_2 respectively. Notice strict inequality maintains in this string of inequalities unless we lose everything in the first inequality because $b_1/b_3 = 1$. Weak inequality is, of course, okay. The last inequality uses (4).

Since we are assuming angle C_2 is acute, we conclude

$$C_4 \leq C_3 = C_2 < \frac{\pi}{2}.$$

It follows then that

$$\sin(C_4) \leq \sin(C_3)$$

and

$$\text{area}(T_4) \leq \frac{1}{2}a_3 b_3 \sin(C_3) = \text{area}(T_3)$$

as desired.

Our final claim then is that if two triangles T_1 and T_4 have sides a_1, b_1, c_1 and a_4, b_4, and c_4 respectively satisfying

$$a_1 = a_4, \quad b_1 = b_4, \quad \text{and} \quad c_1 \leq c_4,$$

and the angle C_4 in triangle T_4 opposite the side of length c_4 is acute, then

$$\text{area}(T_1) \leq \text{area}(T_2).$$

But this is now pretty straightforward because

$$\text{area}(T_1) = \frac{1}{2}a_1 b_1 \sin(C_1) = \frac{1}{2}a_4 b_4 \sin(C_1),$$

and

$$\cos(C_1) = \frac{1}{2a_1b_1}(a_1^2 + b_1^2 - c_1^2) = \frac{1}{2a_4b_4}(a_4^2 + b_4^2 - c_1^2) \geq \frac{1}{2a_4b_4}(a_4^2 + b_4^2 - c_4^2) = \cos(C_4).$$

In particular, $0 < \cos(C_4) \leq \cos(C_1) < 1$, and this means $0 < C_1 \leq C_4 < \pi/2$, angle C_1 is also acute, $\sin(C_1) \leq \sin(C_4)$, and

$$\text{area}(T_1) \leq \text{area}(T_4) \leq \text{area}(T_3) \leq \text{area}(T_2). \qquad \square$$

Follow up/what we used: One thing we didn't use (really) was the coordinatized versions of the triangles. Nevertheless, such a construction can be useful for other purposes, as I hope to demonstrate below. We also didn't use the fact that the angle opposite the side of length a_2 in triangle T_2 was acute. More precisely, given the ordering (which we assumed by rearrangement)

$$0 < \frac{c_1}{c_2} \leq \frac{b_1}{b_2} \leq \frac{a_1}{a_2} \leq 1,$$

we needed the angles C_2 and B_2 opposite the sides (of length) c_2 and b_2 respectively to be acute. No assumption about the angle A_2 opposite the side (of length) a_2 came up.

The string of inequalities in the reduction to consideration of traingle T_4 was difficult (for me) to find, and I found it based on somewhat more general considerations. I will try to describe those considerations now.

The general question may be pharsed as follows: What happens when you scale two sides of a triangle and leave one side fixed? In our case, with reference to the coordinatized triangles suggested at the beginning of my solution (and not used) we can fix the horizontal side of length a and scale the other two sides of lengths b and c as indicated in Figure 2. I was at first primarily interested in scaling the sides (of

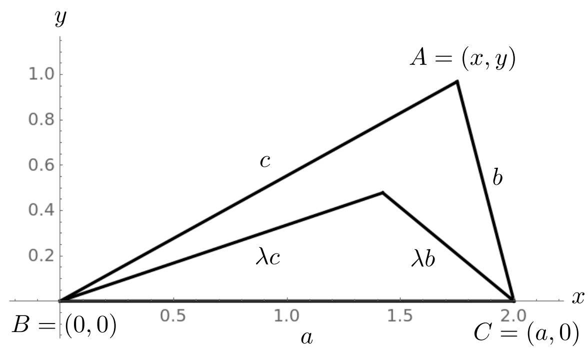

Figure 2: Scaling two sides of a triangle. In this illustration $a = 2$, $b = 1$, $c = 2$, and $\lambda = 1/2$, so the triangle with scaled sides has sides of length $a = 2$, $\lambda b = 1/2$, and $c = 1$.

length) b and c by a scalar $\lambda < 1$. It turns out that, in some way, this actually made the problem more difficult for me as I'll explain below. In any case, I imagined that at least for λ close to 1 you get some other triangle, and that the new angle at vertex C will be smaller as indicated in the figure. And for $\lambda = 1$ you get the same triangle. This turns out to be (basically) correct. But my question was: What conditions on a, b, and c (and λ) are required to ensure the new angle at vertex C is no larger than the original angle C? Put another way, I want to know when the new vertex A lies in the sector determined by sides a and b.

As an aside, I suppose (hope?) you all remember from a good course in Euclidean geometry that it's sort of standard notation to label the sides of a triangle a, b, and c and the respective angles opposite those sides A, B, and C. It's been so long ago that I can't remember if it is standard to denote the corresponding side lengths a, b, and c as well, but I sort of think it was. I guess I could check my Euclidean geometry book I have from highschool—which, yes, I still have, or I could even check Euclid which I have somewhere translated in two or three volumes from Dover press, though I don't seem to see either one on my bookshelf at the moment. Also, I'm not sure Euclid was using what has become standard notation exactly. In any case, I do remember that it was a bit nonstandard to use A, B, and C to denote both the vertices as points (either abstractly as Euclid would have done or in coordinates as Descartes started to do) as well as for the angle measure of the angles at the corresponding vertices. We used something cumbersome like $m\angle A$ to mean the angle measure of the angle at vertex A. I don't see that any great problem arises in using A for $m\angle A$, B for $m\angle B$, etc., so I'm going to (continue to) do that.

