Putnam problem A3 (2020 exam)

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1 Preliminary discussion:

I read over this problem and, as I often do, started to think about it liesurely, over several hours, just going to write down ideas or details when they came to mind. Of course, those taking the exam do not have the luxury to take such an approach, but fortunately, I do not have to actually take the exam in a timed fashion. Unfortunately, in this instance the problem I was thinking about in my head was not the same as the stated problem A3 on the 2020 exam. It was this problem:

Misstatement

If $a_0 = \pi/2$ and $a_n = \sin(a_{n-1})$ for $n = 1, 2, 3, \ldots$, then determine the convergence/divergence of the series

$$\sum_{n=0}^{\infty} a_n. \tag{1}$$

A first thing to note about this series is that $a_1 = \sin(\pi/2) = 1 < \pi/2 \approx 3/2$. It follows that $0 < a_n < \pi/2$ for $n = 1, 2, 3, \ldots$. Therefore, the partial sums

$$\sum_{n=1}^{k} a_n$$

are increasing in k, and the only question is: Are these partial sums bounded above or not?

Perhaps a second thing to note is that the terms are decreasing: $a_n < a_{n-1}$, so the question is, in some sense, how quickly are they decreasing. My first inclination was that probably the series canverged, i.e., was bounded above. This inclination was loosely based on the observation that $\sin x < x$ and the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges for $p > 1$ (and $x^p < x$ for $0 < x < 1$ when $p > 1$). But then I drew this picture:

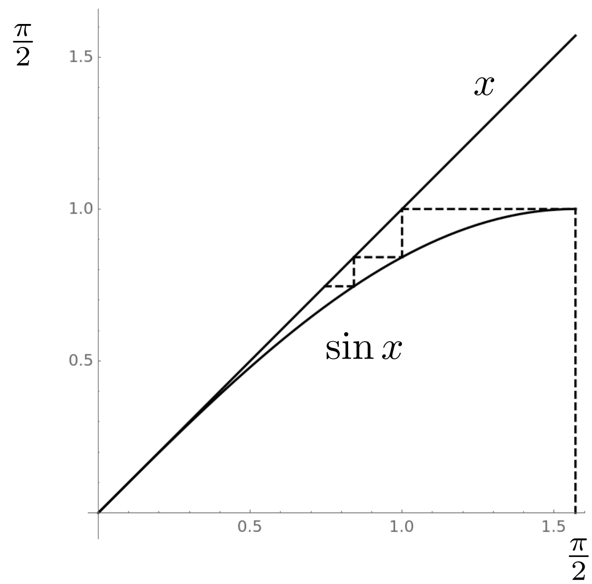

Figure 1: The values of iterates of sine starting with $1 = \sin(\pi/2)$ compared to $y = x$.

This made me realize two things. First of all, these iterates are getting extremely close to $y = x$ and second, I remembered, that for $p > 1$ the function $f(x) = x^p$ is convex and not really close[1] to $y = x$ for $0 < x < 1$, at least in the sense of growth—more precisely, the derivatives $y' \equiv 1$ and $f'(x) = px^{p-1}$ are bounded away from one another near $x = 0$, and $f(x) = x^p$ is much much smaller. This made me skeptical about my first inclination and I started trying to prove divergence to infinity, i.e., that the sequence of partial sums is unbounded.

[1] I've not plotted $f(x) = x^p$ for $p > 1$ in Figure 1, but you might want to draw it in.

