Putnam study meeting notes
Tuesday November 15, 2022

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November 20, 2022

I didn't take good notes for this meeting, but if I remember correctly we discussed basically three problems:

1. A triangle problem related to A2 (2004),
2. A seating arrangement problem posed by Siddharth and solved by Jonathan and Siddharth, and
3. A2 (2007).

1 A2 (2004)

Here is the original statement: Let T_1 and T_2 be triangles in the plane with side lengths a_1, b_1 and c_1 and a_2, b_2 and c_2 respectively. If $a_2 \geq a_1$, $b_2 \geq b_1$ and $c_2 \geq c_1$ and T_2 is an acute triangle, then does this imply $\text{area}(T_2) \geq \text{area}(T_1)$?

Let me try here to review my solution (posted in the notes from our November 1 meeting), generalize the result, and add a few more comments. Perhaps I'll start with the generalization, which is basically along the lines of "you don't really need T_2 to be acute," or "to what extent can you relax the requirement that T_2 is acute?" At the very least, the assertion $\text{area}(T_1) \leq \text{area}(T_2)$ still holds if T_2 happens to be a right triangle. Here is the new statement:

Generalization: Let T_1 and T_2 be triangles in the plane with sides of lengths α_1, β_1 and γ_1 and α_2, β_2 and γ_2 respectively. Assume $\alpha_2 \geq \alpha_1$, $\beta_2 \geq \beta_1$ and $\gamma_2 \geq \gamma_1$ and order the ratios

$$\frac{\alpha_1}{\alpha_2}, \qquad \frac{\beta_1}{\beta_2}, \qquad \text{and} \qquad \frac{\gamma_1}{\gamma_2}$$

between 0 and 1 (possibly including 1 but not including 0). By renaming the sides,[1] we may assume T_1 and T_2 have sides of lengths a_1, b_1 and c_1 and a_2, b_2 and c_2 respectively satisfying

$$0 < \frac{c_1}{c_2} \leq \frac{b_1}{b_2} \leq \frac{a_1}{a_2} \leq 1.$$

Assume the angles opposite sides b_2 and c_2 in triangle T_2 have angle measure less than or equal to $\pi/2$, then (show) $\text{area}(T_1) \leq \text{area}(T_2)$ with strict inequality unless $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, and $\gamma_1 = \gamma_2$.

Note that *at most one angle in any triangle can be non-acute,* that is, at most one angle can have angle measure greater than or equal to $\pi/2$.

Exercise 1 *If the angle A_2 opposite the side of length a_2 in triangle T_2 is non-acute, then the angle A_1 opposite the side of length a_1 in triangle T_1 is non-acute and satisfies*

$$\pi/2 \leq A_2 \leq A_1.$$

Let me outline/review my solution, which I claim (with the help of the exercise above) gives a solution for the more general problem and gives a bit more insight/information into the original solution.

There was an initial step, which was not discussed carefully, but is now made explicit in the generalized problem. This involves relabeling the sides. Let's say the two triangles are as indicated in Figure 1 with In this case we have

$$\frac{\gamma_1}{\gamma_2} < \frac{\alpha_1}{\alpha_2} < \frac{\beta_1}{\beta_2} < 1,$$

so we rename the sides; $a_j = \beta_j$, $b_j = \alpha_j$ and $c_j = \gamma_j$ for $j = 1, 2, 3$. Then

$$\frac{c_1}{c_2} < \frac{b_1}{b_2} < \frac{a_1}{a_2} < 1.$$

In Figure 2 I've rotated (and flipped) these triangles so that the sides with largest ratio appear horizontally at the bottom (and could be put in standard coordinate

