Putnam problem B5 (2020 exam)
Solution

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1 Statement

Given four complex numbers $z_1, z_2, z_3, z_4 \in \mathbb{C}\backslash\{1\}$ with $|z_j| = 1$, $j = 1, 2, 3, 4$, it is not possible that

$$z_1 + z_2 + z_3 + z_4 - z_1 z_2 z_3 z_4 = 3. \tag{1}$$

2 Comments

I've been working on this problem off and on all this semester, and it seems appropriate that I polished off the details of what seems to be a complete and clear solution during the 6 hours of testing during the 2022 exam given on Saturday December 3. I will attempt to present my solution in detail below. This solution is rather different from the published solutions I've tried to read which seem to me to be either incomplete, unclear, or incorrect (I'm not sure which). I'll also prove the generalizations we've discussed before (at least some of them). In particular, I'll show the following:

> Given $n \in \{2, 3, 4, \ldots\}$ and n complex numbers $z_1, z_2, \ldots, z_n$ with $|z_j| = 1$ for $j = 1, 2, \ldots, n$
>
> $$\sum_{j=1}^{n} z_j - \prod_{j=1}^{n} z_j \neq n - 1 \tag{2}$$
>
> unless $\#\{j : z_j \neq 1\} \leq 1$, that is, if equality holds in (2) then all the complex numbers, except at most one, must be $1 \in \mathbb{C}$.

You may recall the even stronger assertion/conjecture from which this will follow:

If $n \geq 2$ and $z_1, z_2, \ldots, z_n \in \partial B_1(0) \in \mathbb{C}$, then

$$\left| n - \sum_{j=1}^{n} z_j \right| \geq \left| 1 - \prod_{j=1}^{n} z_j \right|$$

with equality if and only if $\#\{j : z_j \neq 1\} \leq 1$.

Here $B_1(0) = \{z \in \mathbb{C} : |z| < 1\}$ denotes the open unit disk in the complex plane, and $\partial B_1(0) = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. This formulation has associated with it a nice geometric picture, and we will solve the original problem by proving this.

2.1 Geometric Interpretation(s)

This problem is about the **sum** and **product** of a collection of complex numbers of unit length. The sum of complex vectors $a + bi$ and $c + di$ can be visualized the same way one visualizes vector addition of vectors (a, b) and (c, d) in $\mathbb{R}^2$. Thus, the sum

$$\sum_{j=1}^{n} z_j$$

can be visualized as (the end of) a path starting at the origin and ending somewhere inside (or on) the circle of radius n. By the triangle inequality (and using the case of equality in particular) one can see that the sum will actually be on the boundary circle

$$\{z \in \mathbb{C} : |z| = n\}$$

if and only if all the complex numbers are the same.

Exercise 1 Show that given n complex numbers $z_1, z_2, \ldots, z_n \in \partial B_1(0) \subset \mathbb{C}$,

$$\left| \sum_{j=1}^{n} z_j \right| = n \qquad \text{if and only if} \qquad z_1 = z_2 = \cdots = z_n.$$

We will come back to (and use) the special case where $z_1 = z_2 = \cdots = z_n$ later.

Figure 1 shows a visualization of the sum of 5 complex numbers. The product

$$\prod_{j=1}^{n} z_j = z_1 z_2 \cdots z_n$$

also has a geometric interpretation. It is a little more difficult to "see," and in some sense the product in this problem is the more mysterious and difficult quantity with which one must deal. The geometric interpretation of a product comes, basically, from trigonometric addition formulas. For example, given z_1 and z_2 in $\partial B_1(0) \subset \mathbb{C}$, there are unique angle θ_1 and θ_2 in the interval $[0, 2\pi)$ with

$$z_1 = \cos\theta_1 + i\sin\theta_1 \qquad \text{and} \qquad z_2 = \cos\theta_2 + i\sin\theta_2.$$

If we consider the product $z_1 z_2$ we get

$$\begin{aligned} z_1 z_2 &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2). \end{aligned}$$

Thus, we say "the arguments add up" when we take a product. In the example shown in Figure 1 the arguments of the 5 complex numbers used are

$$\frac{\pi}{6},\ \frac{\pi}{3},\ 0,\ -\frac{\pi}{4},\ \text{and}\ \frac{\pi}{6}.$$

Notice that the sum of these arguments is

$$\frac{5\pi}{12}.$$

In this case, we can imagine the point on $\partial B_1(0)$ representing the partial products

$$\prod_{j=1}^{k} z_j, \qquad j = 1, 2, \ldots, n$$

as moving (back and forth) in the upper half circle and coming to rest at the red point in the figure. In general, the sum of the arguments may be more than 2π, and so we should visualize the partial products as (possibly) winding all the way around $\partial B_1(0)$, maybe several times. The point is, it may be difficult to visualize where the product

$$\prod_{j=1}^{n} z_j$$

ends up on the unit circle.

The product, however will always be on the unit circle, and once we have located the product, the distance from the product to the point $z = 1$ and the distance from

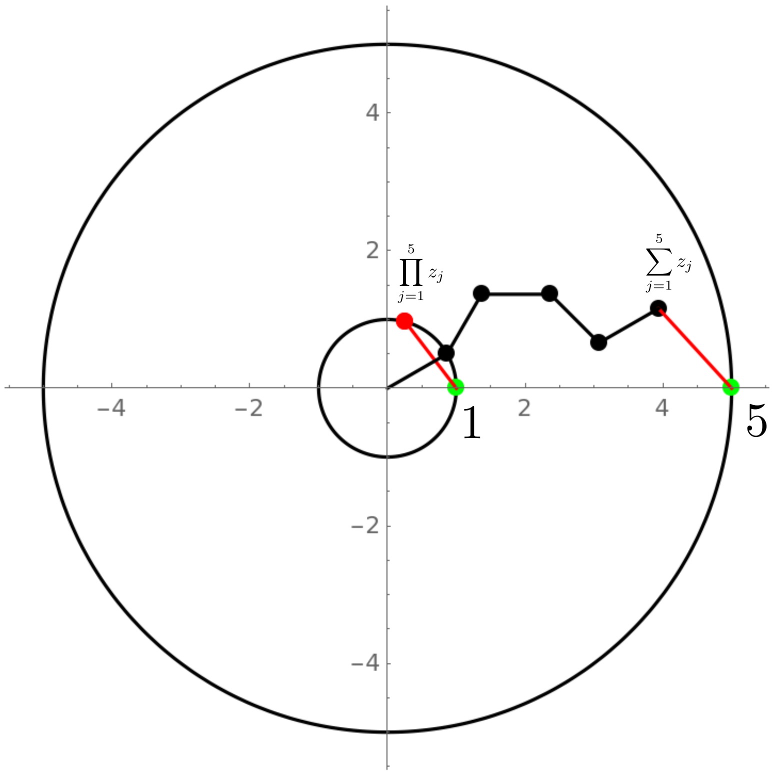

Figure 1: The sum and product of 5 complex numbers and their distances to 5 and 1 respectively.

the sum to the point $z = n$ are easy to visualize; these are the lengths of the red segments in the figure. The conjecture is that the distance from n to the sum is always at least as great as the distance from $z = 1$ to the product. You can see from the figure that even for 5 relatively widely varying arguments, these two lengths can be pretty close to each other, so one might expect this conjecture (if it is true) to be somewhat difficult to prove.