Notice $a_1 = \sin(\pi/2) = 1$. Thus, in some sense, $a_2 = \sin(1)$ is the first interesting term to estimate. I know

$$1 - \frac{x^3}{3!} < \sin x. \tag{2}$$

In particular, $\sin(1) > 5/6$. At this point, I figured I was going to have to (or at least I decided to try to) compare the series in question to the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Certainly it was true that $a_1 = 1$ and $a_2 > 5/6 > 1/2$. So then I tried an induction:

If I know $a_k > 1/k$, then

$$a_{k+1} = \sin(a_k) > \frac{1}{k} - \frac{1}{6k^3} = \frac{6k^2 - 1}{6k^2}.$$

So the question becomes: Is it true that

$$\frac{6k^2 - 1}{6k^3} > \frac{1}{k+1} \qquad ?$$

This inequality would be implied by

$$6k^3 < (k+1)(6k^2 - 1) = 6k^3 + 6k^2 - k - 1$$

which would be implied by the inequality

$$6k^2 > k + 1.$$

This is clearly true for $k \geq 1$ since $6k^2 - k - 1 = (3k+1)(2k-1)$ and both the factors are positive for $k \geq 1$. So I had a (first) solution:

By induction $a_n \geq 1/n$ for $n \geq 1$. Therefore,

$$\sum_{n=0}^{\infty} a_n = \frac{\pi}{2} + \sum_{n=1}^{\infty} a_n \geq \frac{\pi}{2} + \frac{1}{n} = \infty.$$

But this bothered me a bit, though I imagined this was the solution the author of the problem expected. One thing that bothered me was the first term $a_0 = \pi/2$. I wondered, in particular, if I could incorporate the first term, and with that thought came something that seemed interesting:

$$a_0 = \frac{\pi}{2}, \quad a_1 = 1 > \frac{\pi}{2}\frac{1}{2} = \frac{\pi}{4} \approx \frac{3}{4}, \ a_2 = \sin(1) > \frac{5}{6} > \frac{\pi}{2}\frac{1}{3}.$$

These are not tighter inequalities, but they do include a_0 and suggest a pattern. More precisely, I wondered about the (pretty) guess/conjecture:

$$a_n \geq \frac{\pi}{2}\frac{1}{n+1} \qquad \text{for } n = 0, 1, 2, 3, \dots \text{ with strict inequality for } n > 0. \tag{3}$$

Obviously we have our base cases. So now I assume I know

$$a_k \geq \frac{\pi}{2}\frac{1}{k+1}.$$

Then

$$a_{k+1} = \sin(a_k) > \frac{\pi}{2}\left(\frac{1}{k+1} - \frac{\pi^2}{24(k+1)^3}\right) = \frac{\pi}{2}\frac{24k^2 + 48k + 24 - \pi^2}{24(k+1)^3},$$

and the question becomes: Is it true that

$$\frac{24k^2 + 48k + 24 - \pi^2}{24(k+1)^3} > \frac{1}{k+2} \qquad ?$$

Or put another way, is it true that

$$24(k^3 + 3k^2 + 3k + 1) < (k+2)(24(k^2 + 2k + 1) - \pi^2) \qquad ?$$

Since $(k+2)(k^2+2k+1) = k^3+4k^2+5k+2$, this would be implied by the polynomial inequality

$$\pi^2(k+2) < 24k^2 + 48k + 24 \tag{4}$$

which is clearly true for k large enough. In fact, $\pi^2 < 12$, which implies $\pi^2 < 48$ and $2\pi^2 < 24$ so this inequality holds for $k \geq 0$. Of course, estimating $\pi < 4$ or even $\pi < 3.5$ is not good enough to see $\pi^2 < 12$. But it is also true that $\pi < 32/100$ which means $\pi^2 < 1024/100 = 10.24 < 12$.

This establishes (3) and leads to the (perhaps in some ways prettier) solution

$$\sum_{n=0}^{\infty} a_n \geq \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{1}{n+1} = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n} = \infty.$$

When I finished this, I thought also that it would be good enough to show

$$a_n > \frac{1}{n+1} \qquad \text{for } n = 0, 1, 2, 3, \dots,$$

and maybe the induction would be easier. (This is clearly true in view of (3) because $\pi/2 \approx 1.5 > 1$.) Here is the inductive step: Assume

$$a_k > \frac{1}{k+1}.$$

Then

$$a_{k+1} > \frac{1}{k+1}\left(1 - \frac{1}{6(k+1)^2}\right) = \frac{6k^2+12k+5}{6(k+1)^3}.$$

