Problem A1 of the 2022 Putnam Exam

John McCuan

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Original Statement: Determine all ordered pairs of real numbers (a, b) such that the line $y = ax + b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.

Solution/discussion: One can offer some minor criticism of the statement, but what is intended is pretty clear. The curve

$$G = \{(x, \ln(1 + x^2)) \in \mathbb{R}^2 : x \in \mathbb{R}\} \tag{1}$$

is the graph of a function $f : \mathbb{R} \to [0, \infty)$ with values given by $f(x) = \ln(1+x^2)$, and we can begin by using what we know from calculus to get some idea of the shape of this graph/curve. In particular, the function f is smooth and even with $f(0) = 0$, $f(x) > 0$ for $x \neq 0$ so that $f'(0) = 0$ and more generally

$$f'(x) = \frac{2x}{1 + x^2} \tag{2}$$

and

$$f''(x) = \frac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = 2\,\frac{1 - x^2}{(1 + x^2)^2}. \tag{3}$$

As should be expected we can see the first derivative is odd and the second derivative is even. Also $f''(0) = 2 > 0$ corresponding to the absolute (and nondegenerate) minimum value at $x = 0$.

The first derivative has only the one zero corresponding to the global minimum at $x = 0$, and f is strictly decreasing for $x < 0$ and of course strictly increasing, symmetrically with $f'(-x) = -f'(x)$, for $x > 0$.

The second derivative has exactly two zeros at $x = \pm 1$. Thus, the graph of f is "convex up" for $-1 < x < 1$, there are inflection points at $x = \pm 1$ corresponding to the points $(\pm 1, \ln(2))$ on the curve, and the graph of f is "convex down" or "concave" for $x < -1$ and $x > 1$. We note also that $\ln(2) < 1$ since $e \approx 2.718 > 2$.

With this information we can draw a rough picture (or a very precise one as the case may be) of the curve as indicated in Figure 1.

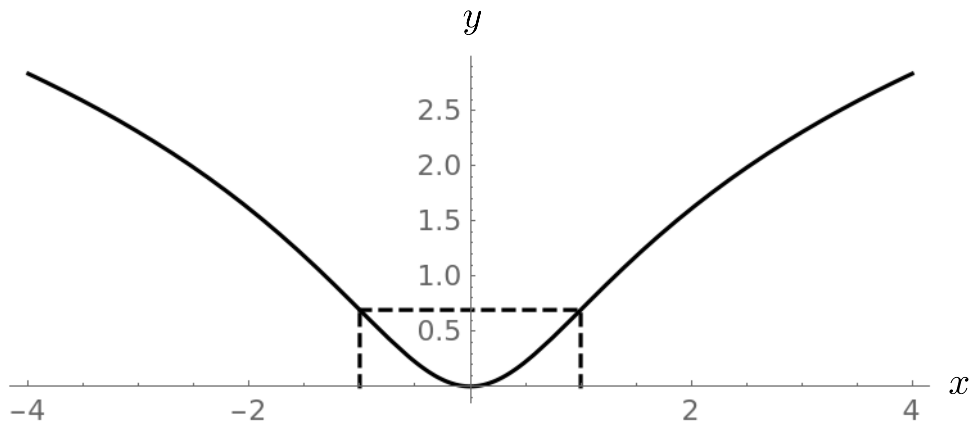

Figure 1: The graph of the function f with values $f(x) = \ln(1 + x^2)$.

With the precise determinate form of the derivative f' and the second derivative f'', we know the local behavior of each tangent line to the graph. In particular, the derivative decreases to a negative minimum of $f'(-1) = -1$ at $x = -1$ and then increases to a positive maximum of $f'(1) = 1$ at $x = 1$. We assume all values strictly between -1 and 0 are taken by $f'(x)$ uniquely for some $x < -1$, and indeed

$$\lim_{x \searrow -\infty} f'(x) = 0.$$

Similarly,

$$\lim_{x \nearrow -1} f'(x) = f(-1) = -1,$$

so all values strictly between 0 and -1 are taken by $f'(x)$ uniquely for some $x < -1$.

Starting with $t < -1$ the tangent line to the graph of f at $(t, f(t)) = (t, \ln(1+t^2))$ is given as the graph of a function $\ell : \mathbb{R} \to \mathbb{R}$ by

$$\ell(x) = \frac{2t}{1+t^2}\,(x - t) + \ln(1 + t^2) = \frac{2t}{1+t^2}\,x + \ln(1 + t^2) - \frac{2t^2}{1+t^2}.$$

Thus, the line

$$L = \{(x, ax + b) \in \mathbb{R}^2 : x \in \mathbb{R}\} \tag{4}$$

with

$$a = a(t) = \frac{2t}{1+t^2} \qquad \text{and} \qquad b = b(t) = \ln(1+t^2) - \frac{2t^2}{1+t^2} \tag{5}$$

for $t < -1$ corresponds to a line that satisfies $\ell(x) \geq f(x)$ for $x \leq -1$ with strict inequality except for $x = t$ by the strict "downward convexity" of the graph of f. In particular,

$$-1 = f'(-1) < \ell'(-1) = a(t) < 0$$

and

$$0 < f(-1) = \ln(2) < \ell(-1) = b(t) - a(t) = \ln(1+t^2) - 2t\frac{t+1}{t^2+1}.$$