Before continuing, let me make a couple more observations about the product and the arguments introduced above in particular: We took the arguments (uniquely) on the interval $[0, 2\pi)$, but one could pick any (half open) interval of length 2π and also get a unique argument for a complex number. In particular, another natural choice is the interval $(-\pi, \pi]$, and this latter choice is more compatible the argument we give below. Also, if we take arguments $\theta_1, \theta_2, \ldots, \theta_n$ associated with our complex numbers

$z_1, z_2, \ldots, z_n$, then Euler's formula holds:

$$z_j = \cos\theta_j + i\sin\theta_j = e^{i\theta_j}.$$

This gives another way to see that the product satisfies

$$\prod_{j=1}^{n} z_j = e^{i\sum_{j=1}^{n}\theta_j} = \cos\left(\sum_{j=1}^{n}\theta_j\right) + i\sin\left(\sum_{j=1}^{n}\theta_j\right).$$

With this notation, notice that we have for the sum

$$\sum_{j=1}^{n} z_j = \sum_{j=1}^{n}\cos\theta_j + i\sum_{j=1}^{n}\sin\theta_j.$$

Finally, the absolute value of a complex number $a+bi$ is the same as the Euclidean norm of the vector $(a,b)\in\mathbb{R}^2$, so we can rephrase our conjecture as a question in which the complex numbers play no explicit role. More precisely, if we consider the function $f = f_n : [\partial B_1(0)]^n \to \mathbb{C}$ and $g = g_n : [\partial B_1(0)]^n \to \mathbb{C}$ by

$$f(z_1, z_2, \ldots, z_n) = \sum_{j=1}^{n} z_j \qquad \text{and} \qquad g(z_1, z_2, \ldots, z_n) = \prod_{j=1}^{n} z_j$$

respectively, then the conjecture asserts that for $n \geq 2$ the function $h : [\partial B_1(0)]^n \to \mathbb{R}$ given by

$$h(z_1, z_2, \ldots, z_n) = |n - f|^2 - |1 - g|^2$$

is nonnegative and vanishes if and only if $\#\{j : z_j \neq 1\} \leq 1$. This is equivalent to asserting (for $n \geq 2$) the function $H = H_n : (-\pi, \pi]^n \to \mathbb{R}$ given by

$$\begin{aligned}
H(\theta_1, \theta_2, \ldots, \theta_n) &= \left(n - \sum_{j=1}^{n}\cos\theta_j\right)^2 + \left(\sum_{j=1}^{n}\sin\theta_j\right)^2 \\
&\qquad - \left[1 - \cos\left(\sum_{j=1}^{n}\theta_j\right)\right]^2 - \sin^2\left(\sum_{j=1}^{n}\theta_j\right) \\
&= n^2 - 2n\sum_{j=1}^{n}\cos\theta_j + \left(\sum_{j=1}^{n}\cos\theta_j\right)^2 + \left(\sum_{j=1}^{n}\sin\theta_j\right)^2 \\
&\qquad - 2 + 2\cos\left(\sum_{j=1}^{n}\theta_j\right) \qquad (3)
\end{aligned}$$

satisfies $H(\theta_1, \theta_2, \ldots, \theta_n) \geq 0$ with equality if and only if $\#\{j : \theta_j \neq 0\} \leq 1$.

While we are on the topic of geometry, let us consider briefly the geometry of the domains of the functions just introduced, namely $[\partial B_1(0)]^n \subset \mathbb{C}^n$, or (geometrically) equivalently $[\partial B_1(0)]^n \subset (\mathbb{R}^2)^n$, and $(-\pi, \pi]^n \subset \mathbb{R}^n$. Actually, we lose nothing by giving up the uniqueness of the argument in the real formulation:

Exercise 2 Consider the extension of the real function H of n variables above given by the same formula (3) to all of $\mathbb{R}^n$ and to the closed cube $[-\pi, \pi]^n \subset \mathbb{R}^n$ in particular. Assume we can show this extension satisfies

$$H(\theta_1, \theta_2, \ldots, \theta_n) \geq 0 \qquad \text{on} \qquad [-\pi, \pi]^n \tag{4}$$

with equality if and only if $\#\{j : \theta_j \neq 0\} \leq 1$. Then show the original conjecture holds. The advantage here is that the extension is a smooth function on all or $\mathbb{R}^n$ and the closed cube $[-\pi, \pi]^n$ is a compact set on which H must attain a minimum. Moreover, all the techniques of minimization from multivariable calculus are available to us in the consideration of this question.

For $n = 1$, the domain $\partial B_1(0)$ considered either as the unit circle in the complex plane $\mathbb{C}$ or the unit circle in the real plane $\mathbb{R}^2$ is quite easy to visualize: It is just a circle. The conjecture is not about the case $n = 1$. In fact, the assertion does not even hold when $n = 1$. The product of two circles $[\partial B_1(0)]^2 = \partial B_1(0) \times \partial B_1(0)$ is topologically an object called a **flat torus**. Unfortunately, there is no nice geometric representation (technically an isometric embedding) of such a torus as a surface in three-dimensional space.[1] The situation concerning geometric visualization only gets worse for $[\partial B_1(0)]^n$ for $n > 2$. Fortunately, our alternative domain $[-\pi, \pi]^n \subset \mathbb{R}^n$ is relatively easy to visualize at least in some respects. In particular, it is very easy to understand this domain when $n = 2$ (a square domain) and when $n = 3$ (a cube). We will use this circumstance to our advantage in the solution.

2.2 Two special cases

Before we briefly review why the generalization/conjecture implies the assertion of the original problem, let us briefly consider two spacial cases. The first special case is when all but (at most) one of the arguments is zero. This is illustrated on the left in Figure 2. In this case, it is geometrically clear (from the figure) that $H(\theta_1, 0, \ldots, 0) = 0$. It is

[1] There is something like a visualization of a "not so nice" embedding of this two-dimensional torus in $\mathbb{R}^3$, but I can't see how it is going to help us much. If you're curious, you can look at a video which has been created of (an approximation of) this torus at `https://www.youtube.com/watch?v=RYH_KXhF1SY`.

