Problem A3 of the 2018 Putnam Exam (currently unsolved)

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October 23, 2022

1 Original statement:

Maximize
$$\sum_{j=1}^{10} \cos(3\theta_j)$$
subject to
$$\sum_{j=1}^{10} \cos(\theta_j) = 0.$$

2 First rephrasing:

Consider the problem as the special case where $n = 10$ of maximizing
$$\sum_{j=1}^{n} \cos(3\theta_j)$$
subject to
$$\sum_{j=1}^{n} \cos(\theta_j) = 0.$$

$$
\begin{aligned}
\cos(3\theta) &= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\cos\theta\sin^2\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\cos\theta(1-\cos^2\theta) \\
&= 4\cos^2\theta - 3\cos\theta.
\end{aligned}
$$

Thus, we are being asked to maximize

$$4\sum_{j=1}^{n}\cos^3\theta_j - 3\sum_{j=1}^{n}\cos\theta_j$$

subject to

$$\sum_{j=1}^{n}\cos(\theta_j) = 0.$$

Given the constraint

$$4\sum_{j=1}^{n}\cos^3\theta_j - 3\sum_{j=1}^{n}\cos\theta_j = 4\sum_{j=1}^{n}\cos^3\theta_j,$$

so the original problem is essentially equivalent to this rephrasing:

Maximize

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} x_j^3 \qquad \text{on } [-1,1]^n$$

subject to

$$\sum_{j=1}^{n} x_j = 0.$$

Assuming a nonnegative maximum value for this problem, the maximum value of the original problem (for general n) will be four times that value.

3 Second rephrasing—substitution

We could approach the problem via Lagrange multipliers. Instead I will substitute for $n \geq 3$

$$x_n = -\sum_{j=1}^{n-1} x_j = -\sum_{j=1}^{n-2} x_j - x_{n-1} = -s - x_{n-1}$$

where

$$s = \sum_{j=1}^{n-2} x_j.$$

In this way, I will obtain an unconstrained problem on a different domain. The new objective function is obtained as follows: For $n \geq 3$

$$\begin{aligned} f &= \sum_{j=1}^{n-2} x_j^3 + x_{n-1}^3 + x_n^3 \\ &= \sum_{j=1}^{n-2} x_j^3 + (x_{n-1} + x_n)(x_{n-1}^2 - x_{n-1}x_n + x_n^2). \end{aligned}$$

With the substitution we write $g = g(x_1, x_2, \dots, x_{n-1})$ as

$$\begin{aligned} g &= \sum_{j=1}^{n-2} x_j^3 - s\left(x_{n-1}^2 + x_{n-1}(s + x_{n-1}) + (s + x_{n-1}^2\right) \\ &= \sum_{j=1}^{n-2} x_j^3 - s\left(s^2 + 3sx_{n-1} + 3x_{n-1}^2\right). \end{aligned}$$

Note: There should be a simpler expression for g. This expression does not make it entirely clear that g is symmetric in all the variables. We know this is the case, however, because

$$g = \sum_{j=1}^{n-1} x_j^3 - \left(\sum_{j=1}^{n-1} x_j\right)^3.$$

Incidentally, in the case $n = 2$, we have

$$f = x_1^3 + x_2^3 = (x_1 + x_2)(x_1^2 - x_1x_2 + x_2^2) \equiv 0 \qquad \text{(given the constraint } x_1 + x_2 = 0\text{)}.$$

In the general case, we wish to maximize g on

$$A = [-1, 1]^{n-1} \cap \left\{ (x_1, \dots, x_{n-1}) : \left| \sum_{j=1}^{n-1} x_j \right| \leq 1 \right\}.$$

Consideration of the cases $n = 3$ and $n = 4$ can help with seeing the structure of the set A.

4 Third rephrasing/first reduction:

I claim that for $n \geq 3$ the function g can have no interior maximum on A. To see this, consider the gradient entries in ∇g:

$$\begin{aligned}
\frac{\partial g}{\partial x_k} &= 3x_k^2 - 3\left(\sum_{j=1}^{n-1} x_j\right)^2 \\
&= -3\left[\left(\sum_{j=1}^{n-1} x_j\right)^2 - x_k^2\right] \\
&= -3\left[\left(\sum_{j=1}^{n-1} x_j - x_k\right)\left(\sum_{j=1}^{n-1} x_j + x_k\right)\right].
\end{aligned}$$

At an interior critical point $(x_1, \ldots, x_{n-1})$ for g, we have the system of equations

$$\left(\sum_{j=1}^{n-1} x_j - x_k\right)\left(\sum_{j=1}^{n-1} x_j + x_k\right) = 0, \qquad k = 1, 2, \ldots, n-1. \tag{1}$$

Lemma 1 *If* $\nabla g(x_1, \ldots, x_{n-1}) = (0, \ldots, 0)$, *then* $g(x_1, \ldots, x_{n-1}) = 0$.

