# Additional notes about Problem 5

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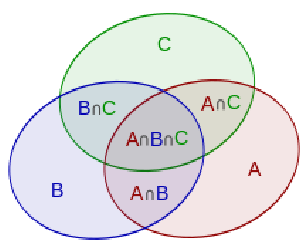
*Given nine planar regions of unit area whose union has area 5, show that some pair of the regions intersect in an area at least 1/9.*

The approach given in class was to incrementally union the different areas and apply the condition in the problem statement. While this is certainly a valid approach, what immediately jumped out to me was an approach using an idea in combinatorics known as the Principle of Inclusion-Exclusion, or PIE. This is a method by which to find the union of sets by adding the individual components and then dealing with the intersections.

To understand this idea, consider the following Venn diagram with three sets, $A, B$, and $C$ (credit to Wikimedia commons):




In order to find $|A \cup B \cup C|$, we would need to sum each of the regions within the diagram. However, let us approach this naively first. If we simply sum the areas of $A, B$, and $C$, then we will overcount, but we will deal with that momentarily. We have:

$$|A \cup B \cup C| = |A| + |B| + |C| - \varepsilon$$

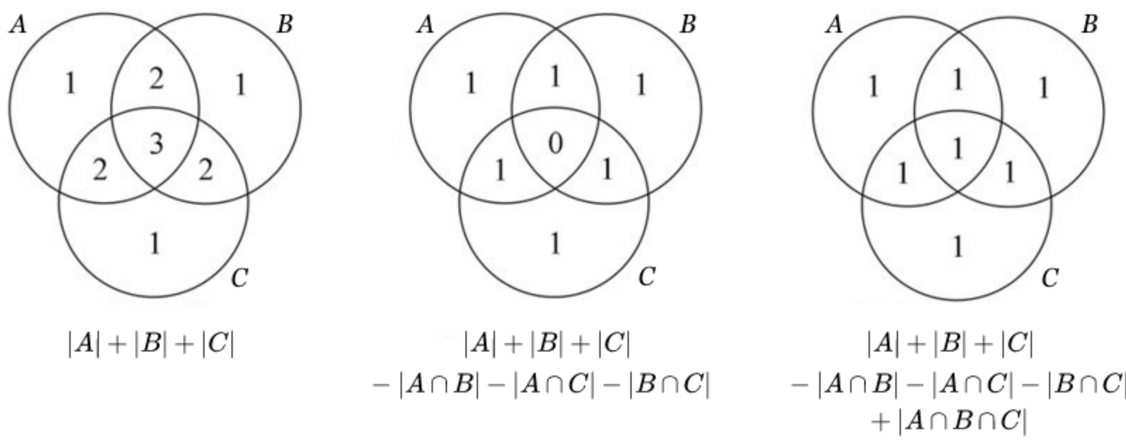
However, here we have overcounted the pairwise intersections of the sets, since $|A| + |B| - |A \cap B| = |A \cup B|$. Thus to compensate, let us subtract the pairwise intersections out. We now have:

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |B \cap C| + |A \cap C|) + \varepsilon'$$

What about the intersection of all three sets? Well consider the LHS and the RHS of the latest equation above. On the left, within the union of all three sets, the intersection $|A \cap B \cap C|$ is "counted" 1 time. On the right side, when we sum $|A|, |B|$, and $|C|$, we "count" $|A \cap B \cap C|$ three times, since it is contained within each of those sets. However, we then subtract the three pairwise intersections of the sets, and thus we subtract $|A \cap B \cap C|$ three times. Therefore, on the right side of our current equation, we are not counting the triple intersection at all! So in order to complete our counting, we can simply add it back in to the RHS as follows:

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |B \cap C| + |A \cap C|) + |A \cap B \cap C|.$$

This is the **Principle of Inclusion-Exclusion**, since at each step we had to decide what to include and what to exclude. A nice way to visualize this if the math above was confusing is the following image (credit to Brilliant):




Again, the number in each subset represents how many times that subset is counted. Here, the overlap of all three subsets must be added back in due to it being removed after the second step.

This formula can be generalized for $n$ sets as follows:

$$\begin{aligned}\left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{S \in (\mathcal{P}(N)/\emptyset)} (-1)^{(|S|+1)} \left|\bigcap_{s \in S} A_s\right| \\ &= (|A_1| + \cdots + |A_n|) - (|A_1 \cap A_2| + \cdots + |A_{n-1} \cap A_n|) + \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|\end{aligned}$$

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We can now finally apply this to the problem at hand. Denote the $9$ regions as $A_1$ through $A_9$. We are given that $|A_1 \cup \cdots \cup A_9| = 5$. By the Principle of Inclusion-Exclusion, we have:

$$|A_1 \cup \cdots \cup A_9| = (|A_1| + \cdots + |A_9|) - (|A_1 \cap A_2| + \cdots + |A_8 \cap A_9|) + \cdots + |A_1 \cap \cdots \cap A_9|$$

$$\begin{gathered} 5 = 9 - (|A_1 \cap A_2| + \cdots + |A_8 \cap A_9|) + \cdots + |A_1 \cap \cdots \cap A_9| \\ \implies |A_1 \cap A_2| + \cdots + |A_8 \cap A_9| = 4 + \varepsilon \end{gathered}$$

where

$$\varepsilon = (|A_1 \cap A_2 \cap A_3| + \dots) - \cdots + |A_1 \cap \cdots \cap A_9|$$

We see that $\varepsilon$ must be positive, since after we take the "triowise" (groups of three sets) intersections and sum them together, which is either $0$ or positive, in each successive step of the inclusion-exclusion process we cannot subtract away everything such that $\varepsilon$ becomes less than $0$. Thus we see that

$$|A_1 \cap A_2| + \cdots + |A_8 \cap A_9| \geq 4$$

From here, there are many ways to proceed (as an exercise, try using the Pigeonhole Principle). For simplicity, we will proceed by contradiction. Suppose the opposite of the problem, i.e. $|A_i \cap A_j| < \frac{1}{9}$ for $1 \leq i, j \leq 9, i \neq j$. We see that the LHS of our inequality above has $\binom{9}{2} = 36$ elements. By the assumption, this means

$$|A_1 \cap A_2| + \cdots + |A_8 \cap A_9| < 36 \cdot \frac{1}{9} = 4$$

Thus, we have a condradiction, since the sum of the pairwise intersections cannot be both larger than $4$ and less than $4$, and we are done.

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