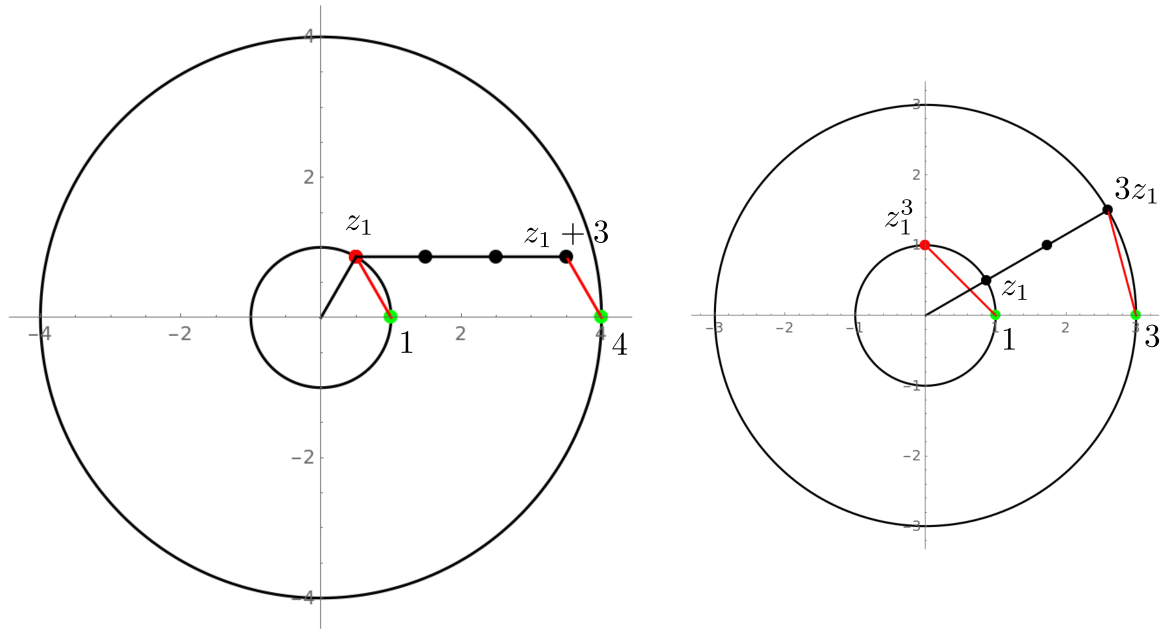

Figure 2: The case when $n = 4$ with $z_2 = z_3 = z_4 = 1$ (left) and the case when $n = 3$ and $z_1 = z_2 = z_3$ (right).

also clear from the formula.

The other special case I want to consider was rather instrumental in some way for me in solving the problem, and it is used in the solution, though most of the approaches it inspired did not work—or I could not get them to work. This is the case when all the arguments are the same, i.e., $z_1 = z_2 = \cdots = z_n$. This is illustrated in Figure 2 on the right, and it is not entirely clear (to me) geometrically that the desired inequality holds. In fact, if one is not careful, then even this special case can be somewhat difficult to see and leads to rather non-obvious trigonometric inequalities, like

$$n^2(1 - \cos\theta) > 1 - \cos(n\theta) \qquad \text{for} \qquad 0 < \theta < \frac{\pi}{2}. \tag{5}$$

Exercise 3 *Verify* (5) *directly.*

If one sticks to the complex formulation, however, the desired result is relatively straightforward. We can start with the observation that

$$\begin{aligned}
|n - nz_1| - |1 - z_1^n| &= n|1 - z_1| - |1 - z_1||1 + z_1 + z_1^2 + \cdots + z_1^{n-1}| \\
&= |1 - z_1|(n - |1 + z_1 + z_1^2 + \cdots + z_1^{n-1}|) \\
&\geq 0
\end{aligned} \tag{6}$$

by the triangle inequality since

$$|1 + z_1 + z_1^2 + \cdots + z_1^{n-1}| \leq \sum_{k=0}^{n-1} |z_1^k| \leq n.$$

It only remains to consider the case of equality. If the quantity in (6) vanishes, then $z_1 = 1$. To see this, note first of all that if the first factor $|1 - z_1|$ vanishes, then clearly $z_1 = 1$. If the second factor vanishes, then we can apply a result similar to that stated in Exercise 1:

Exercise 4 *For $n \geq 2$, if $w_2, w_3, \ldots, w_n \in \partial B_1(0) \subset \mathbb{C}$ and*

$$\left| 1 + \sum_{j=2}^{n} w_j \right| = n, \tag{7}$$

then $w_j = 1$ for $j = 2, 3, \ldots, n$.

Solution: Recall that equality holds in the complex triangle inequality $|z + w| \leq |z| + |w|$ if and only if either one of the numbers z, w is zero or there is some $c > 0$ for which $w = cz$. This means that in the case $n = 2$ we must have $w_2 = c > 0$. That of course means $w_2 = 1$ because $|w_2| = |c| = c = 1$. For $n > 2$, condition (7) implies

$$n = \left| 1 + \sum_{j=2}^{n} w_j \right| \leq \left| \sum_{j=2}^{n} w_j \right| + 1.$$

Thus,

$$\left| \sum_{j=2}^{n} w_j \right| = n - 1$$

and Exercise 1 applies. We conclude

$$w_2 = w_3 = \cdots = w_n, \tag{8}$$

and

$$|1 + (n-1)w_2| = n.$$

Again, the case of equality in the triangle inequality gives us some $c > 0$ for which $(n-1)w_2 = c$. This means of course that $w_2 = 1$, because w_2 is positive and $|w_2| = 1$. In view of (8), we have shown $1 = w_2 = w_3 = \cdots = w_n$ as claimed.

Returning to the case where the second factor in (6) vanishes so that

$$|1 + z_1 + z_1^2 + \cdots + z_1^{n-1}| = n$$

we can apply the assertion of Exercise 4 with

$$w_2 = z_1,\ w_3 = z_1^2, \ldots, w_n = z_1^{n-1}$$

to conclude that, in particular, $z_1 = w_2 = 1$.

2.3 Solution using the conjecture

If we assume, by way of contradiction that there are n complex numbers $z_1, z_2, \ldots, z_n \in \partial B_1(0) \subset \mathbb{C}$ with $n \geq 2$ and

$$\sum_{j=1}^{n} z_j - \prod_{j=1}^{n} z_j = n - 1,$$

then

$$n - \sum_{j=1}^{n} z_j = 1 - \prod_{j=1}^{n} \qquad \text{and} \qquad \left| n - \sum_{j=1}^{n} z_j \right| = \left| 1 - \prod_{j=1}^{n} \right|.$$

By the conjecture this equality implies all but one of the complex numbers are $1 \in \mathbb{C}$. This is a contradiction because it is assumed none of them are $1 \in \partial B_1(0) \subset \mathbb{C}$.

3 Proof of the Conjecture

3.1 The special case $n = 2$

We are going to use induction (and calculus) to analyze the function

$$H : [-\pi, \pi] \to \mathbb{R}$$

defined in (3), and the base case is $n = 2$. The induction is straightforward, and I'll mention it when we come to it, but the more interesting part is a trick involved in the calculus—which was really the key to solving this problem (at least in my solution).

There are several ways to establish

$$|2 - z_1 - z_2| \geq |1 - z_1 z_2| \qquad \text{for} \qquad |z_1| = |z_2| = 1$$

with equality if and only if one of z_1 or z_2 is $1 \in \mathbb{C}$. Let's focus on the equivalent real formulation because that sets us up with terminology and a bit of a feel for how the argument is going to go in the general case. We have then the function

$$H(\theta_1, \theta_2) = 2 - 4(\cos\theta_1 + \cos\theta_2) + (\cos\theta_1 + \cos\theta_2)^2 + (\sin\theta_1 + \sin\theta_2)^2 + 2\cos(\theta_1 + \theta_2).$$

We consider the values of this function on the closed square domain $[-\pi, \pi]^2$ in $\mathbb{R}^2$. The expression for H simplifies as

$$\begin{aligned} H(\theta_1, \theta_2) &= 4 - 4(\cos\theta_1 + \cos\theta_2) + 2\cos(\theta_1 - \theta_2) + 2\cos(\theta_1 + \theta_2) \\ &= 4[1 - (\cos\theta_1 + \cos\theta_2) + \cos\theta_1 \cos\theta_2]. \end{aligned}$$

Note that on the boundary square where $\theta_2 = \pm\pi$ we have

$$H(\theta_1, \pm\pi) = 8(1 - \cos\theta_1) \geq 0$$

with equality only if $\theta_1 = 0$. Similarly, $H(\pm\pi, \theta_2) \geq 0$ with equality only if $\theta_2 = 0$. Thus, on $\partial[-\pi, \pi]^2$ the function H takes strictly positive values except at the center of each side of the square boundary.