Consequently there is some $\epsilon > 0$ for which $\ell(x) > f(x)$ for $-1 \leq x < -1+\epsilon$ as well and also for $t < x < -1+\epsilon$. Since

$$\lim_{x \nearrow \infty} \ell(x) = -\infty$$

and $f(x) \geq 0$ for all x, there is some least $x_1 > -1$ for which $\ell(x_1) = f(x_1)$. Since $\ell(x) > f(x)$ for $t < x < x_1$ it must be the case that $\ell'(x_1) = a(t) \leq f'(x_1)$. If $x_1 < 0$, then by the monotonicity of f' there must hold

$$f'(x) > \ell'(x) = a(t) \qquad \text{for } x_1 < x \leq 0$$

and also for $x_1 < x \leq 1$. This implies $f(x) > \ell(x)$ for $x_1 < x \leq 0$ and for $x_1 < x \leq 1$ as well. In any case, for $x > 0$, we know f is increasing and ℓ is (still) decreasing, so $f(x) > \ell(x)$ for $x > x_1$.

In the case $x_1 \geq 0$, just the monotonicity of f implies the same conclusion:

$$f(x) > \ell(x) \qquad \text{for} \qquad x > x_1.$$

We have established that the particular tangent line L given in (4) with $\ell(t) = f(t)$ and $t < -1$ intersects the graph of f in exactly two points, the point of tangency $(t, \ln(1+t^2))$ and one additional point $(x_1, \ln(1+x_1^2))$ for some $x_1 > -1$. In fact, it can be shown that $f'(x_1) > \ell'(x_1) = a(t)$ so that the second point of intersection is a transverse point of intersection, but we have not shown this yet.

Since there are two points of intersection, the line L determined by $y = \ell(x) = a(t)\ x + b(t)$ in this particular case is not a line corresponding to a point $(a,b) = (a(t), b(t))$ in the set

$$A = \Big\{(a,b) \in \mathbb{R}^2 : \#\{(x, \ln(1+x^2)) \in \mathbb{R}^2 : x \in \mathbb{R},\ \ln(1+x^2) = ax+b\} = 1\Big\}$$

we are asked to determine. Nevertheless, the points

$$T = \{(a(t), b(t)) \in \mathbb{R}^2 : t < -1\} \tag{6}$$

we have discussed and found are of interest, and we proceed to consider the nature and geometry of the set T. Recall that we have T given as a parameterized curve with

$$a = a(t) = \frac{2t}{1+t^2} \qquad \text{and} \qquad b = b(t) = \ln(1+t^2) - \frac{2t^2}{1+t^2}$$

for $t < -1$. The first relation determines t as a function of the slope a:

$$at^2 - 2t + a = 0 \qquad \text{or} \qquad t = \frac{1+\sqrt{1-a^2}}{a} < -1. \tag{7}$$

Note that we take the "+" sign here in solving the quadratic equation because $a < 0$. Notice furthermore that as t ranges over the interval $(-\infty, -1)$, the value $a(t) = f'(t)$ is strictly decreasing with

$$a'(t) = f''(t) = 2\,\frac{1-t^2}{(1+x^2)^2} < 0$$

and takes on precisely the values in the interval $(-1, 0)$ with

$$\lim_{t \searrow -\infty} a(t) = 0 \qquad \text{and} \qquad \lim_{t \nearrow -1} a(t) = -1.$$

Notice this same observation was mentioned/used above as applied directly to $f'(t) = a(t)$.

It is perhaps convenient at this point to interrupt our discussion of the parameterized curve T defined in (6) with several interesting and useful observations. The "other" solution

$$t = t_- = \frac{1-\sqrt{1-a^2}}{a} \tag{8}$$

of the quadratic equation in (7) satisfies $-1 < t_- < 0$ for $-1 < a < 0$. To see that $t_- < 0$ it is enough to observe that for $-1 < a < 0$ one has $1 - \sqrt{1-a^2} > 0$. The inequality

$$t_- = \frac{1-\sqrt{1-a^2}}{a} > -1$$

is equivalent in this case to

$$-a > 1 - \sqrt{1-a^2} \qquad \text{or} \qquad \sqrt{1-a^2} > 1 + a.$$

The last inequality holds because $1+a>0$ and $1-a^2>(1+a)^2=1+2a+a^2$ since $2a^2+2a=2a(a+1)<0$ when $-1<a<0$.