My thinking on this problem (as it progressed) is a little bit embarrassing, but I'm not especially proud and it may be instructive for me to describe it, so I'm going to try. None of you seem to have figured it out either. . . or maybe you are all like Lawrence and just wouldn't say. In any case, when I first thought about scaling these two sides I thought the new top vertex would move along a straight line to a special point on side a. In fact (and this is rather embarrassing) I thought that point would tend to the midpoint of side a. I even told Joseph this on the day we did the practice exam. Of course that can't be the case. The sides (of length) λb and λc keep the same ratio, so when the last vertex reaches side a the sides (of length) $\lambda_{\min} b$ and $\lambda_{\min} c$ must still have the same ratio. That ratio is $1 : 1$ for the midpoint, so clearly the new vertex A does not travel along the median. In fact, one must have

$$\lambda_{\min} b + \lambda_{\min} c = \lambda_{\min}(b + c) = a,$$

and this means

$$\lambda_{\min} = \frac{a}{b+c}, \qquad b_{\min} = \frac{ab}{b+c}, \qquad \text{and} \qquad c_{\min} = \frac{ac}{b+c}.$$

Obviously, if $b \neq c$, then $b_{\min} \neq c_{\min}$. This point $A_{\min} = (c_{\min}, 0)$, however, is some kind of special point, and I persisted in imagining the top vertex would move along a line to this point as λ decreased from 1 to $\lambda_{\min}$. After spending a bit of time trying to prove the triangles with vertices along the median keep the same ratio of lengths (which was ridiculous because they would then end up with ratio $1 : 1$ when $\lambda = \lambda_{\min}$) I tried to prove the ratio would remain the same for points moving along the line from A to $A_{\min} = (c_{\min}, 0)$. Fortunately, I couldn't prove this either, because it's not true.

My son pointed out that it makes good sense to consider $\lambda > 1$, and he somehow immediately intuited that the path of the top vertex would have to "arch over" and come back down to the x-axis outside the side a. I'm not sure how he "saw" this right away because he does not seem exceptionally skilled at writing down and working with ratios. But he was totally correct. Except in the isosceles case in which $b = c$ where the top vertex does indeed travel down the (vertical) median (to the midpoint of side a) as λ decreases from $\lambda = 1$ to $\lambda = \lambda_{\min} = a/(2b)$ and runs straight up the common altitude line as λ increases from $\lambda = 1$ to $+\infty$, the difference between the sides of length λb and λc will grow proportional to the scaling by $\lambda > 1$ and eventually become "too big" forcing the new top vertex back down to the x axis. To see this it is natural to consider cases divided by the isosceles case: If $b < c$ as in Figure 1, then $\lambda b < \lambda c$. One of the triangle inequalities says $c < a + b$ or $c - b < a$, but for these triangles with scaled sides

$$\lambda c - \lambda b = \lambda(c - b)$$

and eventually for λ large enough it will no longer be the case that

$$\lambda c - \lambda b < a \qquad \text{or} \qquad \lambda c < a + \lambda b.$$

Let's consider this "motion" a little more systematically. We start with the three triangle inequalities

$$\begin{aligned} a &< b + c \\ b &< a + c \\ c &< a + b. \end{aligned}$$

The new side lengths a, λb, and λc will be the sides of a triangle if and only if

$$\begin{aligned} a &< \lambda(b + c) \\ \lambda b &< a + \lambda c \\ \lambda c &< a + \lambda b. \end{aligned} \tag{5}$$

The first inequality is violated at $\lambda_{\min} = b/(b + c) > 0$. And for all λ with

$$\lambda_{\min} = \frac{a}{b + c} < \lambda \leq 1, \tag{6}$$

all three inequalities hold. In fact, (6) gives the first inequality immediately. Since $c - b > 0$,

$$-a < 0 < \lambda(c - b) \qquad \text{for all } \lambda > 0$$

so the second inequality always holds. The third inequality is equivalent to

$$\lambda(c - b) < a$$

and holds precisely when

$$0 < \lambda < \lambda_{\max} = \frac{a}{c-b}.$$

Note that since $b > 0$ and $c - b > 0$ in this case we get

$$\frac{a}{b+c} < \frac{a}{c-b}.$$

Thus, for all three inequalities in (5) to hold we need precisely that

$$\lambda_{\min} = \frac{a}{b+c} < \lambda < \frac{a}{c-b} = \lambda_{\max}. \tag{7}$$

Note finally, that since $a < b + c$ and $c - b < a$, the value $\lambda = 1$ is in this interval. In this way, we have a well-defined (or at least better defined) problem to consider: What happens to the angle $\phi = C(\lambda)$ determined by the law of cosines

$$(\lambda c)^2 = a^2 + (\lambda b)^2 - 2a(\lambda b)\cos\phi \qquad \text{or} \qquad \cos\phi = \frac{1}{2a(\lambda b)}\left[a^2 + (\lambda b)^2 - (\lambda c)^2\right] \tag{8}$$

as λ increases along the interval (7) from $\lambda_{\min}$ to $\lambda_{\max}$?

Exercise 2 *Consider the case $b > c$. Determine which values of the scaling parameter $\lambda > 0$ correspond to triangles with side lengths a, λb, and λc and formulate the appropriate question about the angle $\phi = C(\lambda)$ at the vertex C in this case.*

Returning to the case $b < c$ as indicated in Figure 2, note that near $\lambda = \lambda_{\max}$ a triangle is determined with the side (of length) λc very nearly too large, and larger than a in particular as indicated on the right in Figure 3. This confirms my son's intuition that the point $A = A(\lambda)$ must "arch over" and come back down to the axis outside side a. In this case with $b < a$, $A(\lambda)$ comes down to the right of side a as $\lambda \nearrow \lambda_{\max}$.

You can check that if $b > c$, then as λ tends to $\lambda_{\max} = a/(b-c)$, the point $A(\lambda)$ will "arch over" and come back down to the x-axis to the left of side a, or to the left of the origin. Precisely, the limiting point will be $(-\lambda_{\max}c, 0) = (-ac/(b-c), 0)$.