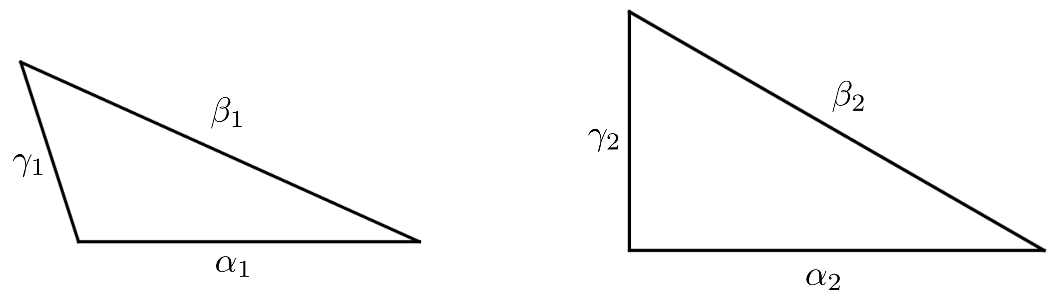

Figure 1: Two triangles; the side lengths of the second triangle dominate those of the first triangle.

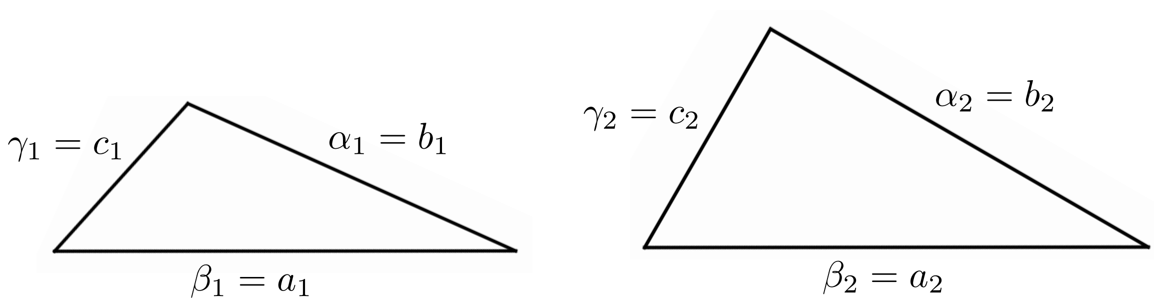

Figure 2: Two triangles congruent to those in Figure 1. The sides are relabeled according to increasing ratios of corresponding sides.

position along the x-axis). It is an assumption at this point that the angles B_2 and C_2 each measure less than or equal to $\pi/2$. As mentioned above, at most one angle in any triangle can measure greater than or equal to $\pi/2$. Thus, if one of the angles B_2 or C_2 does measure $\pi/2$, then the other two angles are acute. There is the possibility however, that the angle at A_2 measures greater than or equal to $\pi/2$. This is the case in the example shown in Figures 1 and 2. According to Exercise 1 when this is the case, one must have the angle A_1 at least the measure of A_2, and we see that is the case in Figure 2.

The next step, then, is to scale triangle T_2 (if necessary) uniformly to attain a

[1]I will continue the practice here of using the same symbols to represent both the sides a, b, and c of a triangle as well as the lengths of those sides. I will also denote the opposite angles A, B, and C and use the same symbols for the angle measures of those angles.

triangle T_3 congruent to T_2 but with $a_3 = a_1$, $b_3 \leq b_1$, and $c_3 \leq c_1$. The scaling constant is a_1/a_2. Naturally, it is very easy to see $\text{area}(T_3) \leq \text{area}(T_2)$. In our