Proof: Each of the equations corresponding to $k = 1, \ldots, n-2$ can be written as

$$x_k^2 = (s + x_{n-1})^2.$$

Thus, at an interior critical point we have

$$\begin{aligned}
g &= \sum_{j=1}^{n-2} (s + x_{n-1})^2 x_j - s\left(s^2 + 3sx_{n-1} + 3x_{n-1}^2\right) \\
&= s(s + x_{n-1})^2 - s\left(s^2 + 3sx_{n-1} + 3x_{n-1}^2\right) \\
&= -s(sx_{n-1} + 2x_{n-1}^2) \\
&= -sx_{n-1}(s + 2x_{n-1}).
\end{aligned}$$

The last such equation corresponding to $k = n-1$ is

$$s(s + 2x_{n-1}) = 0. \qquad \square$$

In view of the lemma, if we can show any value of g is strictly positive, then we can reduce the maximization of g to the maximization of g on ∂A. For $n \geq 3$, g does in fact take positive values. In fact, we can take any two of $x_1, \ldots, x_{n-1}$ to be $-1/2$ and all the rest (if there are any) to satisfy $x_j = 0$. Then

$$\sum_{j=1}^{n-1} x_j = -1 \qquad \text{and} \qquad g = -\frac{1}{8} - \frac{1}{8} - (-1) = \frac{3}{4} > 0.$$

The maximum must occur on ∂A in all cases. Thus, we have reduced to the following problem:

Maximize

$$g = \sum_{j=1}^{n-1} x_j^3 - \left(\sum_{j=1}^{n-1} x_j \right)^3.$$

on ∂A where

$$A = [-1,1]^{n-1} \cap \left\{ (x_1, \ldots, x_{n-1}) : \left| \sum_{j=1}^{n-1} x_j \right| \leq 1 \right\}.$$

5 Fourth rephrasing

The boundary of A contains of $n-1$ faces in the hyperplanes $x_k = -1$ for $k = 1, 2, \ldots, n-1$. Each of these faces is in an $n-2$ dimensional plane and can be said to be of dimension $n-2$. Similarly, there are $n-1$ faces in the hyperplanes $x_k = 1$ for $k = 1, 2, \ldots, n-1$. The remainder of the boundary consists of two remaining $n-2$ dimensional hyperfaces within the hyperplanes

$$\sum_{j=1}^{n-1} x_j = -1 \qquad \text{and|} \qquad \sum_{j=1}^{n-1} x_j = 1.$$

To see this decomposition, draw the sets A in the cases $n = 3$ ($A \subset \mathbb{R}^2$) and $n = 4$ ($A \subset \mathbb{R}^3$). Let's call the $x_k = -1$ hyperfaces F_k^- for $k = 1, \ldots, n-1$ and similarly F_k^+ for the hyperfaces in $x_k = 1$. Finally, let's denote the remaining hyperfaces by $G^\pm$.

Let us consider these cases one by one starting with F_{n-1}^-. Here the problem is the following:

Maximize

$$h = h(x_1, \ldots, x_{n-2}) = \sum_{j=1}^{n-2} x_j^3 - 1 - \left(\sum_{j=1}^{n-2} x_j - 1 \right)^3$$

on

$$H = [-1, 1]^{n-2} \cap \left\{ (x_1, \ldots, x_{n-2}) : 0 \le \sum_{j=1}^{n-2} x_j \le 2 \right\}.$$

The condition defining H comes from the fact that we needed

$$-1 \le \sum_{j=1}^{n-1} x_j \le 1 \qquad \text{and we're taking } x_{n-1} = -1.$$

in A. It is clear that the maximization of g on F_k^- for $k = 1, \ldots, n-2$ will give the same value as obtained from this problem. Thus, these $n-1$ maximum problems are reduced to a single problem.