Now the question is simply, is this quantity greater than $1/(k+2)$? Well,

$$(k+2)(6k^2+12k+5) = 6k^3+24k^2+29k+10, \quad \text{and}$$
$$6(k+1)^3 = 6k^3+18k^2+18k+6,$$

so indeed

$$(k+2)(6k^2+12k+5) > 6(k+1)^3 \qquad \text{for } k \geq 0, \text{ and} \qquad \frac{6k^2+12k+5}{6(k+1)^3} > \frac{1}{k+2}.$$

That's a third somewhat pretty solution:

$$\sum_{n=0}^{\infty} a_n \geq \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Exercise 1 *Does* (3) *follow directly from the fact that* $a_n \geq 1/n$ *for* $n \geq 1$*?*

So I was pretty happy with myself at this point. I'd solved a Putnam problem without looking at the solution. It wasn't too difficult, but it was somewhat difficult, and it has some interesting aspects to it—see the follow-up discussion below. I decided to look at the published solution just to double check my answer and see if I could get any new ideas or better understand what they were expecting as a solution. As I read the solution it became apparent pretty quickly that it was the solution for a different problem. So I went back and read the statement of the problem again. Unfortunately for me, I had misread or misremembered the statement of the problem. Here is the actual statement:

Statement

If $a_0 = \pi/2$ and $a_n = \sin(a_{n-1})$ for $n = 1, 2, 3, \ldots$, then determine the convergence/divergence of the series

$$\sum_{n=0}^{\infty} a_n^2. \tag{5}$$

The terms in the sum are **squared** (!).

Now the proof they had was essentially the same as my proof (for the misstated problem), so that was sort of good. But in principle, the actual problem is much harder than the problem I had solved. More precisely, I don't really have a good intuition as to why to write down the inductive statement/hypothesis

$$a_n \geq \frac{1}{\sqrt{n}} \qquad \text{for } n \geq 1. \tag{6}$$

The solution didn't give any explanation either, but this turns out to be true. Once you have (6) you're back in a position to compare to the harmonic series—so I guess that intuition was correct, but I don't see geometrically (or any other way) that it is obvious that (6) holds. And I don't really see any good reason to argue for divergence to infinity (apriori) for the series in the actual problem.

2 Solution

We should perhaps start with the observations above: We have for all $n = 0, 1, 2, 3, \ldots$

$$0 < a_n \leq \frac{\pi}{2} \qquad \text{with strict inequality for } n \geq 1$$

so that

$$0 < a_{n+1} < a_n \qquad \text{for all } n = 0, 1, 2, 3, \ldots.$$

Claim:

$$a_n \geq \frac{1}{\sqrt{n}} \qquad \text{for } n \geq 1. \tag{7}$$

For $n = 1$, we know $a_1 = 1$, so the inequality holds with equality. Next assume $k \geq 1$ and

$$a_k \geq \frac{1}{\sqrt{k}}.$$

Then

$$a_{k+1} = \sin(a_k) > \sin\left(\frac{1}{\sqrt{k}}\right) > \frac{1}{\sqrt{k}}\left(1 - \frac{1}{6k}\right) = \frac{1}{\sqrt{k}}\,\frac{6k-1}{6k}.$$

So the question is: Is this quantity greater than $1/\sqrt{k+1}$? This would be implied by the inequality

$$(6k-1)\sqrt{k+1} > 6k\sqrt{k},$$

which would be implied by

$$(36k^2 - 12k + 1)(k + 1) > 36k^3,$$

which would be implied by

$$24k^2 - 11k + 1 = 24(k - 11/48)^2 + 1 - (11/48)^2 > 0.$$

This is true for $k \geq 1$ since $1 > 11/48 > (11/48)^2$. Alternatively, $24k^2 - 11k + 1 = (3k - 1)(8k - 1) > 0$ for $k \geq 1$.