Let's continue, however, with the case $b < c$. What happens to the angle? Well, we can write the expression for the cosine of the angle at A given in (8) as

$$\cos\phi = \frac{1}{2ab}\left[\frac{a^2}{\lambda} + \lambda(b^2 - c^2)\right].$$

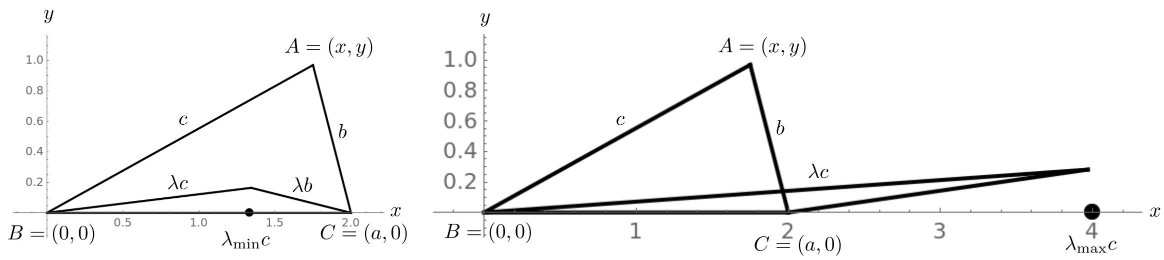

Figure 3: Extremeties of scaling two sides of a triangle when $b < c$. In this illustration $a = 2$, $b = 1$, $c = 2$ as in Figure 1. The extreme values for λ are $\lambda_{\min} = 2/3$ (left) and $\lambda_{\max} = 2$ (right) so the limiting value for $A(\lambda) = (x(\lambda), y(\lambda)$ when λ is close to $\lambda_{\min}$ is $((2/3)a, 0) = (4/3, 0)$ and the limiting value for $A(\lambda) = (x(\lambda), y(\lambda)$ when λ is close to $\lambda_{\max}$ is $(2a, 0) = (4, 0)$.

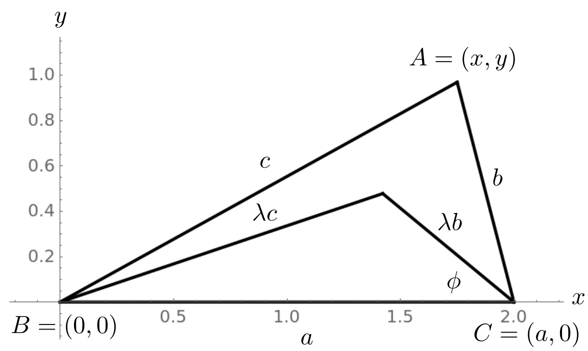

Figure 4: The angle at C when $b < c$.

Note then that

$$2ab\frac{d}{d\lambda}\cos\phi = -\frac{a^2}{\lambda^2} + b^2 - c^2. \tag{9}$$

This doesn't look too difficult to analyze. In Figure 4 I've reproduced Figure 1 with the angle ϕ labeled. The "motion" suggested in Figures 4 and 3 seems like it might have ϕ increasing as a function of λ from $\phi = 0$ when $\lambda = \lambda_{\min}$ to $\phi = \pi$ when $\lambda = \lambda_{\max}$, so let's see if we can prove this. Recall that the graph of cosine is decreasing from $\cos\phi = 1$ to $\cos\phi = -1$ as ϕ increases from $\phi = 0$ to $\phi = \pi$ as indicated in Figure 5, so this means we expect the quantity on the right in (9) to be negative (always). This

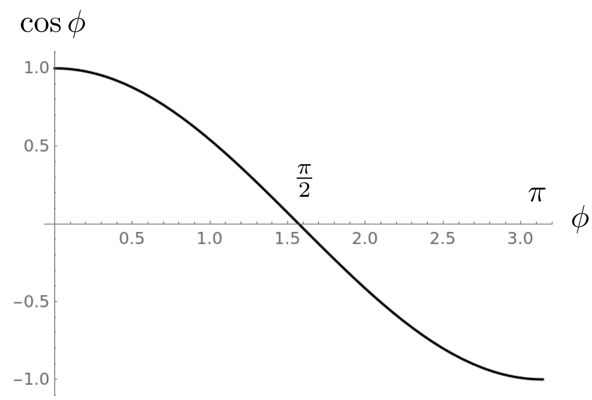

Figure 5: The graph of cosine.

will hold if

$$\frac{a^2}{\lambda^2} + c^2 > b^2 \qquad \text{for} \qquad \frac{a}{b+c} < \lambda < \frac{a}{c-b}.$$

In fact, on this interval we have

$$\frac{a^2}{\lambda^2} + c^2 > \frac{a^2}{a^2/(c-b)^2} + c^2 = (c-b)^2 + c^2 = 2c(c-b) + b^2 > b^2,$$

so indeed, $\cos\phi$ decreases for all values of λ on the interval of interest $(\lambda_{\min}, \lambda_{\max})$ and ϕ increases on the same interval. In particular, the angle ϕ will always be smaller than the original angle A when $\lambda < 1$ as long as a triangle is formed, that is as long as

$$\lambda_{\min} = \frac{a}{b+c} < \lambda < 1.$$

Notice this argument does not say (or require) anything particular about the angles in the original triangle being acute. The condition $b < c$ which we have used to define a particular case, however, does imply $b^2 < a^2 + c^2$ which means the angle at $B = (0,0)$ in this case must be acute. We should expect some kind of restriction on the angle at $B = (0,0)$ to come in explicitly, and that must happen in the other case $b > c$ where such a thing is possible. Note that in this other case, $b > c$, we have $c^2 < a^2 + b^2$, so the angle at $C = (a,0)$ is defacto required to be acute in this case.

Before we move on to the case $b < c$, there are three more interesting things we might do in the case $b > c$:

1. Plot ϕ and $\cos\phi$ as a functions of λ on the interval $(\lambda_{\min}, \lambda_{\max})$.
2. See if we can verify the fact that $\cos\phi$ is decreasing using direct inequalities instead of calculus. (This is sort of important because it is how I got the string of inequalities in the solution above.)
3. Plot the path of the motion of the point $A = A(\lambda) = (x(\lambda), y(\lambda))$ and see just what this "arching over" curve looks like.