Next we differentiate to find interior critical points. In particular, note that if H were anywhere negative, then there would be an interior minimum point and the gradient of H would have to vanish at that point. At all interior critical points $(\theta_1, \theta_2) \in (-\pi, \pi)^2$, there holds

$$\frac{1}{4}\frac{\partial H}{\partial \theta_1} = \sin\theta_1 - \sin\theta_1 \cos\theta_2 = \sin\theta_1(1 - \cos\theta_2) = 0$$

and similarly,

$$\frac{1}{4}\frac{\partial H}{\partial \theta_2} = \sin\theta_2(1 - \cos\theta_1) = 0.$$

If $\sin\theta_j = 0$ for $j = 1$ or $j = 2$, then there is a critical point at (θ_1, θ_2). This means that the entire coordinate axes consist of critical points. We know also that the function H vanishes along the coordinate segments within the square corresponding to either $z_1 = 1$ or $z_2 = 1$. These segments on the coordinate axes have endpoints the centers of the sides of the square boundary where the value of H as zero as we found before. There are no other interior critical points. In fact, if $\theta_1 \neq 0$, then $\sin\theta_1 \neq 0$ and we must have $\cos\theta_2 = 1$. That means $\theta_2 = 0$ and $z_2 = 1$.

So we have a pretty good picture of what happens when $n = 2$. In particular, it gives us a proof that the value of H is strictly positive on the diagonals of the square aside from the point $(\theta_1, \theta_2) = (0, 0)$ at the origin. Explicitly,

$$H(\theta_1, \theta_1) = 4(1 - 2\cos\theta_1 + \cos^2\theta_1) = 4(1 - \cos\theta_1)^2$$

which only vanishes for $-\pi \leq \theta \leq \pi$ when $\theta = 0$.

3.2 symmetry

It will be rather important for us later that the order of the particular values of $\theta_1, \theta_2, \dots, \theta_n$ or correspondingly $z_1, z_2, \dots, z_n$ does not matter in various ways. This is, on the one hand obvious because addition and multiplication are commutative and associative. On the other hand, we may do well to give some formal structure to some of the consequences and record some of the ones we will use below.

Lemma 1 (first symmetry lemma) *Consider the case where θ_{j_0} takes a particular value θ_0 for one index $j_0 \in \{1, 2, \dots, n\}$ or correspondingly z_{j_0} takes a particular value z_0. Let $\mathcal{I} = \{1, 2, \dots, n\}$ denote the collection of all the indices and let $\mathcal{J}_0 = \mathcal{I} \backslash \{j_0\}$. If $\sigma : \mathcal{J}_0 \to \{1, 2, \dots, n-1\}$ is any one-to-one and onto function, then*

$$H(\theta_{\sigma(1)}, \theta_{\sigma(2)}, \dots, \theta_{\sigma(n-1)}, \theta_0) = H(\theta_1, \theta_2, \dots, \theta_n) \text{ and}$$
$$h(z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n-1)}, z_0) = h(z_1, z_2, \dots, z_n).$$

Lemma 2 (second symmetry lemma) *Consider the case where the angles θ_j take one particular value Θ_1 for a certain collection of indices $j \in \mathcal{J} \subset \mathcal{I} = \{1, 2, \dots, n\}$ and one other particular value Θ_2 for all other indices $j \in \mathcal{I} \backslash \mathcal{I}$, or correspondingly, there are two values $Z_1, Z_2 \in \partial B_1(0) \subset \mathbb{C}$ and $(z_1, z_2, \dots, z_n)$ is a point with*

$$z_j = Z_1 \quad \textit{for } j \in \mathcal{J} \textit{ and} \qquad z_j = Z_2 \quad \textit{for } j \in \mathcal{I} \backslash \mathcal{J}.$$

For each $j = 1, 2, \ldots, n$, let $\mathbf{e}_j$ denote the standard unit basis vector in $\mathbb{R}^n$. Then

$$H\left(\sum_{j\in\mathcal{J}} \Theta_1 \mathbf{e}_j + \sum_{j\in\mathcal{I}\setminus\mathcal{J}} \Theta_2 \mathbf{e}_j\right) = H\left(\sum_{j=1}^{k} \Theta_1 \mathbf{e}_j + \sum_{j=k+1}^{n} \Theta_2 \mathbf{e}_j\right) \quad \text{and}$$

$$h\left(\sum_{j\in\mathcal{J}} Z_1 \mathbf{e}_j + \sum_{j\in\mathcal{I}\setminus\mathcal{J}} Z_2 \mathbf{e}_j\right) = H\left(\sum_{j=1}^{k} Z_1 \mathbf{e}_j + \sum_{j=k+1}^{n} Z_2 \mathbf{e}_j\right)$$

where $k = \#\mathcal{J}$.

These results will simply make the computations a little notationally simpler at certain points.

3.3 The boundary values

We are now considering the function $H = H_n$ for $n \geq 3$ on the n-dimensional cube $\mathcal{C} = [-\pi, \pi]^n$ and/or the function $h = h_n$ on $[\partial B_1(0)]^n$. As we did in the case $n = 2$, let us assume $\theta_{j_0} = \pm\pi$ for some $j_0 \in \mathcal{I} = \{1, 2, \ldots, n\}$ corresponding to one of the "faces" of the cube $\mathcal{C}$. It turns out that the algebraic expression is again a little easier to manipulate if we take advantage of the complex notation/version. By the first symmetry lemma the values

$$h(z_1, z_2, \ldots, z_n) \qquad \text{with} \qquad \theta_{j_0} = \pm\pi \text{ and } z_{j_0} = -1$$

attained for $-\pi \leq \theta_j \leq \pi$ for $j \in \mathcal{J}_0 = \mathcal{I}\setminus\{j_0\}$ are the same as the values

$$h(z_1, z_2, \ldots, z_{n-1}, \pm\pi) = \left| n + 1 - \sum_{j=1}^{n-1} z_j \right|^2 - \left| 1 + \prod_{j=1}^{n-1} z_j \right|^2$$

attained for $-\pi \leq \theta_j \leq \pi$ for $j = 1, 2, \ldots, n-1$. Notice first that

$$\left| n + 1 - \sum_{j=1}^{n-1} z_j \right|^2 = \left(n + 1 - \sum_{j=1}^{n-1} \cos\theta_j \right)^2 + \left(\sum_{j=1}^{n-1} \sin\theta_j \right)^2$$

and

$$\left(n + 1 - \sum_{j=1}^{n-1} \cos\theta_j \right)^2 = 4 + 2\left(n - 1 - \sum_{j=1}^{n-1} \cos\theta_j \right) + \left(n - 1 - \sum_{j=1}^{n-1} \cos\theta_j \right)^2.$$