If we extend or adapt the discussion above to points $(t,f(t))=(t_-,f(t_-))$ and the tangent lines at these points with $-1<t<0$ most of the calculations and formulas above apply without change. Specifically, for $-1<t<1$ the tangent line $L=L_-$ given by the expression in (4) with a and b given as functions of t by the same formulas in (5). Furthermore, since $f''(t)>0$ for $-1<t\leq 0$, we see that $a(t)=f'(t)$ increases from $f'(-1)=-1$ to $f'(0)=0$ as t increases from $t=-1$ to $t=0$. Thus, for $-1<t<1$ the value of $a(t)$ takes on precisely the same interval of values $-1<t<0$ though as we have seen $a(t)$ decreases through these values when $-\infty<t<-1$.

Here the point of tangency $(t,f(t))=(t,\ln(1+t^2))$ has $-1<t<0$, and while we can take the slope

$$a=f'(t)=\frac{2t}{1+t^2}$$

to have the same value a given in (5) for $t<-1$ precisely by taking $t=t_-$ to be the "other" root given in (8), the value of b given by the same formula, namely,

$$b=\ln(1+t^2)-\frac{2t^2}{1+t^2}$$

with $t=t_-$ will necessarily be different from $b(t_+)$ where

$$t=t_+=\frac{1+\sqrt{1-a^2}}{a}$$

is given in (7) precisely because we have taken $t=t_-$. From the plot in Figure 2 it looks like there should hold

$$\ell_+(x)=a(t_+)\,x+b(t_+)>a(t_-)\,x+b(t_-)=\ell_-(x)\qquad\text{for all } x\in\mathbb{R}$$

and hence

$$b(t_-)<b(t_+), \tag{9}$$

but we should verify this assertion analytically (rather than rely on the illustration). To this end, note that

$$t_\pm^2=\frac{2-a^2\pm 2\sqrt{1-a^2}}{a^2}$$

so that

$$1+t_\pm^2=2\,\frac{1\pm\sqrt{1-a^2}}{a^2}\qquad\text{and}\qquad\frac{2t_\pm^2}{1+t_\pm^2}=at_\pm=1\pm\sqrt{1-a^2}.$$

Therefore, we can write

$$b(t_\pm) = \ln\left(2\,\frac{1 \pm \sqrt{1-a^2}}{a^2}\right) \mp \sqrt{1-a^2} - 1. \tag{10}$$

In particular,

$$\begin{aligned} b(t_+) - b(t_-) &= \ln\left(2\,\frac{1+\sqrt{1-a^2}}{a^2}\right) - \ln\left(2\,\frac{1-\sqrt{1-a^2}}{a^2}\right) - 2\sqrt{1-a^2} \\ &= \ln\left(\frac{1+\sqrt{1-a^2}}{1-\sqrt{1-a^2}}\right) - 2\sqrt{1-a^2}. \end{aligned}$$

Consider then $\phi : [0, 1) \to \mathbb{R}$ by

$$\phi(\xi) = \ln\left(\frac{1+\xi}{1-\xi}\right) - 2\xi.$$

We have then $\phi(0) = 0$ and

$$\lim_{\xi \nearrow 1} \phi(\xi) = +\infty.$$

Also,

$$\phi'(\xi) = \frac{1-\xi}{1+\xi}\,\frac{1-\xi+1+\xi}{(1-\xi)^2} - 2 = 2\left(\frac{1}{1-\xi^2} - 1\right) = \frac{2\xi^2}{1-\xi^2} > 0.$$

Thus, $\phi(\xi) \geq 0$ for $0 \leq \xi < 1$ with strict inequality only for $\xi = 0$. Since $\xi = \sqrt{1-a^2}$ is a decreasing function of a taking the values on the interval $0 \leq \xi \leq 1$ when $-1 < a < 0$, we see

$$b(t_+) - b(t_-) = \phi(\sqrt{1-a^2}) > 0$$

as suggested by the illustration.

There is, in fact, another way to see this assertion which yields an additional interesting and perhaps important piece of information. Recall that we have shown the line

$$L = L_+ = \{(x, ax + b) : x \in \mathbb{R}\},$$

given in (4) with a satisfying $-1 < a < 0$ and $b = b(t_+)$, contains (exactly) two points of intersection with the graph G. These are the point of tangency at $(t_+, \ln(1 + t_+^2))$ with $t_+ < -1$ and the point $(x_1, \ln(1 + x_1^2))$ with $x_1 > -1$. We also know

$$f(x) = \ln(1 + x^2) < \ell_+(x) = ax + b(t_+) \qquad \text{for} \qquad t_+ < x < x_1.$$

By the mean value theorem, there is some point t_* with $t_+ < t_* < x_1$ for which

$$f'(t_*) = \frac{f(x_1) - f(t_+)}{x_1 - t_+} = \frac{\ell_+(x_1) - \ell_+(t_+)}{x_1 - t_+} = a.$$

The only value t_* for which this can hold is the unique value t_- with $-1 < t_- < 0$. Thus, first of all, $\ell_-(x) = ax + b(t_-)$ satisfies

$$\ell_-(t_-) = f(t_-) < \ell_+(t_-) \qquad \text{or} \qquad at_- + b(t_-) < at_- + b(t_+)$$

and $b(t_+) > b(t_-)$ as claimed. Furthermore, if $x_1 \le 0$, then since f' is increasing for $t_- \le x < 0$, we know $f'(x_1) > a = \ell'_+(x_1)$. Thus, the intersection $(x_1, f(x_1))$ of the tangent line at $(t_+, f(t_+))$ with the graph of f is a transverse intersection.