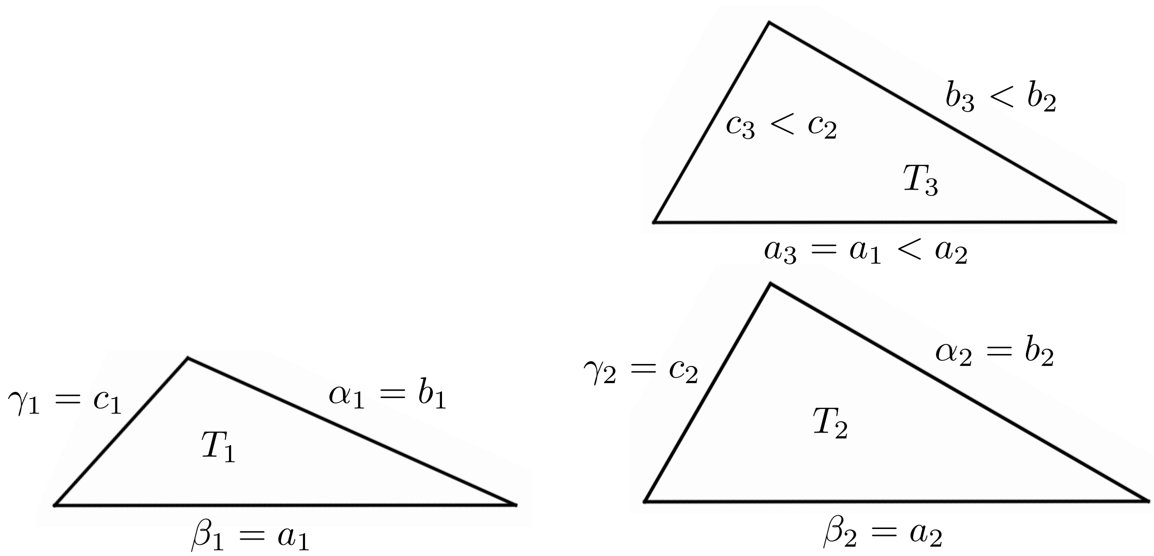

Figure 3: Triangle T_3 is similar to triangle T_2, but (in this example) each side length in T_3 is shorter than the corresponding side length in T_2. Most importantly, note that $b_3 \geq b_1$ and $c_3 \geq c_1$.

example case here $\text{area}(T_3) < \text{area}(T_2)$.

The next step is to scale sides b_3 and c_3 of triangle T_3 by some factor $b_1/b_3 \leq 1$ while leaving the third side length $a_3 = a_1$ fixed to obtain a triangle T_4.

Exercise 2 *Show the triangle T_4 exists and satisfies $a_4 = a_1$, $b_4 = b_1$ and $c_4 \leq c_1$ and* $\text{area}(T_4) \leq \text{area}(T_3)$.

This is an enhanced version of the assertion used in my solution of the problem. The triangle T_4 obtained in our example is indicated in Figure 4. In summary

$$\text{area}(T_1) \leq \text{area}(T_4) \leq \text{area}(T_3) \leq \text{area}(T_2).$$

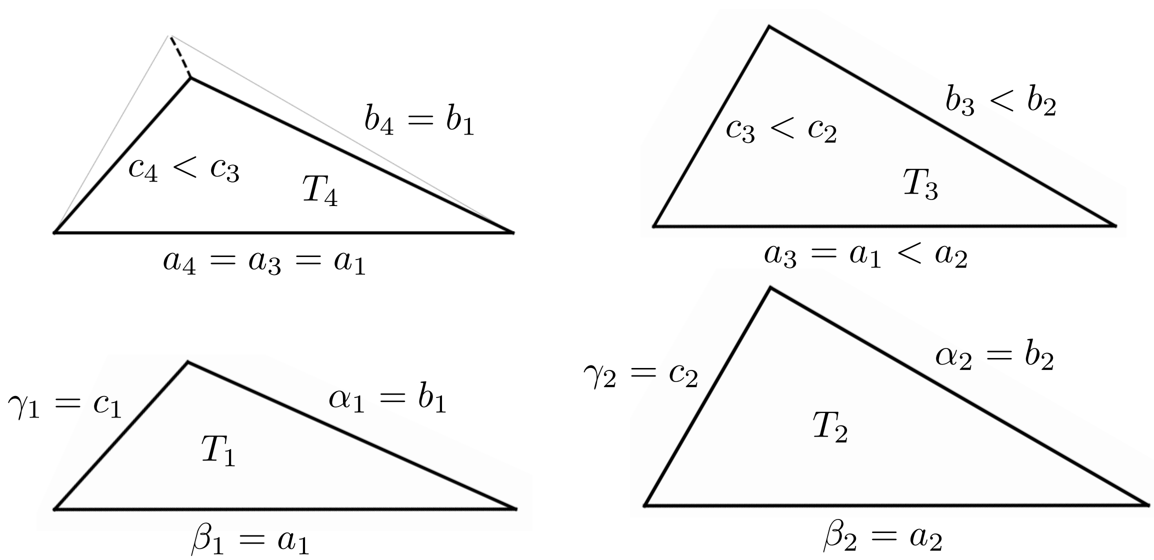

Figure 4: Triangle T_4 is obtained by scaling two sides of triangle T_3 and has area no more than that of triangle T_3. Furthermore triangle T_4 shares two side lengths with triangle T_1: $a_4 = a_1$ and $b_4 = b_1$, but $c_4 \geq c_1$. Under these conditions, and since angle C_4 has measure less than or equal to $\pi/2$, one concludes area$(T_1) \leq$ area(T_4).