Therefore, for $n \geq 3$ we an aply the inductive hypothesis

$$\begin{aligned}\left(n-1-\sum_{j=1}^{n-1}\cos\theta_j\right)^2+\left(\sum_{j=1}^{n-1}\sin\theta_j\right)^2 &\geq \left[1-\cos\left(\sum_{j=1}^{n-1}\theta_j\right)\right]^2+\sin^2\left(\sum_{j=1}^{n-1}\theta_j\right)\\ &= 2\left[1-\cos\left(\sum_{j=1}^{n-1}\theta_j\right)\right].\end{aligned}$$

to conclude

$$\begin{aligned}&\left|n+1-\sum_{j=1}^{n-1}z_j\right|^2\\ &\quad\geq 4+2\left(n-1-\sum_{j=1}^{n-1}\cos\theta_j\right)+2\left[1-\cos\left(\sum_{j=1}^{n-1}\theta_j\right)\right].\end{aligned}$$

On the other hand,

$$\begin{aligned}\left|1+\prod_{j=1}^{n-1}z_j\right|^2 &= \left[1+\cos\left(\sum_{j=1}^{n-1}\theta_j\right)\right]^2+\sin^2\left(\sum_{j=1}^{n-1}\theta_j\right)\\ &= 2\left[1+\cos\left(\sum_{j=1}^{n-1}\theta_j\right)\right].\end{aligned}$$

From this and the fact that for $n \geq 3$

$$n-1-\sum_{j=1}^{n-1}\cos\theta_j \geq 0, \tag{9}$$

we see

$$h(z_1, z_2, \ldots, z_{n-1}, \pm\pi) \geq 4+2\left(n-1-\sum_{j=1}^{n-1}\cos\theta_j\right)-4\cos\left(\sum_{j=1}^{n-1}\theta_j\right) \geq 0.$$

Finally, the equality $h(z_1, z_2, \ldots, z_{n-1}, \pm\pi) = 0$ implies equality in (9) and we can then conclude $\cos\theta_j = 1$ and

$$\theta_1 = \theta_2 = \cdots = \theta_{n-1} = 0 \qquad \text{or} \qquad z_1 = z_2 = \cdots = z_{n-1} = 1. \tag{10}$$

In fact, if for some j_0 we have $\cos\theta_{j_0} < 1$, then we get from equality in (9) that

$$0 = n - 2 - \sum_{j\neq j_0} \cos\theta_j + 1 - \cos\theta_{j_0} \geq 1 - \cos\theta_{j_0} > 0$$

(which is a contradiction).

Notice that the equality condition expressed by (10) specifies that the value of H will be positive on the entire face $\theta_n = \pm\pi$ with the exception of the intersection of that face with the coordinate axes passing through it. A similar assertion holds then for each of the other faces $\theta_{j_0} = \pm\pi$ of $\partial[-\pi,\pi]^n$.

We have established at this point that if H take any negative value (on the interior $(-\pi,\pi)^n$ of the cube $\mathcal{C} = [-\pi,\pi]]^n$) then a minimum negative value must be attained at some interior point in $(-\pi,\pi)^n$, and the gradient of H must vanish at this point.

4 Interior values

At this point it is natural to compute the gradient of H and attempt to show the value of H is nonnegative at all critical points and only takes the value $H = 0$ on the coordinate axes within the cube $\mathcal{C} = [-\pi,\pi]]^n$. We will in fact show this, but the approach is somewhat non-standard. In any case, we start by computing the components of the gradient. Differentiating the expression in (3) with respect to θ_ℓ we find

$$\frac{1}{2}\frac{\partial H}{\partial\theta_\ell} = n\sin\theta_\ell - \sin\theta_\ell \sum_{j=1}^n \cos\theta_j + \cos\theta_\ell \sum_{j=1}^n \sin\theta_j - \sin\left(\sum_{j=1}^n \theta_j\right).$$

Thus, if these values vanish for $\ell = 1,2,\ldots,n$ we obtain the relations

$$\sin\theta_\ell\left(n - \sum_{j=1}^n \cos\theta_j\right) + \cos\theta_\ell \sum_{j=1}^n \sin\theta_j = \sin\left(\sum_{j=1}^n \theta_j\right) \qquad \text{for} \qquad \ell = 1,2,\ldots,n.$$

We know this system of equations has solutions along the axes where any $n-1$ of the components in $(\theta_1,\theta_2,\ldots,\theta_n)$ vanish. If we knew these were the only critical points, then we would be done. Unfortunately, I do not know how to prove the solution set of this system of equations is precisely along the axes. (I do not know if that is true at the moment, but see my further comments after the solution is complete.)

Let us think a little bit carefully about the relation

$$\sin\theta_\ell\left(n-\sum_{j=1}^n\cos\theta_j\right)+\cos\theta_\ell\sum_{j=1}^n\sin\theta_j=\sin\left(\sum_{j=1}^n\theta_j\right)$$

for a fixed ℓ. This is a rather crucial point in the solution, specifically because I don't know how to solve the system. We can think of this relation as saying there is a fixed vector

$$(a,b)=\left(\sum_{j=1}^n\sin\theta_j,\, n-\sum_{j=1}^n\cos\theta_j\right)\in B_{2n}(\mathbf{0})\subset\mathbb{R}^2$$

and a fixed number

$$c=\sin\left(\sum_{j=1}^n\theta_j\right)\in[-1,1]\subset\mathbb{R}$$

for which the unit vector $(\cos\theta_\ell,\sin\theta_\ell)$ satisfies

$$(\cos\theta_\ell,\sin\theta_\ell)\cdot(a,b)=c. \tag{11}$$

Of course, a, b, and c depend on θ_ℓ, but nevertheless these are the same numbers for each $\ell\in\{1,2,\ldots,n\}$. Notice, furthermore, that if the vector $(a,b)=(0,0)$, then

$$n-\sum_{j=1}^n\cos\theta_j=0,$$

and as we have noted before this means $(\theta_1,\theta_2,\ldots,\theta_n)$ is the interior point $\mathbf{0}\in\mathbb{R}^n$. We already know this is a critical point and $H(\mathbf{0})=0$, so we can set this point aside and assume $(a,b)\neq(0,0)$. Then we have a well-defined unit vector

$$\mathbf{u}=\frac{(a,b)}{\sqrt{a^2+b^2}},$$

and we can write the relation (11) as

$$(\cos\theta_\ell,\sin\theta_\ell)\cdot\mathbf{u}=\frac{c}{\sqrt{a^2+b^2}}. \tag{12}$$

Geometrically, we conclude that if there exists a critical point $(\theta_1,\theta_2,\ldots,\theta_n)$ for H away from the origin in the interior $(-\pi,\pi)^n$ of the cube $\mathcal{C}$, then there must exist a well-defined unit vector $\mathbf{u}\in\mathbb{R}^2$ and a number

$$\alpha=\frac{c}{\sqrt{a^2+b^2}}\in[-1,1]\subset\mathbb{R}$$

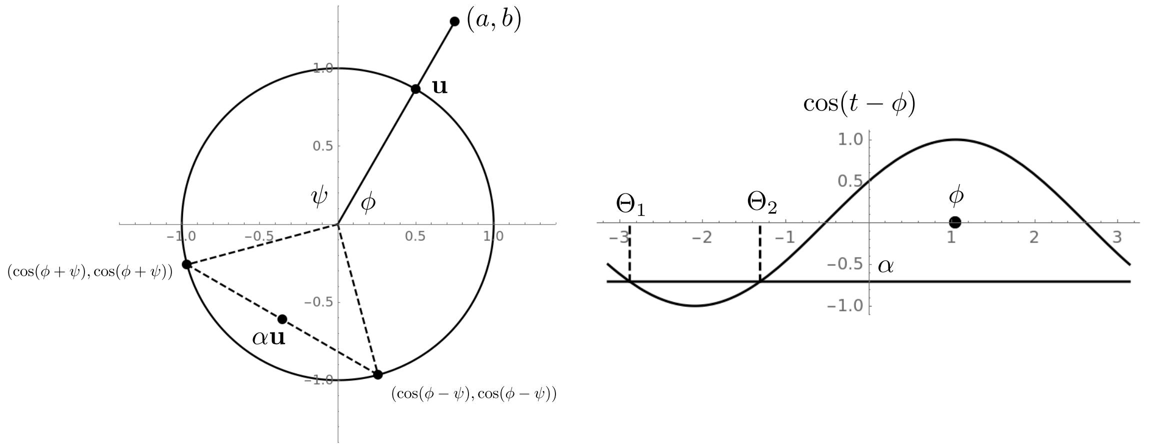

Figure 3: Finding θ_ℓ geometrically (left) and analytically (right).