I'm going to do two of those now. Figure 6 gives the plots of $\cos\phi$ and $\sin\phi$ in our example case with $a = c = 2$ and $b = 1$. As should be expected, there is a special

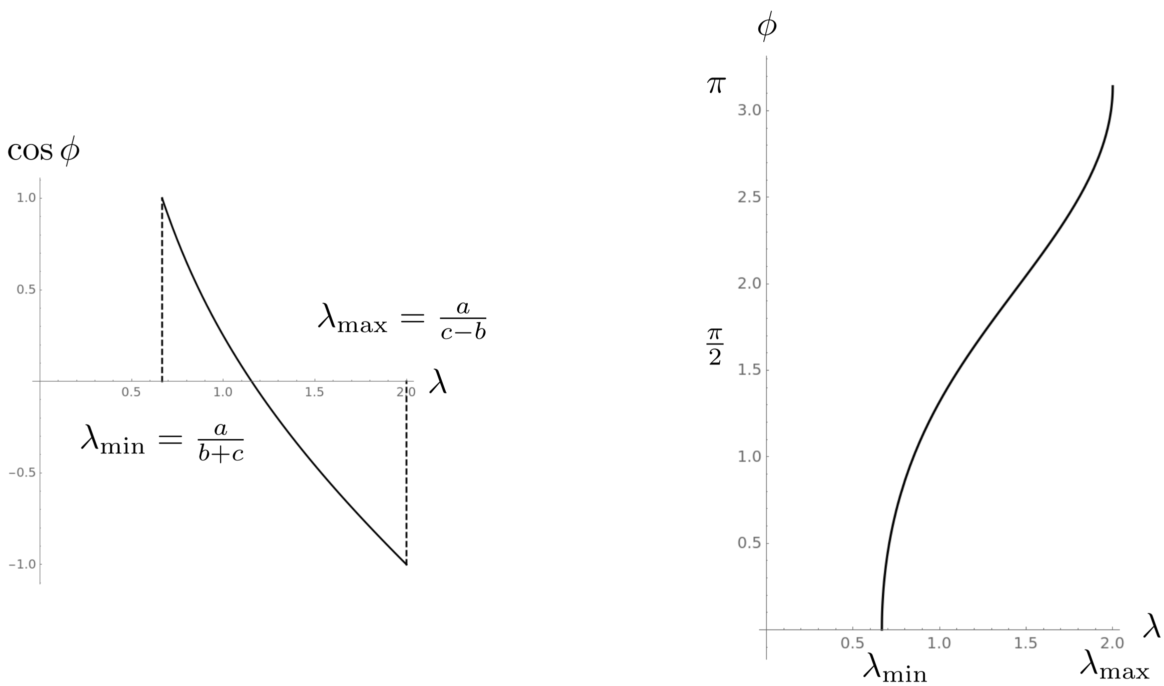

Figure 6: $\cos\phi$ and ϕ as functions of λ when $a = 2$, $b = 1$ and $c = 2$.

value of λ where the angle at $C = C(\lambda)$ becomes $\pi/2$ and the point $A = (x(\lambda), y(\lambda)$ is directly above $C = (a, 0)$. This value $\lambda = \lambda_c$ is indicated on the left in Figure 6 where the graph crosses the λ-axis and is given explicitly from the condition

$$a^2 + (\lambda_c b)^2 = (\lambda_c c)^2 \qquad \text{or} \qquad \lambda_c = \frac{a}{\sqrt{c^2 - b^2}}.$$

In general, the value of λ_c may be less than $\lambda = 1$ or greater than $\lambda = 1$ depending on the initial angle at A.

We didn't check the monotonicity of the angle at the "top" point $A(\lambda) = (x(\lambda), y(\lambda))$, but its limiting values (from the figures at least) are π at $\lambda = \lambda_{\min}$ and 0 at $\lambda = \lambda_{\max}$, so there should be at least one value for which this angle is $\pi/2$, and in fact there is a unique such value $\lambda = \lambda_a$ determined by

$$a^2 = (\lambda_a b)^2 + (\lambda_a c)^2 \qquad \text{or} \qquad \lambda_a = \frac{a}{\sqrt{b^2 + c^2}}.$$

We should expect, and indeed it is obviously the case that, $\lambda_a < \lambda_c$. Again, where λ_a falls relative to $\lambda = 1$ depends in general on the lengths a, b, and c in the original triangle. In our example triangle with all acute angles we have $1 < \lambda_a < \lambda_c$. Nothing particularly special (apparently) seems to happen in the "motion" at either $\lambda = \lambda_a$ or $\lambda = \lambda_c$. In general, in the case $b < c$, the angle at $B = (0, 0)$ must always remain acute; in both the limits $\lambda = \lambda_{\min}$ and $\lambda = \lambda_{\max}$ we have the angle at $B = (0, 0)$ tending to $\theta = 0$. This means of course that this angle has some positive (acute) maximum during the motion.

Exercise 3 *In the case $b < c$, determine the maximum value of the angle at $B = B(\lambda) = (0, 0)$ and all values of λ for which this maximum is attained.*

The second task on our "to do" list is to try to show $\cos\phi$ is a (strictly) decreasing function of λ directly using inequalities rather than calculus. Here is how to do that in reverse: **We expect** for $\lambda_{\min} < \lambda_1 < \lambda_2 < \lambda_{\max}$ there holds $\cos\phi(\lambda_2) < \cos\phi(\lambda_1)$, that is,

$$\frac{a^2}{\lambda_2} + \lambda_2(b^2 - c^2) < \frac{a^2}{\lambda_1} + \lambda_1(b^2 - c^2).$$

Indeed, this doesn't look too terrible, and it is equivalent to

$$a^2\lambda_1 + (b^2 - c^2)\lambda_1\lambda_2^2 < a^2\lambda_2 + (b^2 - c^2)\lambda_1^2\lambda_2$$

