Putnam problem B5 (2020 exam)

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September 28, 2022

1 Statement

Given four complex numbers $z_1, z_2, z_3, z_4 \in \mathbb{C}\backslash\{1\}$ with $|z_j| = 1$, $j = 1, 2, 3, 4$, it is not possible that

$$z_1 + z_2 + z_3 + z_4 - z_1 z_2 z_3 z_4 = 3. \tag{1}$$

2 Comments

Generally, it seems reasonable to proceed by contradiction, so unless otherwise stated, let's assume (1) holds. This means the following two (real) equations hold

$$\text{Re}(z_1) + \text{Re}(z_2) + \text{Re}(z_3) + \text{Re}(z_4) - \text{Re}(z_1 z_2 z_3 z_4) = 3 \tag{2}$$

and

$$\text{Im}(z_1) + \text{Im}(z_2) + \text{Im}(z_3) + \text{Im}(z_4) - \text{Im}(z_1 z_2 z_3 z_4) = 0. \tag{3}$$

Every complex number z in the unit circle, i.e., satisfying $|z| = 1$ has $-1 \le \text{Re}(z) \le 1$. If $z \ne 1$, then the inequality on the right is strict. Thus, we can start with a basic estimate:

$$\text{Re}(z_1) + \text{Re}(z_2) + \text{Re}(z_3) + \text{Re}(z_4) - \text{Re}(z_1 z_2 z_3 z_4) < 4 - \text{Re}(z_1 z_2 z_3 z_4). \tag{4}$$

Since $|z_1 z_2 z_3 z_4| = |z_1||z_2||z_3||z_4| = 1$, the product $w = z_1 z_2 z_3 z_4$ satisfies $|w| = 1$, and we can say we have five complex numbers in the unit circle, i.e., satisfying $|z| = 1$, with

$$z_1 + z_2 + z_3 + z_4 - w = 3.$$

Of course, maybe $w = 1$. For example if $z_1 = z_2 = z_3 = z_4 = i$, then $w = 1$. In this case, however, $z_1 + z_2 + z_3 + z_4 - z_1z_2z_3z_4 = z_1 + z_2 + z_3 + z_4 - w = 4i - 1$ is not even real. We could also take $z_1 = z_2 = i$ and $z_3 = z_4 = -i$, so that again $w = 1$ and $z_1 + z_2 + z_3 + z_4 - z_1z_2z_3z_4 = -1 \neq 3$.

In any case, we conclude $\mathrm{Re}(w) \leq 1$. In fact, $-1 \leq \mathrm{Re}(w) \leq 1$. And for the other four numbers we have

$$-1 \leq \mathrm{Re}(z_j) < 1.$$

The estimate (4) also gives

$$\mathrm{Re}(z_1) + \mathrm{Re}(z_2) + \mathrm{Re}(z_3) + \mathrm{Re}(z_4) - \mathrm{Re}(z_1z_2z_3z_4) < 4 - \mathrm{Re}(w) \leq 5$$

with equality on the right only if $w = -1$. This is, of course, a long way from contradicting (2). Nevertheless, this kind of estimate can rule out some possibilities.

The possibility that $z_k = -1$ for any k can be ruled out: If for example $z_4 = -1$, then the relation (1) becomes

$$z_1 + z_2 + z_3 + z_1z_2z_3 = z_1 + z_2 + z_3 - w = 4.$$

Thus,

$$4 = \mathrm{Re}(z_1) + \mathrm{Re}(z_2) + \mathrm{Re}(z_3) - \mathrm{Re}(w) \leq \mathrm{Re}(z_1) + \mathrm{Re}(z_2) + \mathrm{Re}(z_3) + 1 < 4.$$

We conclude $z_j \in \mathbb{C}\backslash\{\pm 1\}$ for $j = 1, 2, 3, 4$. In particular,

$$-1 < \mathrm{Re}(z_j) < 1 \qquad \text{for } j = 1, 2, 3, 4.$$

Similarly, if $\mathrm{Re}(z_3), \mathrm{Re}(z_4) \leq 0$, then

$$3 = \mathrm{Re}(z_1) + \mathrm{Re}(z_2) + \mathrm{Re}(z_3) + re(z_4) - \mathrm{Re}(w) \leq \mathrm{Re}(z_1) + \mathrm{Re}(z_2) - \mathrm{Re}(w) < 3.$$