Our discussion above has established that the tangent lines corresponding to points $(t, f(t))$ with $t < 0$ are as indicated in Figure 2. In view of this discussion, it is natural to extend the notation involving "$\pm$" subscripts to the set of interest T given in (6) and write

$$T_+ = \{(a(t), b(t)) \in \mathbb{R}^2 : t < -1\}$$

as well as

$$T_- = \{(a(t), b(t)) \in \mathbb{R}^2 : -1 < t < 0\}.$$

Before we continue with the effort to understand the set T_+, and now the set T_- as well, let us attempt to establish that the points in T_- also correspond to points outside the set A of primary interest. Specifically, we wish to show there exists some unique $x_2 < t_+$ such that

$$L_- \cap G = \{(x_2, f(x_2)), (t_-, f(t_-))\} \tag{11}$$

and the intersection at $(x_2, f(x_2))$ is a transverse intersection with $f'(x_2) > a = \ell'_-(x_2)$. The key is the growth rate of $f(x) = \ln(1 + x^2)$ as $x \searrow -\infty$. In fact,

$$\lim_{x \searrow -\infty} [\ln(1 + x^2) - \ell_-(x)] = \lim_{x \searrow -\infty} [\ln(1 + x^2) - ax - b(t_-)] = -\infty$$

because

$$e^{\ln(1+x^2) - ax} = \frac{1 + x^2}{e^{ax}},$$

and by L'Hopital's rule this ∞/∞ indeterminate form is comparable to

$$\frac{2x}{ae^{ax}}$$

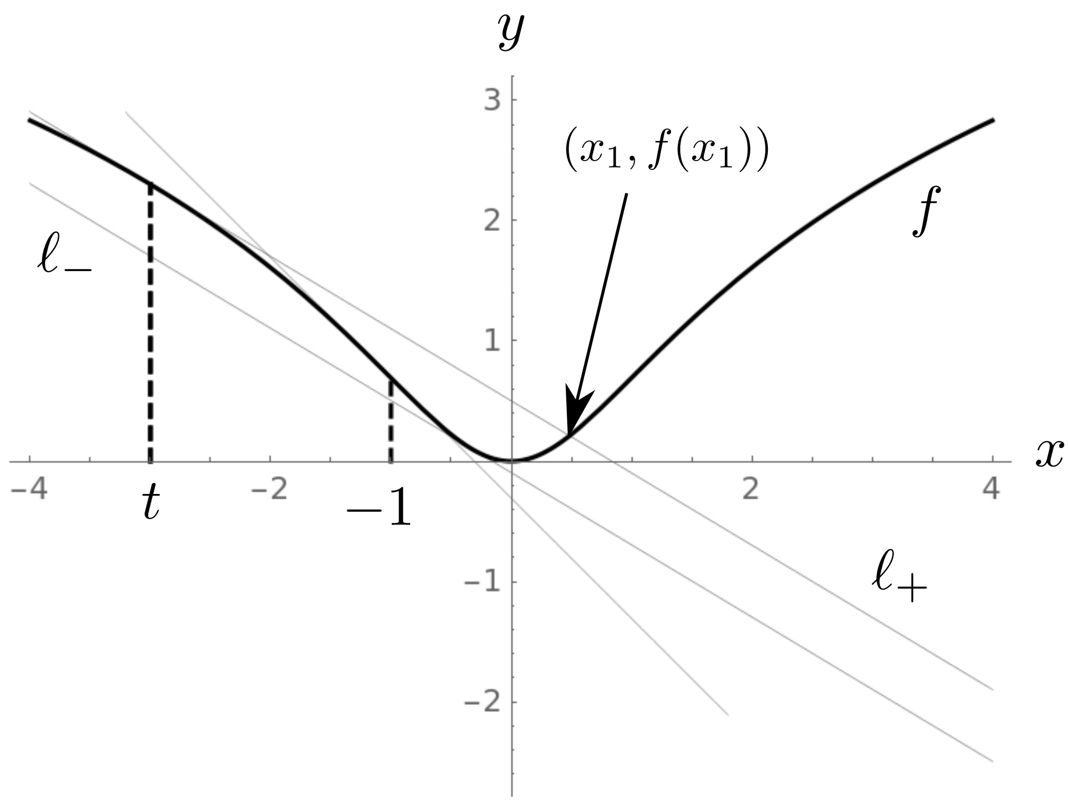

Figure 2: Tangent lines to the graph of the function f with values $f(x) = \ln(1 + x^2)$. The two parallel lines are the only two lines tangent to the graph sharing a fixed slope $-1 < a < 0$. We have labeled the points of tangency $(t_\pm, \ell_\pm(t_\pm))$. The unique tangent line with slope $a = -1$ and passing through $(-1, \ln(2))$ is also shown.