In view of (7) we have

$$\sum_{n=0}^{\infty} a_n^2 \geq \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \qquad \square$$

Incidentally, my "pretty stuff" doesn't work here. Take the hypothesis

$$a_n \geq \frac{\pi}{2} \frac{1}{\sqrt{n+1}}.$$

For $n = 0$ this holds, but for $n = 1$ you're asking for

$$1 \geq \frac{\pi}{2\sqrt{2}} \qquad \text{or} \qquad 8 \geq \pi^2,$$

and this of course is not true.

3 Follow-up

Even before I saw/comprehended the real problem, much less saw the solution, I had a couple follow-up or warm-up questions/topics to go along with this problem.

harmonic series

One of the topics I had in mind is the demonstration that the harmonic series diverges to infinity.

Exercise 2 *Show*

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Solution: I like the estimate

$$\sum_{n=1}^{2^k} \frac{1}{n} > 1 + \frac{k}{2} = \frac{k+2}{2}. \tag{8}$$

To see this, let n be arbitrary and consider the terms in the series following $1/j$:

$$\frac{1}{j+1} + \frac{1}{j+2} + \frac{1}{j+3} + \cdots + \frac{1}{j+m}.$$

Each of these terms is at least $1/(j+m)$. All but one of them is strictly greater, and there are m of them. Therefore, the sum of these terms satisfies

$$\sum_{n=j+1}^{j+m} \frac{1}{n} \geq \frac{m}{j+m}.$$

In particular, taking $m = 2j$ we find

$$\sum_{n=j+1}^{2j} \frac{1}{n} \geq \frac{1}{2}.$$

Take $j = 1$ to get

$$\sum_{n=1}^{2} \frac{1}{n} = 1 + \frac{1}{2}.$$

Then assuming

$$\sum_{n=1}^{2^\ell} \frac{1}{n} > 1 + \frac{\ell}{2},$$

we can consider $j = 2^\ell$ and the next $2^{\ell+1}$ terms:

$$\sum_{n=1}^{2^{\ell+1}} \frac{1}{n} = \sum_{n=1}^{2^\ell} \frac{1}{n} + \sum_{n=2^\ell+1}^{2^{\ell+1}} \frac{1}{n} > 1 + \frac{\ell}{2} + \frac{1}{2} = 1 + \frac{\ell+1}{2}.$$

Thus, (8) follows by induction. In particular,

$$\sum_{n=1}^{\infty} \frac{1}{n} > 1 + \frac{k}{2} \qquad \text{for every } k = 1, 2, 3, \ldots$$

and the harmonic series is not bounded above. Of course, the posted solution of problem A3 (2020) assumed the divergence of the harmonic series.

Estimating sine with the Taylor polynomial

The posted solution, on the other hand, gave a proof of my assertion (2), which I would have assumed could be quoted without proof. On the other hand, something much more general is true which is fun to know about. Let's start with the published proof since that uses Taylor's formula with remainder which we've recently considered. The third order formula with remainder for the sine function is

$$\sin x = x - \frac{x^3}{3!} + \frac{d^4}{dt^4} \sin t\Big|_{t=x^*} \frac{x^4}{4!}.$$

In this case, the fourth derivative of $\sin t$ is $\sin t$, so the remainder term is

$$\sin x^* \, \frac{x^4}{4!}$$

for some x^* between 0 and x. Thus, if x is between 0 and π, which in this application we know happens to be between 0 and $\pi/2$, then the remainder is positive, and we get

$$\sin(x) > x - \frac{x^3}{3!} \qquad \text{for } 0 < x < \pi.$$

In fact, the inequality (2) holds as follows:

$$\sin(x) > x - \frac{x^3}{3!} \qquad \text{for } x > 0.$$

This does require some proof. Let's start with a preliminary exercise:

Exercise 3 *Write down the Taylor expansions at $x = 0$ for sine and cosine.*

Solution:

$$\sin x = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \, x^{2j+1},$$

$$\cos x = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \, x^{2j}.$$

It will be noted that each of these series is alternating, and the claim is that each additional nonzero term of the series gives a **global monotone estimate** for the value of the respective trigonometric function. This does not quite work for the first term of the series for cosine, but we do get

$$\cos x \leq 1 \qquad \text{a global estimate from above for all } x \in \mathbb{R}. \tag{9}$$