(which is obtained by multiplying through by $\lambda_1\lambda_2$). Rearranging terms, we have a proposed inequality

$$a^2(\lambda_2 - \lambda_1) > (b^2 - c^2)\lambda_1\lambda_2(\lambda_2 - \lambda_1)$$

or

$$a^2 > (b^2 - c^2)\lambda_1\lambda_2. \tag{10}$$

In this case, we already have $b^2 < a^2 + c^2$ because $b < c$, so this means we actually have

$$a^2 > b^2 - c^2. \tag{11}$$

Notice that (10) follows form (11) if $\lambda_1\lambda_2 \le 1$. In particular, if we restrict to the interval $\lambda_{\min} < \lambda_1 < \lambda_2 \le 1$ (which actually contains the values we are really interested in for the original problem anyway) then

$$a^2 > b^2 - c^2 > (b^2 - c^2)\lambda_1\lambda_2.$$

Once we have established (10) the string of inequalities above is reversible, and we get $\cos\phi(\lambda_2) < \cos\phi(\lambda_1)$ directly, at least for $\lambda_1, \lambda_2 \le 1$.

If you take the special case of the discussion I have just given with $\lambda_1 = b_1/b_3 < 1$ and $\lambda_2 = 1$, reversing the inequalities as suggested, then you get pretty much the string of inequalities I used in the main solution. You should give it a try. Of course, things need to be "adjusted" a little more to make sure the case $b > c$ is taken into account.

Exercise 4 *Can you get the inequality* $\cos\phi(\lambda_2) < \cos\phi(\lambda_1)$ *directly in the case* $b < c$ *even for* $\lambda_1 \le 1 < \lambda_2$ *or* $1 < \lambda_1 < \lambda_2$*?*

Let's take a look at the case $b > c$. Hopefully, you've figured out that the situation is quite different. Nominally, the example we've considered above illustrates what happens if we switch b and c and/or consider what happens to the angle at the origin. Doing the former leads to Figure 7.

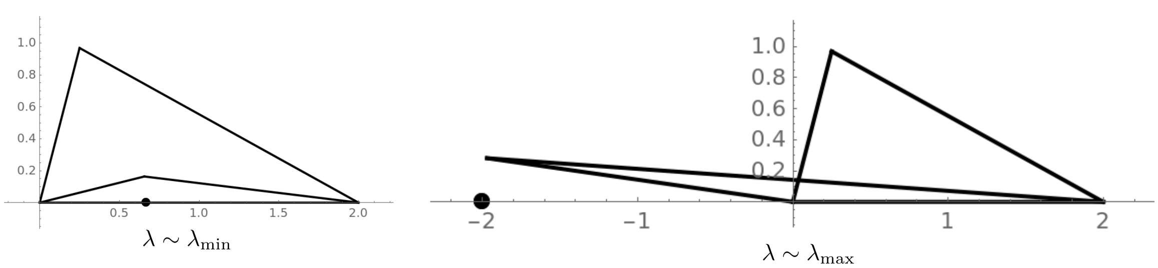

Figure 7: Extreme values when $b > c$.

As you may note, the angle at C limits to $\phi = 0$ at both extremes, so we do not expect monotonicity. In fact, if we plot $\cos\phi$ and ϕ in the our example with

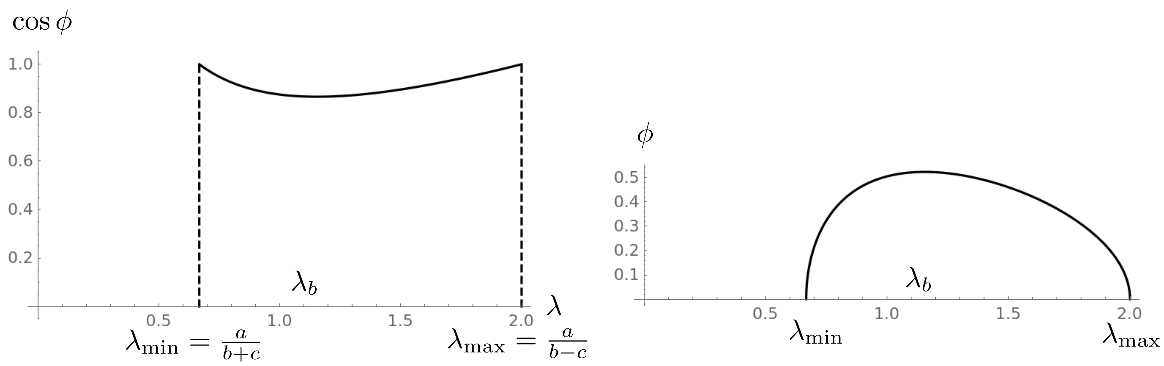

Figure 8: $\cos\phi$ and ϕ as functions of λ when $a = 2 = b$ and $c = 1$.

$a = 2 = b$ and $c = 1$, we obtain the plots in Figure 8. If we check the derivative we see, as indicated in the figures, there is a unique value of λ for which

$$2ab\frac{d}{d\lambda}\cos\phi = -\frac{a^2}{\lambda^2} + b^2 - c^2$$

vanishes, namely

$$\lambda_b = \frac{a}{\sqrt{b^2 - c^2}}.$$

What is this value? First of all, note that this values falls in our interval corresponding to triangles (Exercise 2):

$$\lambda_{\min} = \frac{a}{b+c} < \frac{a}{\sqrt{b^2 - c^2}} < \frac{a}{b-c} = \lambda_{\max}.$$

The first inequality is equivalent to $(b+c)^2 > b^2 - c^2$ or $2c(b+c) > 0$. The second inequality is equivalent to $b^2 - c^2 > (b-c)^2$ or $2c(b-c) > 0$. Finally, noting that when $\lambda = \lambda_b$ there holds

$$a^2 + (\lambda_b c)^2 = (\lambda_b b)^2$$

we see that λ_b is the unique value for which there is a right angle at vertex $B = (0,0)$. Thus, $\cos\phi$ is not decreasing across the entire interval $\lambda_{\min} < \lambda < \lambda_{\max}$ when $b > c$. Rather $\cos\phi(\lambda)$ decreases (so that ϕ increases as a function of λ (and ϕ becomes smaller with smaller λ) for $\lambda_{\min} < \lambda < \lambda_b$. There is a horizontal tangent to the graphs of both $\cos\phi$ and ϕ at $\lambda = \lambda_b$ and then $\cos\phi$ increases as a function of λ back to $\cos\phi(\lambda_{\max}) = 1$.