Exercise 3 *Given a circle $\partial B_r(0,0)$ of radius r centered at the origin and any point $(R,0)$ with $R > 0$, there is a unique point $(\rho,0)$ with $0 < \rho < r$ such that $\partial B_r(0,0)$ is the locus of points $\mathbf{p}$ whose distances to $(\rho,0)$ and $(R,0)$ have ratio satisfying*

$$\frac{\mathrm{dist}(\mathbf{p},(\rho,0))}{\mathrm{dist}(\mathbf{p},(R,0))} = \frac{r-\rho}{R-r}.$$

(True or false?)

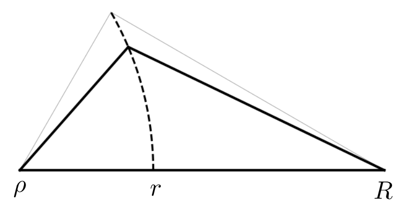

Figure 5: As noted elsewhere, the path of the vertex between two sides of a triangle that moves under scaling of those side lengths by the same value in the path of a circle.

2 Knights of the round table

Actually, lots of boys and girls at a round table: If 25 girls and 25 boys sit at a round table, then show some child must have a girl on both the right and the left.

First Solution: (Jonathan) Assume, by way of contradiction, that no child has a girl on both the right and the left. Then, in particular, every boy is sitting next to at least one boy. This means there are some number k of well-defined groups of boys, and each such group has at least two boys in it, where a "group" is defined to be a collection of boys sitting next to each other consecutively around the table. There can be at most 12 such groups because $(2)(13) = 26 > 25$. Now one can consider the girls after each group of boys. Again, there can be no more than 12 groups of girls. If each such group had two or fewer girls, then there would be $(2)(12) = 24$ or fewer girls, which is a contradiction. We conclude that some group of girls has at least three girls in it, and one of those girls has a girl on both her left and right.

Second solution: (Siddharth) Consider the children in two groups H_1 and H_2 of 25 children each as follows: Start with any one child as the first child, and go around the table including every other child. So one group, say a group called H_1, consists of the first, third, fifth, and so on up to the forty-ninth child. The second group H_2 is the complement of the first group (containing the even numbered children). Since each of these groups includes an odd number of children, namely 25 children, neither group can have the same number of boys as girls. In fact, one group, call it G will have more girls than boys, and since there are, in total, equal numbers of boys and girls, the other group, call it group B must have more boys than girls. If one goes around the table considering the children in group G starting with a girl, and we imagine that one finds after each girl in group G a boy coming next in group G, then when one completes the circuit around the table reaching the starting girl, one will have found/passed/considered at least as many boys in G as girls. But we know there are more girls in this group G than boys. This means that at some point one must find two girls in a row in group G. The child in group B sitting between these two girls. . . is sitting between two girls.

3 A2 (2007)

I suggested this problem:

Find the minimum (possible) area of a convex set in the plane containing a point in each of the two branches of each of the two hyperbolas $xy = 1$ and $xy = -1$.

Preliminary discussion: The problem gave an explanation of what it means to be a convex set. The explanation was that given any two points in the set, the segment connecting those two points is also in the set. Equivalently, one can say (more symbolically) C is a convex set if $\mathbf{x}, \mathbf{y} \in C$ implies $(1-\lambda)\mathbf{x} + \lambda\mathbf{y} \in C$ for all λ with $0 \le \lambda \le 1$.

Let us say the points $(x_1, 1/x_1)$, $(-x_2, 1/x_2)$, $(-x_3, -1/x_3)$, and $(x_4, -1/x_4)$ are in a convex set C for some $(x_1, x_2, x_3, x_4) \in (0, \infty)^4$.