so that the projection of the unit vector $(\cos\theta_\ell, \sin\theta_\ell)$ onto the line determined by $\mathbf{u}$ is precisely $\alpha\mathbf{u}$. This situation is illustrated on the left in Figure 3. Given the projection $\alpha\mathbf{u} \in \overline{B_1(0,0)}$ on the line determined by $\mathbf{u}$, there can be at most two unit vectors $(\cos\theta_\ell, \sin\theta_\ell)$ satisfying the relation (12); there must be at least one. Furthermore, the two possibilities for $(\cos\theta_\ell, \sin\theta_\ell)$ have the form

$$(\cos(\phi \pm \psi), \sin(\phi \pm \psi))$$

where ϕ is the argument of $\mathbf{u}$

$$\mathbf{u} = (\cos\phi, \sin\phi)$$

and we can take $\phi \in (-\pi, \pi]$ and where ψ is some angle with $\psi \in [0, \pi]$. If $\psi \in (0, \pi)$, then there are two possibilities, if $\psi = 0$, then $\theta_\ell = \phi \in (-\pi, \pi)$ for all ℓ, and if $\psi = \pi$ there is only one possible choice for $\theta_\ell \in (-\pi, \pi)$.

Notice that we have not concluded (and cannot conclude) from the equality

$$(\cos\theta_\ell, \sin\theta_\ell) = (\cos(\phi \pm \psi), \sin(\phi \pm \psi)$$

that $\theta_\ell \in \{\phi \pm \psi\}$. We can conclude, however, that

$$\theta_\ell \in \{\phi \pm \psi + 2\pi k : k \in \mathbb{Z}\}$$

and from the point of view of calculating values of H, this amounts to the same thing: At a critical point $(\theta_1, \theta_2, \ldots, \theta_\ell) \in (-\pi, \pi)^n \backslash \{\mathbf{0}\}$ the values of θ_ℓ all lie in a set containing at most two angles: If there is a nonzero interior critical point, then there exist fixed angles Θ_1 and Θ_2 satisfying $-\pi < \Theta_1 \leq \Theta_2 < \pi$ such that

$$\theta_\ell \in \{\Theta_1, \Theta_2\} \qquad \text{for} \qquad j = 1, 2, \ldots, n.$$

We should reach this same conclusion analytically. A way to do that is as follows: Define the angle $\phi \in (-\pi, \pi]$ by

$$\cos\phi = \frac{a}{\sqrt{a^2+b^2}} \qquad \text{and} \qquad \sin\phi = \frac{b}{\sqrt{a^2+b^2}}.$$

That is, ϕ is the argument of $\mathbf{u}$. Then the relation (12) can be written as

$$\cos(\theta_\ell - \phi) = \alpha.$$

Recall that we know $|\alpha| \leq 1$. We claim there are at most two solutions (and at least one solution) of the equation

$$\cos(t - \phi) = \alpha$$

with $-\pi < t < \pi$. We can write these solutions as $\Theta_1 \leq \Theta_2$ with equality corresponding to the case when there is only one solution. This conclusion is illustrated on the right in Figure 3. Notice also the symmetry of the function $\cos(t - \phi)$ with respect to ϕ; there is a unique maximum of 1 at $t = \phi$. This tells us the form of and how to find the solutions $\Theta_1 \leq \Theta_2$. Namely, there exist angles t_1 and t_2 with

$$t_1 = \phi - \psi_* \leq \phi \leq \phi + \psi_* = t_2 \qquad \text{for some } \psi_* \in [-\pi/2, \pi/2]$$

such that

$$\cos(t_j - \phi) = |\alpha| \qquad \text{for} \qquad j = 1, 2.$$

These angles t_1 and t_2 are not indicated in Figure 3 and they may not be in the interval $(-\pi, \pi)$. However, if $t_j \notin (-\pi, \pi)$ for either $j = 1$ or $j = 2$, then exactly one of the angles $t_j \pm 2\pi$ is in $[-\pi, \pi]$. Call this angle t_j^* with $t_j^* = t_j$ if $t_j \in (-\pi, \pi)$ for $j = 1, 2$. That is to say, for $j = 1, 2$, set

$$t_j^* = \begin{cases} t_j, & \text{if } t_j \in (-\pi, \pi) \\ t_j - 2\pi, & \text{if } t_j - 2\pi \in [-\pi, \pi] \\ t_j + 2\pi, & \text{if } t_j + 2\pi \in [-\pi, \pi]. \end{cases}$$

Now we consider various possibilities. If $\phi = 0$ and $\alpha \geq 0$, then $-\pi/2 \leq t_1 = -\psi_* \leq \psi_* = t_2 \leq \pi/2$. We can take $\Theta_1 = t_1$ and $\Theta_2 = t_2$ to conclude $\theta_\ell = \pm\psi_*$ for all ℓ.

Similarly, if $\phi = 0$ and $\alpha < 0$, then $t = \Theta_1 = t_2 - \pi$ and $t = \Theta_1 = t_1 + \pi$ are the unique solutions of $\cos(t - \phi) = \alpha$ and they must satisfy $-\pi < \Theta_1 < -\pi/2 < \pi/2 < \Theta_2 < \pi$. We conclude that $\theta_\ell = \pm(\psi_* - \pi)$. Notice that in this case we cannot have $\alpha = -1$ because then there could be no $t \in (-\pi, \pi)$ for which $\cos(t - \phi) = \cos t = \alpha = -1$ and hence no crtical point.

We have considered all possibilities in which $\phi = 0$.

If $0 < \phi < \pi$, then $t_1 \in (-\pi, \pi)$. If in this case we also have $\alpha \geq 0$, then one of the solutions of $\cos(t - \phi) = \alpha$ is $t = t_1 = \phi - \psi_*$. If $t_2 < \pi$, then $t = t_2 = \phi + \psi_*$ is the other solution of $\cos(t-\phi) = \alpha$ in $(-\pi, \pi)$ is t_2. We take in this case $\Theta_1 = t_1 < t_2 = \Theta_2$ and we have reached the desired conclusion

$$\theta_\ell \in \{\Theta_1, \Theta_2\}. \tag{13}$$

Notice that the determination of Θ_1 and Θ_2 here depended only on the disposition of the aggregate values ϕ and α, i.e., on a, b, and c, but not individually on θ_ℓ.