For good measure, let us note that

$$2ab\frac{d^2}{d\lambda^2}\cos\phi = \frac{2a^2}{\lambda^3} > 0.$$

Let us pause to apply what we have found to the original problem: If we want the angle at $C = C(\lambda)$ to increase as a function of λ for $\lambda_{\min} < \lambda < 1$, i.e., to become smaller as λ decreases from 1 and in particular to be smaller than the original angle C in the original triangle corresponding to $\lambda = 1$, then in the case $b < c$, there is no further restriction needed. This condition, however, implies the angle at $B = (0,0)$ is acute. If $b = c$, then also

$$2ab\cos\phi = \frac{a^2}{\lambda}$$

is clearly decreasing in λ, which means ϕ is increasing. In the final case, we need $\lambda_b > 1$. That is,

$$\frac{a}{\sqrt{b^2 - c^2}} > 1 \qquad \text{or} \qquad b^2 < a^2 + c^2,$$

that is, we need the original angle at $B = (0,0)$ to be acute. In this case, we can also get the inequality directly without differentiation: We want (and expect if $\cos\phi$ is a decreasing function of λ) for $\lambda_1 < \lambda_2$ to have

$$\frac{a^2}{\lambda_2} + (b^2 - c^2)\lambda_2 < \frac{a^2}{\lambda_1} + (b^2 - c^2)\lambda_1. \tag{12}$$

As long as $\lambda_1, \lambda_2 > 0$ and certainly if $0 < \lambda_1 \le \lambda_2 \le 1$, this is equivalent to

$$a^2\lambda_1 + (b^2 - c^2)\lambda_1\lambda_2^2 < a^2\lambda_2 + (b^2 - c^2)\lambda_1^2\lambda_2 \tag{13}$$

or

$$(b^2 - c^2)\lambda_1\lambda_2(\lambda_2 - \lambda_1) < a^2(\lambda_2 - \lambda_1). \tag{14}$$

This last inequality is equivalent to

$$(b^2 - c^2)\lambda_1\lambda_2 < a^2 \tag{15}$$

if $\lambda_1 < \lambda_2 \le 1$, and this inequality follows if $b^2 - c^2 < a^2$, meaning the angle at B is acute, and $0 < \lambda_1 < \lambda_2 \le 1$. In our application we have $0 < \lambda_{\min} < \lambda_1 \le \lambda_2 \le 1$ and consider also the case $\lambda_1 = \lambda_2 = 1$.

I'm now going to go through the process of "reversing" the inequalities, that is to say going through the string of inequalities in reverse order twice. The first time I'll do it just as they stand. The second time, I'll take the special case $0 < \lambda_1 \le \lambda_2 = 1$, and this should be essentially the same as the string of inequalities in my solution.

We start with the assumption that the angle at B is acute which gives

$$b^2 < a^2 + c^2 \qquad \text{or} \qquad b^2 - c^2 < a^2.$$

If $0 < \lambda_1 \le \lambda_2 \le 1$, then we get

$$\lambda_1\lambda_2(b^2 - c^2) \le (b^2 - c^2) < a^2.$$

Thus (15) holds. At this point we need to multiply both sides by $\lambda_2 - \lambda_1$ which may be zero if $\lambda_1 = \lambda_2$, so we lose (14), but we do get

$$(b^2 - c^2)\lambda_1\lambda_2(\lambda_2 - \lambda_1) \le a^2(\lambda_2 - \lambda_1)$$

with equality in the case both sides are dead zero. Next we rearrange to get the weak version of (13):

$$a^2\lambda_1 + (b^2 - c^2)\lambda_1\lambda_2^2 \le a^2\lambda_2 + (b^2 - c^2)\lambda_1^2\lambda_2$$

which is also a strict inequality unless $\lambda_1 = \lambda_2$. Finally, we divide by the product $\lambda_1\lambda_2 > 0$ to get the weak version of (12) which again is a strict inequality unless $\lambda_1 = \lambda_2$:

$$\frac{a^2}{\lambda_2} + (b^2 - c^2)\lambda_2 \le \frac{a^2}{\lambda_1} + (b^2 - c^2)\lambda_1.$$

Dividing both sides by $2ab$ gives

$$\cos\phi(\lambda_2) \le \cos\phi(\lambda_1) \qquad \text{for } \lambda_{\min} < \lambda_1 \le \lambda_2 \le 1$$

with strict inequality unless $\lambda_1 = \lambda_2$.

Now, let us repeat this with $0 < \lambda = \lambda_1 \le \lambda_2 = 1$. Again, we must start with the assumption that the angle at B is acute so

$$\lambda(b^2 - c^2) \le b^2 - c^2 < a^2 \tag{16}$$

with strict inequality unless $\lambda = 1$. Even in the case of weak inequality in (16) we have strict inequality

$$\lambda(b^2 - c^2) < a^2.$$

Now, we multiply both sides by $1 - \lambda$, and we can no longer preserve strict inequality under these assumptions, but we get

$$(b^2 - c^2)\lambda(1 - \lambda) \le a^2(1 - \lambda)$$

with equality when $\lambda = 1$ and we are henceforth manipulating the equation $0 = 0$. Next rearrangement gives

$$a^2\lambda + (b^2 - c^2)\lambda \leq a^2 + (b^2 - c^2)\lambda^2$$

which is also a strict inequality unless $\lambda = 1$. Dividing by $2ab\lambda > 0$ we conclude

$$\cos\phi = \frac{1}{2a}(a^2 + b^2 - c^2) \leq \frac{1}{2ab}\left[\frac{a^2}{\lambda^2} + (b^2 - c^2)\lambda\right] = \cos\phi(\lambda)$$

with strict inequality unless $\lambda = 1$. Putting $\lambda = b_1/b_3$, this is precisely the string of inequalities giving the application in my solution.