Now consider the estimate for sine using the first term:

$$\sin x \le x.$$

We can get this from integrating (9), and this tells us two interesting things. First the estimate holds for all $x \ge 0$ and is strict for $x > 0$:

$$\sin x < x \qquad \text{for } x > 0. \tag{10}$$

Second, taking the positive difference $g(x) = x - \sin x$ and differentiating we see the derivative $g'(x) = 1 - \cos x \ge 0$ with only isolated zeros at $x = 2k\pi$, $k = 0, 1, 2, 3, \ldots$. This implies a rather strong version of (10):

$$0 < x - \sin x < \max_{0<t\le x_0} (t - \sin t) = x_0 - \sin x_0 \qquad \text{for } 0 \le x < x_0.$$

Notice that $x > \sin x$ and the difference $g(x)$ is strictly monotone increasing, even though the derivative g' sometimes vanishes. This is what we mean by a global monotone estimate.

Exercise 4 *Plot the first few terms of the Taylor series for* $\sin x$ *and* $\cos x$ *in compraison with* $\sin x$ *and* $\cos x$ *respectively.*

Of course, things only get better. Integrating (10) we find

$$\int_0^x \sin t\, dt < \int_0^x t\, dt \qquad \text{or} \qquad -\cos x + 1 < \frac{x^2}{2} \qquad \text{for } x > 0.$$

That is,

$$\cos x > 1 - \frac{x^2}{2} \qquad \text{globally for } x > 0. \tag{11}$$

Exercise 5 *State (and prove) a stronger version of* (11) *as a global monotone estimate.*

Integrating estimate (11) brings us to the global estimate (2) I quoted at the outset of my solution to the misstated problem:

$$\sin x > x - \frac{x^3}{6} \qquad \text{globally for } x > 0.$$

This estimate, and every such estimate for cosine and sine using the Taylor expansions, is a global monotone estimate with the "worst" approximation precisely at the right endpoint.

Exercise 6 *State and prove an inductive hypothesis giving all the global monotone estimates for cosine and sine in terms of Taylor polynomials.*

As a final note, when and if one would actually want to use a Taylor expansion to approximate the value of sine or cosine[2] at a particular value of $x > 0$, then using the symmetry and periodicity of the trigonometric functions one really only needs to consider these polynomial approximations on the interval $[0, \pi/2]$, and the error values are given by the next nonzero remainder. For example, if one wishes to approximate $\sin x$ for $x = 7\pi + 0.1$, then one can use the value approximating $-\sin(0.1)$ or

$$\sin(7\pi + 0.1) \approx -0.1 + \frac{(0.1)^3}{6}$$

and know one is off (in fact, above or to the right) by no more than

$$\frac{(0.1)^5}{5!} = \frac{(0.1)^5}{120}.$$

More generally, if one wants to approximate the value of $\sin x$ on the entire interval $7\pi \leq x \leq 7\pi + \pi/2$, then one can use the value

$$\sin x \approx -(x - 7\pi) + \frac{(x - 7\pi)^3}{6}$$

and know the approximation is off (in the same direction as the one at $7\pi + 0.1$) by no more than

$$\frac{(x - 7\pi)^5}{120} < \frac{\pi^5}{2^5(120)}$$

with the latter estimate applying globally on the entire interval. Thus, generally speaking, the global error values

$$\frac{\pi^n}{2^n(n!)}$$

are of interest. You can approximate sine or cosine on the entire real line using reduction formulas and a polynomial of order $n - 2$ to this accuracy.

Exercise 7 *Compile a table of how many $(n - 2)$ terms you need in a Taylor expansion for sine (and one for cosine) in order to get an accuracy*

$$\frac{\pi^n}{2^n(n!)} < 0.1, 0.001, 0.0001, 0.00001.$$

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[2]Incidentally, this is precisely how early computers or calculators would calculate such a value; I've seen it translated from machine language.