We should check our geometric intuition associated with the illustration on the left of Figure 3: We have here $\Theta_1 = \phi - \psi_*$ and $\Theta_2 = \phi + \psi_*$ and θ_ℓ is, up to an additive multiple of 2π, of the form $\phi \pm \psi$.

If $\phi > 0$, $\alpha \geq 0$ and $t_2 = \pi$, then there is only one solution of $\cos(t - \phi) = \alpha$ with $-\pi < t < \pi$, namely t_1, so we take $\Theta_1 = t_1$ and reach the same conclusion (13). In this case our geometric intuition holds as well: We have $\theta_\ell = \phi - \psi_* = \phi - \psi$, and it is just the case that $\phi + \psi = \pi$ and so is excluded.

If $\phi > 0$, $\alpha \geq 0$ and $t_2 > \pi$, then $t_2^* = t_2 - 2\pi$ satisfies $-\pi < t_2^* \leq t_1 < \pi$ and $\cos(t_2^* - \phi) = \alpha$. The two solutions of $\cos(t - \phi) = \alpha$ are t_2^* and t_1. We take $\Theta_1 = t_2^*$ and $\Theta_2 = t_1$ and (13) then holds.

As for the form $\theta_\ell = \phi \pm \psi \pm 2k\pi$, we have $\Theta_1 = \phi + \psi_* - 2\pi$ and $\Theta_2 = \phi - \psi_*$.

If $\phi > 0$ and $\alpha < 0$ as shown in Figure 3, then $t_3 = t_1 - \pi$ and $t_4 = t_2 - \pi$ are solutions of $\cos(t - \phi) = \alpha$. These solutions are symmetrically spaced with respect to the minimum at $\phi - \pi$ with $t_3 = \phi - \pi - \psi_*$, and $t_4 = \phi - \pi + \psi_* \in (-\pi, \pi)$. If also $t_3 > -\pi$ as shown in the illustration, then we can take $\Theta_1 = t_3 \leq t_4 = \Theta_2$ and (13) holds with

$$\Theta_1 = \phi - \psi_* - \pi = \phi + (\pi - \psi_*) - 2\pi \qquad \text{and} \qquad \Theta_2 = \phi + \psi_* - \pi = \phi - (\pi - \psi_*).$$

If it happens that $t_3 = -\pi$, then there is only one possibility for θ_ℓ, namely, $\Theta_2 = t_4 = \phi + \psi_* - \pi$. If $t_3 = t_1 - \pi < -\pi$, then $t_1 < 0$ and we take $\Theta_1 = t_4 = \phi + \psi_* - \pi$ and

$$\Theta_2 = t_3 + 2\pi = t_1 + \pi < \pi.$$

In this case,

$$\Theta_1 = \phi + \psi_* - \pi = \phi - (\pi - \psi_*) \qquad \text{and} \qquad \Theta_2 = \phi - \psi_* + \pi = \phi + (\pi - \psi_*).$$

It remains to consider the cases when $-\pi < \phi < 0$. These are, of course, somewhat symmetric to the cases we have considered above. We omit the details.

The important conclusion is this: If there is an interior critical point $(\theta_1, \theta_2, \ldots, \theta_n) \in (-\pi, \pi)^n$ for H, then there exist angles Θ_1 and Θ_2 satisfying

$$-\pi < \Theta_1 \leq \Theta_2 < \pi$$

and there exists a subcollection $\mathcal{J}$ of the indices $\mathcal{I} = \{1, 2, \cdots, n\}$ such that

$$\begin{aligned} \theta_\ell &= \Theta_1 \quad \text{for } \ell \in \mathcal{J}, \text{ and} \\ \theta_\ell &= \Theta_2 \quad \text{for } \ell \in \mathcal{I} \backslash \mathcal{J}. \end{aligned}$$

By the second symmetry lemma, the possible values taken H restricted to the set

$$\begin{aligned} \Gamma &= \Gamma_{\mathcal{J}, \Theta_1, \Theta_2} \\ &= \{(\theta_1, \theta_2, \ldots, \theta_n) \in (-\pi, \pi) : \theta_\ell = \Theta_1 \text{ for } \ell \in \mathcal{J}, \text{ and } \theta_\ell = \Theta_2 \text{ for } \ell \in \mathcal{I} \backslash \mathcal{J}\} \end{aligned}$$

are precisely the same as the values taken by $G : (-\pi, \pi)^2 \to \mathbb{R}$ by

$$G(\Theta_1, \Theta_2) = H\left(\sum_{j=1}^{k} \Theta_1 \mathbf{e}_j + \sum_{j=k+1}^{n} \Theta_2 \mathbf{e}_j\right)$$

where $k = \#\mathcal{J}$ and $\mathbf{e}_j$ is the j-th standard unit basis vector in $\mathbb{R}^n$.

In this way, we are reduced to considering the values of $G : [-\pi, \pi]^2 \to \mathbb{R}$ on the square domain $[-\pi, \pi]^2 \subset \mathbb{R}^2$ for each $k = 0, 1, \ldots, n$. Actually, we probably only need to consider at least a full half of the values of k, but the important point is that these values will include the values of H at each interior critical point. If we can show they are all non-negative (with appropriate equality conditions) then we will have proved the conjecture. We already know the boundary values of G are among the boundary values of H and satisfy

$$G(\pm\pi, \Theta_2) \geq 0 \qquad \text{and} \qquad G(\Theta_1, \pm\pi) \geq 0 \qquad \text{for } (\Theta_1, \Theta_2) \in [-\pi, \pi]^2.$$

Thus, we turn to the gradient and look for critical points. Note first that

$$\begin{aligned} G(\Theta_1, \Theta_2) = {} & [n - k\cos\Theta_1 - (n-k)\cos\Theta_2]^2 + [k\sin\Theta_1 + (n-k)\sin\Theta_2]^2 \\ & - (1 - \cos[k\Theta_1 + (n-k)\Theta_2])^2 - \sin^2[k\Theta_1 + (n-k)\Theta_2]. \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2k}\frac{\partial G}{\partial \Theta_1} &= [n - k\cos\Theta_1 - (n-k)\cos\Theta_2]\sin\Theta_1 + [k\sin\Theta_1 + (n-k)\sin\Theta_2]\cos\Theta_1 \\
&\qquad - (1 - \cos[k\Theta_1 + (n-k)\Theta_2])\sin[k\Theta_1 + (n-k)\Theta_2] \\
&\qquad\qquad - \sin[k\Theta_1 + (n-k)\Theta_2]\cos[k\Theta_1 + (n-k)\Theta_2] \\
&= [n - (n-k)\cos\Theta_2]\sin\Theta_1 + (n-k)\sin\Theta_2\cos\Theta_1 \\
&\qquad - (1 - \cos[k\Theta_1 + (n-k)\Theta_2])\sin[k\Theta_1 + (n-k)\Theta_2] \\
&\qquad\qquad - \sin[k\Theta_1 + (n-k)\Theta_2]\cos[k\Theta_1 + (n-k)\Theta_2],
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2(n-k)}\frac{\partial G}{\partial \Theta_2} &= [n - k\cos\Theta_1 - (n-k)\cos\Theta_2]\sin\Theta_2 + [k\sin\Theta_1 + (n-k)\sin\Theta_2]\cos\Theta_2 \\
&\qquad - (1 - \cos[k\Theta_1 + (n-k)\Theta_2])\sin[k\Theta_1 + (n-k)\Theta_2] \\
&\qquad\qquad - \sin[k\Theta_1 + (n-k)\Theta_2]\cos[k\Theta_1 + (n-k)\Theta_2] \\
&= [n - k\cos\Theta_1]\sin\Theta_2 + [k\sin\Theta_1\cos\Theta_2 \\
&\qquad - (1 - \cos[k\Theta_1 + (n-k)\Theta_2])\sin[k\Theta_1 + (n-k)\Theta_2] \\
&\qquad\qquad - \sin[k\Theta_1 + (n-k)\Theta_2]\cos[k\Theta_1 + (n-k)\Theta_2].
\end{aligned}$$