There was maybe one more thing to do: What is the actual path of the top vertex under scaling of the two sides? As mentioned at the very beginning of my solution, the top vertex is determined uniquely in coordinates as the intersection of two circles, namely

$$\begin{aligned} x^2 + y^2 &= c^2 \\ (x - a)^2 + y^2 &= b^2. \end{aligned}$$

Subtracting the second equation from the first gives $2ax - a^2 = c^2 - b^2$ or

$$x = \frac{1}{2a}(a^2 + c^2 - b^2).$$

The first equation then gives the height of the triangle[2]

$$y = \sqrt{c^2 - \frac{1}{4a^2}(a^2 + c^2 - b^2)^2}.$$

More generally, for $\lambda_{\min} < \lambda < \lambda_{\max}$ where $\lambda_{\min} = a/(b + c)$ and $\lambda_{\max} = a/|b - c|$ if $b \neq c$ and $\lambda_{\max} = \infty$ otherwise, we get a point

$$A(\lambda) = (x(\lambda), y(\lambda)) = \left(\frac{1}{2a}[a^2 + \lambda^2(c^2 - b^2)], \sqrt{\lambda^2 c^2 - \frac{1}{4a^2}[a^2 + \lambda^2(c^2 - b^2)]^2}\right).$$

These expressions can be plotted for our example triangles with $a = 2$, $b = 1$ and $c = 2$ and $a = 2 = b$ and $c = 3$. What we see in Figure 9 came as a bit of a surprise to me. By the symmetry of the intersection of the circles top-to-bottom it was clear

[2]It was my initial strategy to keep track of the area of the triangle(s) using this quantity, though in the end I just used $\text{area}(T) = ab\sin C/2$.

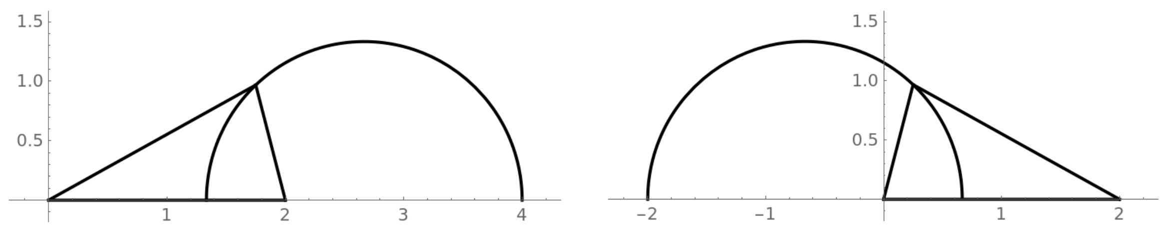

Figure 9: The path of the top point $A = A(\lambda)$ as function of λ when $a = 2 = c$ and $b = 1$ (left) and when $a = 2 = b$ and $c = 1$ (right).

that these curves should cross the x-axis at a right angle, but circles. . . really? Okay, if this is the case we should be able to check it. Let's take the first case with $b < c$, then the left limiting point has

$$x(\lambda_{\min}) = x\left(\frac{a}{b+c}\right) = \frac{a}{2}\left[1 + \frac{c-b}{c+b}\right] = \frac{ac}{c+b}$$

and $y(\lambda_{\min}) = 0$ while the right limiting point has

$$x(\lambda_{\max}) = x\left(\frac{a}{c-b}\right) = \frac{a}{2}\left[1 + \frac{c+b}{c-b}\right] = \frac{ac}{c-b}$$

and $y(\lambda_{\min}) = 0$. This means that if we have a circle, then the center must be

$$\left(\frac{x(\lambda_{\min}) + x(\lambda_{\max})}{2}, 0\right) = \left(\frac{ac^2}{c^2 - b^2}, 0\right)$$

and the radius must be

$$\frac{x(\lambda_{\max}) - x(\lambda_{\min})}{2} = \frac{abc}{c^2 - b^2}.$$

Let's see if this works.

$$
\begin{aligned}
\left(x(\lambda) - \frac{ac^2}{c^2-b^2}\right)^2 &+ y(\lambda)^2 \\
&= \left(x(\lambda) - \frac{ac^2}{c^2-b^2}\right)^2 + \lambda^2 c^2 - x(\lambda)^2 \\
&= \left(\frac{ac^2}{c^2-b^2}\right)^2 - 2x(\lambda)\frac{ac^2}{c^2-b^2} + \lambda^2 c^2 \\
&= \frac{a^2c^4}{(c^2-b^2)^2} - \frac{c^2}{c^2-b^2}[a^2 + \lambda^2(c^2-b^2)] + \lambda^2 c^2 \\
&= \frac{a^2c^4}{(c^2-b^2)^2} - \frac{a^2c^2}{c^2-b^2} \\
&= \frac{a^2c^2}{(c^2-b^2)^2}[c^2 - (c^2-b^2)] \\
&= \frac{a^2b^2c^2}{(c^2-b^2)^2}.
\end{aligned}
$$

Indeed, the path is a circle with the center and radius prescribed above. There is no need to check the case $b > c$ as the diagram illustrates (that case follows from this one by symmetry).