Similarly, maximizing g on F_k^+ for $k = 1, \ldots, n-1$ gives the same result as this problem:

Maximize

$$h = h(x_1, \ldots, x_{n-2}) = \sum_{j=1}^{n-2} x_j^3 + 1 - \left(\sum_{j=1}^{n-2} x_j + 1 \right)^3$$

on

$$H = [-1, 1]^{n-2} \cap \left\{ (x_1, \ldots, x_{n-2}) : -2 \le \sum_{j=1}^{n-2} x_j \le 0 \right\}.$$

The two problems for the faces parallel to the coordinate hyperplanes are also related. Denoting the first objective function as $h^- : H^- \to \mathbb{R}$ and the second objective function as $h^+ : H^+ \to \mathbb{R}$ we see

$$H^+ = \{-\mathbf{x} : \mathbf{x} \in H^-\} \qquad \text{and} \qquad h^+(-\mathbf{x}) = -h^-(\mathbf{x}).$$

Consequently, if $h^-(\mathbf{x}_0) = \min_{\mathbf{x} \in H^-} h^-(\mathbf{x})$, then $h^+(-\mathbf{x}_0) = \max_{\mathbf{x} \in H^+} h^+(\mathbf{x})$. Similarly, if $h^+(\mathbf{x}_0) = \min_{\mathbf{x} \in H^+} h^+(\mathbf{x})$, then $h^-(-\mathbf{x}_0) = \max_{\mathbf{x} \in H^-} h^-(\mathbf{x})$. Either way, if we understand the maximum and minimum for one of these two problems on the coordinate faces, then we understand the maximization problem on the union of all the coordinate faces.

Let us now turn to the non-coordinate hyperfaces. Maximizing g on G^- is the following problem:

Maximize

$$h(x_1, \ldots, x_{n-2}) = \sum_{j=1}^{n-2} x_j^3 + 1 - \left(\sum_{j=1}^{n-2} x_j + 1 \right)^3$$

on

$$H = [-1, 1]^{n-2} \cap \left\{ (x_1, \ldots, x_{n-2}) : -2 \leq \sum_{j=1}^{n-2} x_j \leq 0 \right\}.$$

This will be recognized as identical to the problem of maximizing h^+ on H^+. Effectively, the face G^- is parameterized on H^+ with g taking precicely the same values as h^+, that is to say the same values g takes on F_{n-1}^+.

Similarly, g takes the same values on G^+ that g takes on F_{n-1}^-, so no higher values are attained on G^+. We have reduced the maximization of g on ∂A to the following:

Let $M = \max_{\mathbf{x} \in H} h(\mathbf{x})$ and $m = \min_{\mathbf{x} \in H} h(\mathbf{x})$ where

$$h = h(x_1, \ldots, x_{n-2}) = \sum_{j=1}^{n-2} x_j^3 - 1 - \left(\sum_{j=1}^{n-2} x_j - 1 \right)^3$$

and

$$H = [-1, 1]^{n-2} \cap \left\{ (x_1, \ldots, x_{n-2}) : 0 \leq \sum_{j=1}^{n-2} x_j \leq 2 \right\}.$$

Then

$$\max_{\mathbf{x} \in A} g(\mathbf{x}) = \max_{\mathbf{x} \in \partial A} g(\mathbf{x}) = \max\{M, -m\}.$$

6 Comments about h

If we seek interior critical points for $h : H \to \mathbb{R}$, then noting that

$$h = \sum_{j=1}^{n-2} x_j^3 - 1 - \left(\sum_{j=1}^{n-2} x_j - 1 \right)^3 = \sum_{j=1}^{n-2} x_j^3 - 1 - (s-1)^3,$$

we have

$$\frac{\partial h}{\partial x_k} = 3x_k^2 - 3(s-1)^2 \qquad \text{for } k = 1, \ldots, n-2.$$

Thus, if $\nabla h = \mathbf{0}$ at an interior point, then we must have

$$x_k^2 = (s-1)^2 \qquad \text{for } k = 1, \ldots, n-2,$$

and (at the critical point)

$$h = (s-1)^2 s - 1 - (s-1)^3 = s(s-2).$$

Since $0 \leq s \leq 2$ on H we see these values must be non-positive with the least possible value taken for $n \geq 4$ at

$$x_1 = x_2 = \cdots = x_{n-2} = \frac{1}{n-2}$$

with $s = 1$. For $n \geq 4$, we clearly get an interior point in H where the value

$$h = -1 \qquad \text{is attained.}$$

For $n = 3$, as explained in more detail below, the relation determining the critical point

$$x_1^2 = (s-1)^2 = (x_1 - 1)^2$$

already specifies $x_1 = 1/2$. This is an interior point of H, but the minimum value is not $h = -1$ but only $h(x_1) = -3/4$.