If both of these quantities vanish, we can set them equal to each other, and this gives, noticing the last terms (associated with the product) are identical and after simplification

$$n\sin\Theta_1 - (n-k)\sin(\Theta_1 - \Theta_2) \qquad = n\sin\Theta_2 + k\sin(\Theta_1 - \Theta_2)$$

or

$$\sin\Theta_1 - \sin\Theta_2 = \sin(\Theta_1 - \Theta_2).$$

This can be rewritten as

$$\sin\Theta_1(1 - \cos\Theta_2) = \sin\Theta_2(1 - \cos\Theta_1).$$

If either $\Theta_1 = 0$ or $\Theta_2 = 0$, we can assume by symmetry that $\Theta_n = 0$ and conclude for $n \geq 3$ more generally that

$$H(\theta_1, \theta_2, \ldots, \theta_{n-1}, 0) = \left| n - 1 - \sum_{j=1}^{n-1} e^{i\theta_j} \right|^2 - \left| 1 - \prod_{j=1}^{n-1} e^{i\theta_j} \right|^2$$

is nonnegative and vanishes only if at least all but one of $\theta_1, \theta_2, \ldots, \theta_{n-1}$ vanishes. This is by induction.

Consequently, we can turn to the case where neither Θ_1 nor Θ_2 vanishes and write

$$\frac{\sin \Theta_1}{1 - \cos \Theta_1} = \frac{\sin \Theta_2}{1 - \cos \Theta_2}. \tag{14}$$

Now the graph of the function

$$f(\theta) = \frac{\sin \theta}{1 - \cos \theta}$$

on $(-\pi, \pi)$ is shown in Figure 4, and it can be plainly seen that f is decreasing and if (14) holds for some nonzero Θ_1 and Θ_2, then we must have $\Theta_1 = \Theta_2$. In fact,

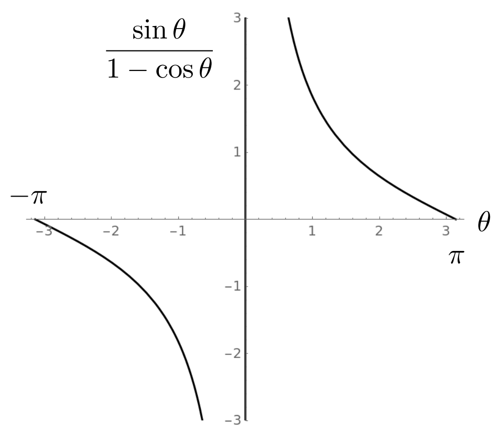

Figure 4: The graph of $\sin \theta / (1 - \cos \theta)$.

$$f'(\theta) = -\frac{1}{1 - \cos \theta}$$

so it is easy to see the value of f decreases from $f(-\pi) = 0$ to $-\infty$ as θ approaches $\theta = 0$ from the left, and $f(\theta)$ approaches $+\infty$ as θ approaches $\theta = 0$ from the right and decreases to $f(\pi) = 0$ (just as indicated in the illustration).

This tells us that if there is a critical point for G, it has to be on the diagonal

$$\{(\Theta, \Theta) : -\pi < \Theta < \pi\}$$

of the square $(-\pi,\pi)^2$. Also, any critical point of H would correspond to/result in a critical point of G, so we know also that all critical points of H must lie on the diagonal axes

$$\{(\theta,\theta,\ldots,\theta) : -\pi < \theta < \pi\}$$

of the cube $\mathcal{C} = (-\pi,\pi)^n$.

We happen to have already checked these values as a special case associated with the illustration on the right of Figure 2 in section 2.2 above. In fact we saw there that all the diagonal values were positive except at the origin.

Thus we have shown $G \geq 0$ with equality only if $k = 1$ or $k = n-1$ and $\Theta_1 = 0$ in the former case and $\Theta_2 = 0$ in the latter case. That is, H is positive except for precisely on the coordinate axes in the cube $[-\pi,\pi]^n$, and the conecture holds. $\square$

5 Follow up

There are mainly two things to say. First of all, it would be nice to know there are no critical points on the diagonal axes. I suspect this is true, but I don't see that I've actually shown it. I just showed that the values are positive along the diagonal axes, except at the origin. Maybe there is a local maximum or saddle point on a diagonal... but I doubt it.

Second, there were a number of other conjectures (some of which seemed pretty interesting) and were nominally stronger, i.e., harder to prove, than the conjecture we've proved above. I guess those are true, and the methods used above can be used to verify at least some of them. But I haven't done that. These conjectures, remember, are contained in the first set of notes I posted about this problem before I could solve it.

This is IMHO a pretty neat inequality involving complex numbers. The field of "several complex variables" is a notoriously difficult field. Whatever the case, I'm glad to have laid this one to rest.

6 One more comment

Given a set $\mathcal{J} \subset \mathcal{I} = \{1, 2, \ldots, n\}$ the "diagonal sets"

$$\mathcal{D} = \mathcal{D}_{\mathcal{J}} = \{(\theta_1,\theta_2,\ldots,\theta_n) \in [-\pi,\pi]^n : \theta_i = \theta_j \text{for } i,j \in \mathcal{J} \text{ and } \theta_i = \theta_j \text{ for } i,j \in \mathcal{I}\backslash\mathcal{I}\}$$

used in the proof of the conjecture above are interesting sets to consider. Just to be clear about the analytic expression here, a point is in $\mathcal{C}$ if there are numbers Θ_1 and

Θ_2 in $[-\pi, \pi]$ for which

$$\theta_j = \Theta_1 \text{ for } j \in \mathcal{J}, \text{ and}$$
$$\theta_j = \Theta_2 \text{ for } j \in \mathcal{I} \backslash \mathcal{J}.$$

The numbers Θ_1 and Θ_2 depend on the point $(\theta_1, \theta_2, \ldots, \theta_n) \in \mathcal{D}$, but the set $\mathcal{D}$ does not depend on the two numbers.

Exercise 5 *Draw all the diagonal sets in the cube $\mathcal{C} = [-\pi, \pi]^3 \subset \mathbb{R}^3$ and analyze the values of H on these subsets of $\mathcal{C}$.*

As far as the values go, due to the second symmetry lemma above we need only consider the special diagonal sets

$$\mathcal{D} = \mathcal{D}_k = \{(\theta_1, \theta_2, \ldots, \theta_n) \in [-\pi, \pi]^n : \theta_1 = \theta_2 = \cdots = \theta_k, \ \theta_{k+1} = \cdots = \theta_n\}$$

for $k = 1, 2, \ldots, n$.

Incidentally, I'm not entirely happy with the statement(s) and application(s) of the first and second symmetry lemmas above. I think they are at least sort of correct, but I'm guessing there's a much better way to say (precisely) what I'm trying to say there. If you have any ideas, let me know.