Exercise 5 *How (on earth!) can you see immediately that the path of the top point is a circle. Note: This construction—the appearance of (half) circles, the involvement of scaling sides of triangles, the case $b = c$ where the circle becomes a straight line, etc., suggests several topics in somewhat other (and perhaps some somewhat advanced) subjects in mathematics, namely, conic sections, complex analysis, Riemann surfaces, three-dimensional spherical geometry. It would be very interesting if some or all of these could come together to allow one to "see" what is going on here. A starting reference is the book* Geometry and the Imagination *by David Hilbert and Stefan Cohn-Vossen.*

Final note: The discussion above (I believe) essentially gives the (unusual) characterizes of circles as the locus of points with distances to two fixed points having the same (fixed) ratio b/c. Recall that the standard characterization of an ellipse is the locus of points having the sum of the distances from two fixed points constant. In the case of an ellipse, the two fixed points are called the **focal points**. Similar characterizations are well-known for parabolas and hyperbolas. In the context of these characterizations, the circle is distinguished as the ellipse with its two focal points coinciding. I do not know what the two special points would be called in this (unusual) characterization of circles. There are various "hits" referring to this result when one does an internet search (not using the "only do evil" search engine that shall not be named of course) for something like "locus with distances to two fixed points having a constant ratio." I didn't find any special name however.

5 A3 (2004)

Define a seqeunce $\{u_n\}_{n=0}^{\infty}$ as follows:

$$u_0 = u_1 = u_2 = 1.$$

For $n \geq 0$,

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!.$$

Show u_n is an integer for all n.

Note: At least Drake, Joeseph, and Lawrence got this one. Drake made a striking observation:

$$\begin{aligned} u_3 &= 0! + 1 = 2 \\ u_4 &= 1! + 2 = 3 \\ u_5 &= 2! + 6 = 8 = (4)(2) \\ u_6 &= (3! + 24)/2 = 15 = (5)(3) \\ u_7 &= (4! + 120)/3 = 48 = (6)(4)(2) \end{aligned}$$

and in general for $n \geq 2$

$$u_n = \begin{cases} (2k-1)(2k-3)\cdots(1), & n = 2k \text{ (even)} \\ (2k)(2k-2)\cdots(2), & n = 2k+1 \text{ (odd)}. \end{cases}$$

With this observation, the solution can be given using a relatively straightforward induction argument. The inductive assertion can be written as

$$u_n = \begin{cases} \dfrac{(2k)!}{2^k k!}, & n = 2k \text{ (even)} \\ \\ 2^k k!, & n = 2k+1 \text{ (odd)}. \end{cases}$$

It has been checked above that this is true for $n = 2, \ldots, 7$. If we assume the inductive hypothesis for $n \le m = 2\ell + 1$, then the next value to check is $u_{m+1} = u_{2\ell+2}$. The determinant relation gives

$$u_{m+1} = \frac{(2\ell - 1)! + u_{2\ell} u_{2\ell+1}}{u_{2\ell-1}}.$$

By the inductive hypothesis

$$u_{2\ell-1} = 2^{\ell-1}(\ell - 1)!,$$

$$u_{2\ell} = \frac{(2\ell)!}{2^\ell \ell!},$$

and

$$u_{2\ell+1} = 2^\ell \ell!.$$

Notice that

$$u_{2\ell} u_{2\ell+1} = \frac{(2\ell)!}{2^\ell \ell!} \; 2^\ell \ell! = (2\ell)!.$$

Therefore,

$$\begin{aligned} u_{m+1} &= \frac{(2\ell - 1)! + (2\ell)!}{2^{\ell-1}(\ell - 1)!} \\ &= \frac{(2\ell + 1)(2)(2\ell - 1)!}{2^\ell(\ell - 1)!} \\ &= \frac{[2(\ell + 1)]!}{2^{\ell+1}(\ell + 1)!} \qquad \text{for } m + 1 = 2(\ell + 1) \text{ (even).} \end{aligned}$$

This is the first formula in the inductive assertion we need to prove:

$$u_{m+1} = \begin{cases} \dfrac{[2(\ell + 1)]!}{2^{\ell+1}(\ell + 1)!}, & m = 2(\ell + 1) \text{ (even)} \\ \\ 2^{\ell+1}(\ell + 1)!, & m = 2(\ell + 1) + 1 \text{ (odd).} \end{cases}$$

The next term $u_{m+2} = u_{2\ell+3}$ is given by the determinant formula as

$$u_{m+2} = \frac{(2\ell)! + u_{2\ell+1} u_{2\ell+2}}{u_{2\ell}}.$$

By the inductive hypothesis

$$u_{2\ell} = \frac{(2\ell)!}{2^{\ell}\ell!},$$
$$u_{2\ell+1} = 2^{\ell}\ell!,$$

and

$$u_{2\ell+2} = \frac{[2(\ell+1)]!}{2^{\ell+1}(\ell+1)!}.$$

The product $u_{2\ell+1}u_{2\ell+2}$ simplifies:

$$u_{2\ell+1}u_{2\ell+2} = \frac{[2(\ell+1)]!}{2(\ell+1)} = (2\ell+1)!.$$

Therefore,

$$\begin{aligned} u_{m+2} &= \frac{(2\ell)! + (2\ell+1)!}{(2\ell)!}\, 2^{\ell}\ell! \\ &= 2(\ell+1)2^{\ell}\ell! \\ &= 2^{\ell+1}(\ell+1)! \qquad \text{for } m+2 = 2(\ell+1)+1 \text{ (odd).} \end{aligned}$$

This is the second formula in the inductive assertion and completes the induction.

6 A4 (2004)

Show that for any positive integer n, the n-th order polynomial $x_1x_2\cdots x_n$ can be written in the form

$$x_1x_2\cdots x_n = \sum_{j=1}^{N} c_j(a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n)^n$$

where N is a positive integer, the numbers $c_1, \ldots, c_N$ are rational numbers and $a_{jk} \in \{0, \pm 1\}$ for $j = 1, \ldots, N$ and $k = 1, \ldots, n$.
Note: Drake had an idea on this one, and we talked through ironing out some of the details, but I can't say we nailed it all down. I'm not going to type up the idea, though with some effort I might be able to do so.