which tends to zero from the right as $x \searrow -\infty$ either because one knows the exponential denominator outpaces the linear numerator or by a second application of L'Hopital's rule yielding

$$\lim_{x \searrow -\infty} \frac{1 + x^2}{e^{ax}} = \lim_{x \searrow -\infty} \frac{2x}{ae^{ax}} = \lim_{x \searrow -\infty} \frac{2}{a^2 e^{ax}} = 0.$$

Thus,

$$\begin{aligned}\lim_{x\searrow-\infty}[\ln(1+x^2)-\ell_-(x)] &= b(t_-)+\lim_{x\searrow-\infty}\ln\left(e^{\ln(1+x^2)-ax}\right)\\ &= b(t_-)+\lim_{v\searrow 0}\ln v\\ &= -\infty.\end{aligned}$$

We conclude that for large negative values of x there holds $\ell_-(x) > f(x)$. On the other hand, since $f''(t_-) > 0$ and $f''(x) > 0$ for $-1 < x < 1$ we know

$$\ell_-(x) \le f(x) \qquad \text{for} \qquad -1 \le x \le 1$$

with strict inequality except for $x = t_-$. Thus, there is at least one value x_2 with $x_2 < -1$ and $\ell_-(x_2) = f(x_2)$. That is, $(x_2, \ln(1 + x_2^2))$ is a second intersection point of the tangent line L_- with the graph G, and $(a, b(t_-)) \notin A$ for $-1 < a < 0$.

In fact, since $\ell_-(-1) < f(-1)$ and $f'(x) < a = \ell'_-(x)$ for $t_+ < x < -1$, we know

$$\ell_-(x) < f(x) \qquad \text{for} \qquad t_+ < x < t_-.$$

Furthermore, $\ell'_-(x) = a < f'(x)$ for $-\infty < x < t_+$. Therefore, there can be only one point x_2 with $-\infty < x_2 < t_+$ and $\ell_-(x_2) = f(x_2)$. Since there also holds $\ell'_-(x_2) = a < f'(x_2)$, we know the intersection of L_- with G at $(x_2, f(x_2))$ is a transverse intersection. Finally, since $\ell_-(0) < f(0)$, or alternatively $\ell_-(1) < f(1)$ and $f'(x) > 0$ for $x > 0$ while $\ell'_-(x) \equiv a < 0$, we know there are no intersection points $(x, f(x))$ of L_- with G satisfying $x \ge 0$. This is a convenient time to include the details of the little calculus argument upon which this last assertion relies: For $x > 0$

$$\ell_-(x) = \int_0^x a\,d\xi < \int_0^x f'(\xi)\,d\xi = f(x).$$

We have established (11) and all the associated assertions.

We return now to the direct consideration of $T_\pm$. Perhaps the direct consideration of $\gamma(t) = (a(t), b(t))$ with $a(t) = f'(t)$ and $b(t) = f(t) - tf'(t)$ as a parametric curve with domain the appropriate interval is the easiest way to do this. For T_+, we consider $-\infty < t < 1$, and we have

$$\gamma'(t) = (f''(t), -tf''(t)) = f''(t)(1, -t).$$

Since $f''(t) < 0$ for $-\infty < t < 1$, this tells us right away that the first component function $a(t)$ of γ is decreasing, and T_+ is a graph over the interval $-1 < a < 0$. We know furthermore that $b'(t) = -tf''(t) < 0$, so the curve T_+ is the graph of a smooth **increasing** function $b = b_+(a)$ of a (in the a, b-plane). Note next that

$$\lim_{t \searrow -\infty} a(t) = \lim_{t \searrow -\infty} \frac{2t}{1+t^2} = 0.$$

Also,

$$\lim_{t \searrow -\infty} f(t) = \lim_{t \searrow -\infty} \ln(1+t^2) = +\infty, \qquad \text{and} \qquad \lim_{t \searrow -\infty} tf'(t) = \lim_{t \searrow -\infty} \frac{2t^2}{1+t^2} = 2.$$

Therefore,

$$\lim_{t \searrow -\infty} b(t) = +\infty,$$

and the function $b_+(a)$ increases to $+\infty$ as $a \nearrow 0$. At the other end of the interval(s)

$$\lim_{t \searrow -1} a(t) = \lim_{t \searrow -1} \frac{2t}{1+t^2} = -1, \qquad \text{and} \qquad \lim_{t \searrow -1} b(t) = \lim_{t \searrow -1} [f(t) - tf'(t)] = \ln 2 - 1 < 0.$$