We conclude at most one of the numbers z_1, z_2, z_3, z_4 has a non-positive real part. Without loss of generality

$$0 < \mathrm{Re}(z_j) < 1 \qquad \text{for } j = 1, 2, 3.$$

We can write

$$z_j = e^{i\theta_j} = \cos\theta_j + i\sin\theta_j \qquad \text{with } -\pi < \theta_j < \pi \text{ for } j = 1, 2, 3, 4.$$

Also, we have $\theta_j \neq 0$ for $j = 1, 2, 3, 4$. Alternatively, we can take $0 < \theta_j < 2\pi$ with $\theta_j \neq \pi$ for $j = 1, 2, 3, 4$.

In this polar form $w = e^{\theta_1+\theta_2+\theta_3+\theta_4}$, and (2) and (3) take the form(s)

$$\cos(\theta_1) + \cos(\theta_2) + \cos(\theta_3) + \cos(\theta_4) - \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) = 3$$

and

$$\sin(\theta_1) + \sin(\theta_2) + \sin(\theta_3) + \sin(\theta_4) - \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) = 0.$$

One is tempted to use the method of Lagrange multipliers to maximize

$$F(\theta_1, \theta_2, \theta_3, \theta_4) = \cos(\theta_1) + \cos(\theta_2) + \cos(\theta_3) + \cos(\theta_4) - \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4)$$

on $[0, 2\pi]^4 = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$ subject to the constraint

$$G(\theta_1, \theta_2, \theta_3, \theta_4) = \sin(\theta_1) + \sin(\theta_2) + \sin(\theta_3) + \sin(\theta_4) - \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) = 0. \quad (5)$$

We have

$$DF = \begin{pmatrix} -\sin\theta_1 + \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ -\sin\theta_2 + \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ -\sin\theta_3 + \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ -\sin\theta_4 + \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) \end{pmatrix}$$

and

$$DG = \begin{pmatrix} \cos\theta_1 - \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ \cos\theta_2 - \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ \cos\theta_3 - \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ \cos\theta_4 - \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) \end{pmatrix}.$$

These lead to the system of equations

$$\begin{aligned} \sin\theta_1 + \lambda\cos\theta_1 &= \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) + \lambda\cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ \sin\theta_2 + \lambda\cos\theta_2 &= \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) + \lambda\cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ \sin\theta_3 + \lambda\cos\theta_3 &= \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) + \lambda\cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ \sin\theta_4 + \lambda\cos\theta_4 &= \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) + \lambda\cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) \end{aligned}$$

for θ_j, $j = 1, 2, 3, 4$ and $\lambda \in \mathbb{R}$. Of course, we also have the constraint equation (5).

The idea would be to show this system of five equations has no interior solution $(\theta_1, \theta_2, \theta_3, \theta_4, \lambda) \in (0, 2\pi)^4 \times \mathbb{R}$ or perhaps more properly in some other (more complicated) set obtained by removing the points where one of the angles θ_j satisfies $\theta_j = \pi$ for some $j = 1, 2, 3, 4$—these points should also be considered boundary values. Then one would need to consider the boundary values and show $F(\theta_1, \theta_2, \theta_3, \theta_4) \leq 3$ for all the boundary values. I don't see what to do with the system of equations, but let's look a little bit at the latter question. If $\theta_4 = 0$ or 2π, then we are looking at

$$F(\theta_1, \theta_2, \theta_3, 0) = \cos(\theta_1) + \cos(\theta_2) + \cos(\theta_3) + 1 - \cos(\theta_1 + \theta_2 + \theta_3)$$

for $(\theta_1, \theta_2, \theta_3) \in (0, 2\pi)^3$. Thus, we would like to show

$$F_3(\theta_1, \theta_2, \theta_3) \leq 2 \qquad \text{for } (\theta_1, \theta_2, \theta_3) \in [0, 2\pi]^3$$

where

$$F_3(\theta_1, \theta_2, \theta_3) = \cos(\theta_1) + \cos(\theta_2) + \cos(\theta_3) - \cos(\theta_1 + \theta_2 + \theta_3).$$

We have

$$DF_3 = \begin{pmatrix} -\sin\theta_1 + \sin(\theta_1 + \theta_2 + \theta_3) \\ -\sin\theta_2 + \sin(\theta_1 + \theta_2 + \theta_3) \\ -\sin\theta_3 + \sin(\theta_1 + \theta_2 + \theta_3) \end{pmatrix}.$$

If this vanishes, then $\sin\theta_j = \sin(\theta_1 + \theta_2 + \theta_3)$ for $j = 1, 2, 3$. Again, I'm not sure what to do with this system, but it seems like we've achieved (or come close to achieving) some kind of reduction.