Note the following: There is some $t_0 \in (0, 1)$ for which $(1 - t_0)x_1 - t_0 x_2 = 0$, namely,

$$t_0 = \frac{x_1}{x_1 + x_2}.$$

This means the point

$$\left(0, (1-t_0)\frac{1}{x_1} + t_0\frac{1}{x_2}\right) = \left(0, \frac{1}{x_1 + x_2}\left(\frac{x_2}{x_1} + \frac{x_1}{x_2}\right)\right) = \left(0, \frac{x_1^2 + x_2^2}{x_1 x_2 (x_1 + x_2)}\right) \in C$$

as well as the entire segment

$$\left\{(1-t)\left(x_1, \frac{1}{x_1}\right) + t\left(x_2, \frac{1}{x_1}\right) : 0 \le t \le 1\right\}.$$

Similar assertions hold for the points $(-x_3, -1/x_3)$ and $(x_4, -1/x_4)$: The point

$$\left(0, -\frac{x_3^2 + x_4^2}{x_3 x_4 (x_3 + x_4)}\right) \in C$$

and so is the segment

$$\left\{(1-t)\left(-x_3, -\frac{1}{x_3}\right) + t\left(x_4, -\frac{1}{x_4}\right) : 0 \le t \le 1\right\}.$$

Notice that the origin is on the segment from

$$\left(0, \frac{x_1^2 + x_2^2}{x_1 x_2 (x_1 + x_2)}\right) \qquad \text{to} \qquad \left(0, -\frac{x_3^2 + x_4^2}{x_3 x_4 (x_3 + x_4)}\right).$$

Thus, the triangle T_1 with vertices $(0,0)$, $(x_1, 1/x_1)$, and $(-x_2, 1/x_2)$ is entirely in C. There are also three more non-overlapping triangles all sharing the origin as a vertex and all entirely in C including the triangle T_3 with vertices $(0,0)$, $(-x_3, -1/x_3)$, and $(x_4, -1/x_4)$. Let other triangle containing $(-x_2, 1/x_2)$ be T_2 and the remaining triangle containing $(-x_3, -1/x_3)$ be T_3, then

$$\text{area}(C) \geq \text{area}(T_1) + \text{area}(T_2) + \text{area}(T_3) + \text{area}(T_4).$$

Triangle T_2 has area

$$A_1 = \frac{1}{2}x_1 \frac{x_1^2 + x_2^2}{x_1 x_2 (x_1 + x_2)} + \frac{1}{2}x_2 \frac{x_1^2 + x_2^2}{x_1 x_2 (x_1 + x_2)} = \frac{x_1^2 + x_2^2}{2(x_1 + x_2)} \left(\frac{1}{x_1} + \frac{1}{x_2} \right) = \frac{x_1^2 + x_2^2}{2 x_1 x_2}.$$

This function of x_1 and x_2 considered for $(x_1, x_2) \in (0, \infty)^2$ has gradient

$$\nabla A_1 = \frac{1}{2x_1^2 x_2^2} \begin{pmatrix} x_2(x_1^2 - x_2^2) \\ x_1(x_1^2 + x_2^2) \end{pmatrix}.$$

Thus, the only interior critical points are along $x_1 = x_2$ where the area of T_1 satisfies identically $A_1(x_1, x_1) = \text{area}(T_1) = 1$.

More generally, the other values of A_1 may be expressed in polar coordinates with $r = \sqrt{x_1^2 + x_2^2} \in (0, \infty)$ and $\theta = \tan^{-1}(x_2/x_1) \in (0, \pi/2)$ as

$$A_1 = \frac{1}{2\cos\theta\sin\theta} = \frac{1}{\sin(2\theta)} \geq 1.$$

Thus, $\text{area}(C) \geq 4$, and this is clearly achieved by, for example, the square with vertices $(\pm 1, \pm 1)$ and/or any of the rectangles aligned with the coordinate axes having vertices $(x_1, 1/x_1), (-x_1, 1/x_1), (-x_1, -1/x_1), (x_1, -1/x_1)$.

I guess that's a solution. $\smile$