Thus, at the left endpoint $a = -1$, the parametric curve has a finite limit

$$\lim_{t \nearrow -1} \gamma(t) = \gamma(-1) = (a(-1), b(-1)) = (-1, \ln 2 - 1)$$

as indicated in Figure 3. We have also drawn in Figure 3 the unit tangent vector in the direction of parameterization of the curve at the point $\gamma(-1) = (-1, \ln 2 - 1)$. Notice that this vector must be obtained as a limit because the nominal velocity vector

$$\gamma'(t) = f''(t)(1, -t)$$

satisfies $\gamma'(-1) = (0, 0)$ since $f''(-1) = 0$. Specifically, for $-\infty < t < -1$ we can define

$$\mathbf{u}(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{f''(t)(1, -t)}{\sqrt{f''(t)^2}\sqrt{1+t^2}} = \frac{(-1, t)}{\sqrt{1+t^2}}. \tag{12}$$

Notice that $f''(t) < 0$ for $-\infty < t < -1$ so that

$$\sqrt{f''(t)^2} = -f''(t).$$

Let me pause at this point to specify carefully the two functions currently under consideration and specifically their domains. First we have a parameterization $\gamma : (-\infty, -1] \to \mathbb{R}^2$ of T_+ where γ is smooth up to the endpoint $t = -1$ but has vanishing

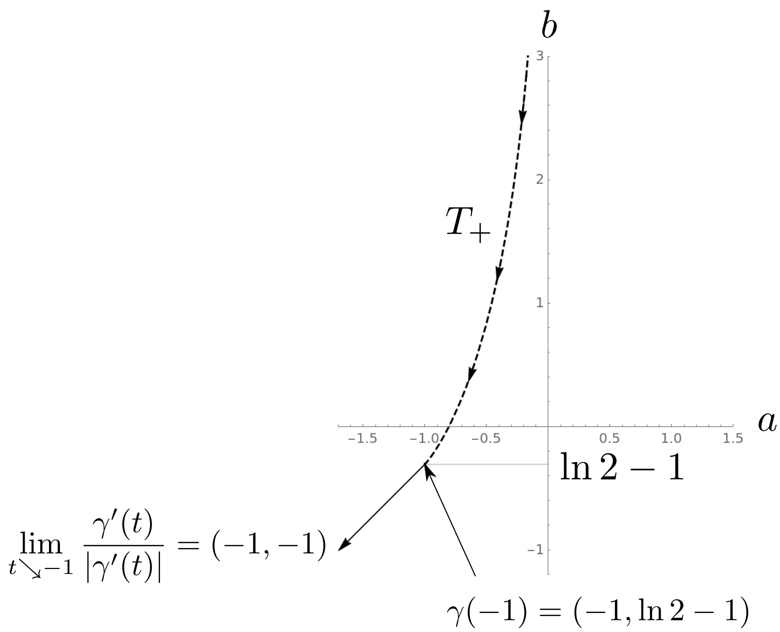

Figure 3: The curve of interest T_+ in the a, b-plane.

derivative there. As a parameterized curve, the vanishing of the derivative $\gamma'(-1)$ is considered to be a singularity or to distinguish $t = -1$ as a singular point for the parameterization. This singular behavior was evidenced in the extra effort required to extract the tangent direction at $\gamma(-1) = (-1, \ln 2 - 1)$. On the other hand, we have a real valued function $b_+ : [-1, 0) \to \mathbb{R}$ with $b_+(a) = b(t_+(a))$ whose graph is T_+. While the parameterization traces T_+ from right to left with $a = a(t)$ and $b = b(t)$ both decreasing as indicated by the arrows in Figure 3, the function b_+ is increasing from $b(-1) = \ln 2 - 1 < 0$ to $+\infty$.

In order to understand/see analytically the convexity of the curve T_+ one can consider the changing values of the unit tangent vector (or unit tangent field) along T_+. This is probably one of the easiest ways to proceed because of the very simple form of the unit tangent field $\mathbf{u} : (-\infty, -1] \to \mathbb{R}^2$ given in (12). The necessary calculation can be made even simpler by introducing an angle $\theta : (-\infty, -1] \to (0, \pi/4]$ given by

$$\theta = \cot^{-1}\left(\frac{b'(t)}{a'(t)}\right) = -\cot^{-1} t$$

as indicated in Figure 4. In this particular illustration we have zoomed in slightly, and the particular point we have used for the illustration is $\gamma(t) = \gamma(-3)$.