In general, we have shown that for $n \geq 4$

$$\inf_{\mathbf{x} \in \mathrm{int}(H)} h \leq -1.$$

Also, for $n \geq 4$ the interior supremum value

$$\sup_{\mathbf{x} \in \mathrm{int}(H)} h \geq 0$$

with the value $h = 0$ attained at all points $\mathbf{x} \in \mathrm{int}(H)$ with $x_1 = -x_2$ and $x_j = 0$ for $j = 3, \ldots, n-2$. When $n = 3$, as discussed below, the actual maximum value is $h = 0$ and is attained on the boundary at $x_1 = 0$ and/or $x_1 = 1$.

7 Special cases

Recall that when $n = 2$ the function f is identically zero and g is not well-defined.

When $n = 3$, the set H is $[-1, 1] \cap [0, 2] = [0, 1]$, and we wish to both minimize and maximize

$$h(x_1) = x_1^3 - 1 - (x_1 - 1)^3 = 3(x_1^2 - x_1) = 3x_1(x_1 - 1).$$

On the interval $[0, 1]$ this function has maximum $M = 0$ and minimum $m = -3/4$ at $x_1 = 1/2$. Thus, the maximum value of g on A is

$$\max\{0, 3/4\} = 3/4$$

and the maximum for the original problem when $n = 3$ is 3.

For $n \geq 4$, we know the following

$$\textit{There is no interior critical point with value outside } [0, 1], \tag{2}$$

and

$$m = \min_{\mathbf{x} \in H} h(\mathbf{x}) = \inf_{\mathbf{x} \in \mathrm{int}(H)} h(\mathbf{x}) \leq -1 \leq 0 \leq \sup_{\mathbf{x} \in \mathrm{int}(H)} h(\mathbf{x}) = \max_{\mathbf{x} \in \mathrm{int}(H)} h(\mathbf{x}). \tag{3}$$

When $n = 4$, the set H is a triangle in the x_1, x_2-plane determined by the lines $x_1 = -x_2$, $x_1 = 1$, and $x_2 = 1$. Checking the values of h around ∂H the implication of (2-3) can be used to solve the problem. In fact, for $x_1 = -x_2$, we get

$$h = x_1^3 + x_2^3 - 1 - (x_1 + x_2 - 1)^3 \equiv 0.$$

For $x_1 = 1$, we have

$$h(1, x_2) = 1 + x_2^3 - 1 - (1 + x_2 - 1)^3 \equiv 0.$$

And similarly (or by symmetry) $h(x_1, 1) \equiv 0$. Thus, for $n = 4$ we have

$$h\Big|_{\mathbf{x} \in \partial H} \equiv 0.$$

Therefore, $M = 0$ and $m = -1$. The maximum values of g and f are 1, and the original problem for $n = 4$ has answer $4 \max\{0, 1\} = 4$ taken at $\cos\theta_1 = x_1 = 1/2 = x_2 = \cos\theta_2$, $\cos\theta_3 = x_3 = -1$ and $\cos\theta_4 = x_4 = -(x_1 + x_2 + x_3) = 0$. This means when $n = 4$, the maximum is attained at $\theta_1 = \theta_2 = \pi/3$, $\theta_3 = \pi$, and $\theta_4 = \pi/2$.

In summary, we have reduced the general problem to determining the maximum and minimum values of

$$h = h(x_1, \ldots, x_{n-2}) = \sum_{j=1}^{n-2} x_j^3 - 1 - \left(\sum_{j=1}^{n-2} x_j - 1 \right)^3$$

on the region

$$H = [-1,1]^{n-2} \cap \left\{ (x_1, \ldots, x_{n-2}) : 0 \le \sum_{j=1}^{n-2} x_j \le 2 \right\}.$$

Unfortunately, consideration of the case $n = 5$, in which the region H can be visualized in $\mathbb{R}^3$, suggests that the region H grows in complexity to a certain extent with dimension. There should still be a decomposition into faces parallel to the coordinate axes along with two additional faces in the (hyper)planes

$$\sum_{j=1}^{n-2} x_j = 0 \qquad \text{and} \qquad \sum_{j=1}^{n-2} x_j = 2.$$