3 An aside

Casting about for simpler versions of the problem, I noted that generally it seems like if we could show (1) fails, that is, either

$$z_1 + z_2 + z_3 + z_4 - z_1 z_2 z_3 z_4 \neq \operatorname{Re}(z_1) + \operatorname{Re}(z_2) + \operatorname{Re}(z_3) + \operatorname{Re}(z_4) - \operatorname{Re}(z_1 z_2 z_3 z_4)$$

or

$$\operatorname{Re}(z_1) + \operatorname{Re}(z_2) + \operatorname{Re}(z_3) + \operatorname{Re}(z_4) - \operatorname{Re}(z_1 z_2 z_3 z_4) \neq 3,$$

it is likely that for the (real) numbers $x_j = \operatorname{Re}(z_j)$ and $\xi = \operatorname{Re}(w)$ for which

$$z_1 + z_2 + z_3 + z_4 - z_1 z_2 z_3 z_4 = x_1 + x_2 + x_3 + x_4 - \xi$$

there holds

$$x_1 + x_2 + x_3 + x_4 - \xi < 3.$$

Now, I know it is not the case that $\xi = \operatorname{Re}(w) = \operatorname{Re}(z_1 z_2 z_3 z_4)$ is equal to the product of the real parts $x_1 x_2 x_3 x_4$. Nevertheless, I asked the following question as a modification of the problem:

New problem: If $x_j \in (-1,1)$ for $j = 1,2,3,4$, then show

$$x_1 + x_2 + x_3 + x_4 - x_1x_2x_3x_4 < 3.$$

I was able to show this and a rather significant generalization of it. If I've got the assertions correct, I think you can show the following using induction:

Consider $f_n : \mathbb{R}^n \to \mathbb{R}$ and $g_n : \mathbb{R}^n \to \mathbb{R}$ by

$$f_n(\mathbf{x}) = \sum_{j=1}^{n} x_j - \prod_{j=1}^{n} x_j \qquad \text{and} \qquad g_n(\mathbf{x}) = \sum_{j=1}^{n} x_j + \prod_{j=1}^{n} x_j.$$

For $n \geq 2$ there holds

1. $-n-1 \leq f_n(\mathbf{x}) \leq n-1$ for $\mathbf{x} \in [-1,1]^n$ with equality on the left if and only if

   $$x_j = -1, \quad j = 1,2,\ldots,n \tag{6}$$

   and equality on the right if and only if there exists some $m \in \{1,2,\ldots n\}$ such that

   $$x_j = 1, \quad j \neq m. \tag{7}$$

2. $-n+1 \leq g_n(\mathbf{x}) \leq n+1$ for $\mathbf{x} \in [-1,1]^n$ with equality on the left if and only if there exists some $m \in \{1,2,\ldots n\}$ such that

   $$x_j = -1, \quad j \neq m \tag{8}$$

   and equality on the right if and only if

   $$x_j = 1, \quad j = 1,2,\ldots,n. \tag{9}$$

Corollary 1 *For $n \geq 2$*

$$f_n(\mathbf{x}) \leq n-1 \qquad \textit{for } \mathbf{x} \in [-1,1]^n$$

with equality if and only if there exists some $m \in \{1,2,\ldots,n\}$ such that

$$x_j = 1, \quad j \neq m.$$

In particular, $f_n(\mathbf{x}) < n-1$ for $\mathbf{x} \in [-1,1)^n$, and in particular

$$f_4(\mathbf{x}) = x_1x_2x_3x_4 - x_1x_2x_3x_4 < 3 \qquad \text{if } -1 \leq x_j < 1,\ j = 1,2,3,4.$$