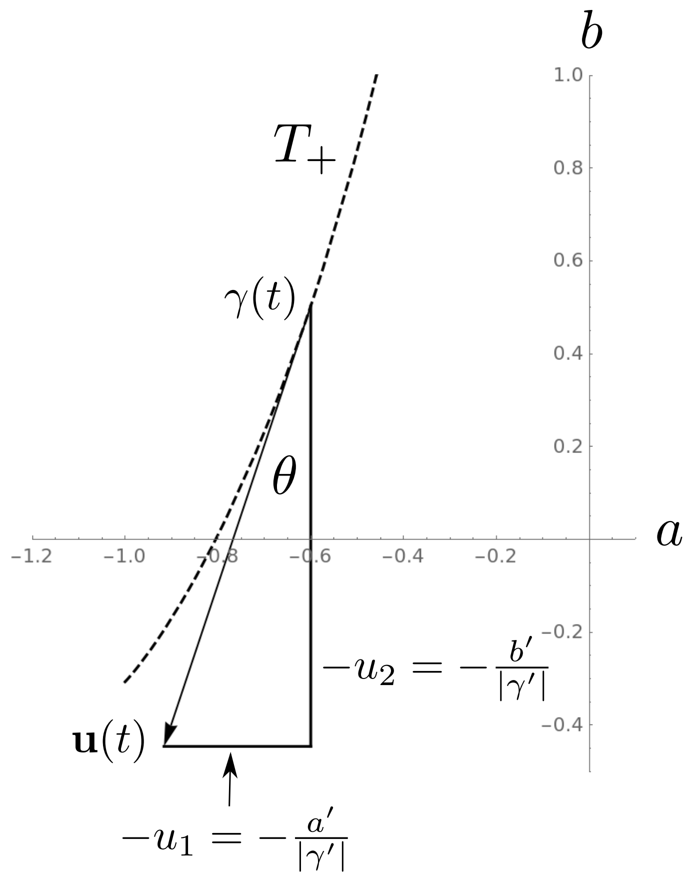

Figure 4: An angle θ associated with the unit tangent along T_+. Curvature/convexity is determined by how the angle θ changes along the curve.

Since

$$\frac{d\theta}{dt} = \frac{1}{1+t^2} > 0,$$

we see the curve T_+ is convex. An alternative approach would be to consider the function $b_+ : [-1, 0) \to \mathbb{R}$ directly with

$$b_+(a) = b(t_+) = \ln\left(2\,\frac{1+\sqrt{1-a^2}}{a^2}\right) - \sqrt{1-a^2} - 1.$$

as given in (10). Then we can try to differentiate:

$$\begin{aligned}
b'_+ &= \frac{a^2}{1+\sqrt{1-a^2}}\;\frac{-a^2\frac{a}{\sqrt{1-a^2}} - 2a\left(1+\sqrt{1-a^2}\right))}{a^4} + \frac{a}{\sqrt{1-a^2}} \\
&= -\frac{1}{a\left(1+\sqrt{1-a^2}\right)}\;\frac{a^2+2\sqrt{1-a^2}+2+-2a^2}{\sqrt{1-a^2}} + \frac{a}{\sqrt{1-a^2}} \\
&= -\frac{2\sqrt{1-a^2}+2-a^2-a^2-a^2\sqrt{1-a^2}}{a\left(1+\sqrt{1-a^2}\right)\sqrt{1-a^2}} \\
&= -\frac{(2-a^2)\sqrt{1-a^2}+2(1-a^2)}{a\left(1+\sqrt{1-a^2}\right)\sqrt{1-a^2}} \\
&= -\frac{(2-a^2)+2\sqrt{1-a^2}}{a\left(1+\sqrt{1-a^2}\right)} \qquad (13) \\
&> 0.
\end{aligned}$$

We can also check that

$$b'_+(-1) = 1 \qquad \text{and} \qquad \lim_{a\nearrow 0} b'_+(a) = +\infty.$$

This is all in agreement with what we expect, so this is at least some indication that the formula (13) may be correct. For the convexity we should compute the second derivative b''_+ which looks not so pleasant to compute at least with $b'_+(a)$ in the form we have given in (13). As an alternative, let us write

$$b'_+(a) = \frac{a}{1+\sqrt{1-a^2}} - \frac{2}{a}.$$

Then we can compute

$$b''_+ = \frac{1+\sqrt{1-a^2}+a^2/\sqrt{1-a^2}}{\left(1+\sqrt{1-a^2}\right)^2} + \frac{2}{a^2} > 0.$$

That was aapparently a lot easier than it looked at first, and it may even be correct. Given that this seems to have gone so well, perhaps we can go with the same non-parametric approach to understanding the set T_- consisting of points (a, b) with $-1 < a < 0$ corresponding to tangent lines

$$L_- = \{(x, ax+b) : x \in \mathbb{R}\}$$

with points of tangency $(t, \ln(1+t^2))$ having $-1 < t < 0$. In this case,

$$T_- = \{(a, b_-(a)) : -1 < a < 0\}$$

where according to (5) and (8)

$$\begin{aligned} b_-(a) &= b(t_-(a)) \\ &= \ln(1+t_-^2) - \frac{2t_-^2}{1+t_-^2} \\ &= \ln\left(2\,\frac{1-\sqrt{1-a^2}}{a^2}\right) - 1 + \sqrt{1-a^2} \end{aligned}$$

since

$$1 + t_-^2 = 1 + \frac{2 - a^2 - 2\sqrt{1-a^2}}{a^2} = 2\,\frac{1-\sqrt{1-a^2}}{a^2}$$

and

$$\frac{2t_-^2}{1+t_-^2} = t_- f'(t_-) = a t_- = a\,\frac{1-\sqrt{1-a^2}}{a}.$$

The left endpoint (limiting) value in this case may be obtained by direct evaluation with

$$b_-(-1) = \ln 2 - 1.$$

Note that this matches the left endpoint value $b_+(-1)$. At $a = 0$, we observe

$$\lim_{a \nearrow 0} \frac{1-\sqrt{1-a^2}}{a^2} = \lim_{a \nearrow 0} \frac{a/\sqrt{1-a^2}}{2a} = \frac{1}{2},$$

so

$$\lim_{a \nearrow 0} b_-(a) = \ln 1 = 0.$$

For the derivative we find

$$\begin{aligned}
b'_-(a) &= \frac{a^2}{1-\sqrt{1-a^2}} \, \frac{a^3/\sqrt{1-a^2} - 2a + 2a\sqrt{1-a^2}}{a^4} - \frac{a}{\sqrt{1-a^2}} \\
&= \frac{a^2 - 2\sqrt{1-a^2} + 2 - 2a^2}{a\left(1-\sqrt{1-a^2}\right)\sqrt{1-a^2}} - \frac{a}{\sqrt{1-a^2}} \\
&= \frac{2 - a^2 - 2\sqrt{1-a^2} - a^2 + a^2\sqrt{1-a^2}}{a\left(1-\sqrt{1-a^2}\right)\sqrt{1-a^2}} \\
&= \frac{2(1-a^2) - (2-a^2)\sqrt{1-a^2}}{a\left(1-\sqrt{1-a^2}\right)\sqrt{1-a^2}} \\
&= \frac{2\sqrt{1-a^2} - (2-a^2)}{a\left(1-\sqrt{1-a^2}\right)} && (14) \\
&= \frac{a}{1-\sqrt{1-a^2}} - \frac{2}{a}. && (15)
\end{aligned}$$

Again, at the left endpoint we have by evaluation

$$b'_-(-1) = -1 + 2 = 1.$$

At the right endpoint $a = 0$ we take a limit starting with the expression (14):

$$\begin{aligned}
b'_-(0) &= \lim_{a \nearrow 0} \frac{-2a/\sqrt{1-a^2} + 2a}{1 - \sqrt{1-a^2} + a^2/\sqrt{1-a^2}} \\
&= \lim_{a \nearrow 0} 2a \, \frac{\sqrt{1-a^2} - 1}{\sqrt{1-a^2} - 1 + 2a^2} \\
&= 0
\end{aligned}$$

since

$$\lim_{a \nearrow 0} \frac{\sqrt{1-a^2} - 1}{\sqrt{1-a^2} - 1 + 2a^2} = \lim_{a \nearrow 0} \frac{-2a/\sqrt{1-a^2}}{-2a/\sqrt{1-a^2} + 4a} = -1.$$

Notice that alternatively (for computing the last limit), we can write

$$\frac{\sqrt{1-a^2} - 1}{\sqrt{1-a^2} - 1 + 2a^2} = \left(1 + \frac{2a^2}{\sqrt{1-a^2} - 1}\right)^{-1}$$

and

$$\lim_{a \nearrow 0} \frac{2a^2}{\sqrt{1-a^2} - 1} = \lim_{a \nearrow 0} \frac{4a}{-2a/\sqrt{1-a^2}} = -2.$$

At any rate, with a little more work we can show b_- is differentiable (at least from the left) at $a = 0$ with $b_-'(0) = 0$. Finally, we have the expression (15) from which we can compute

$$\begin{aligned} b_-''(a) &= \frac{1 - \sqrt{1-a^2} - a^2/\sqrt{1-a^2}}{\left(1-\sqrt{1-a^2}\right)^2} + \frac{2}{a^2} \\ &= \frac{\sqrt{1-a^2} - 1}{\left(1-\sqrt{1-a^2}\right)^2 \sqrt{1-a^2}} + \frac{2}{a^2} \\ &= -\frac{1}{\left(1-\sqrt{1-a^2}\right)\sqrt{1-a^2}} + \frac{2}{a^2} \\ &= \frac{1}{1-a^2-\sqrt{1-a^2}} + \frac{2}{a^2} \\ &= \frac{a^2 + 2 - 2a^2 - 2\sqrt{1-a^2}}{a^2\left(1-a^2-\sqrt{1-a^2}\right)} \\ &= \frac{2 - a^2 - 2\sqrt{1-a^2}}{a^2\left(1-a^2-\sqrt{1-a^2}\right)} \\ &< 0 \end{aligned}$$

because

$$1 - a^2 - \sqrt{1-a^2} = -\left(1-\sqrt{1-a^2}\right)\sqrt{1-a^2} < 0$$

and (we claim)

$$a^2 + 2\sqrt{1-a^2} < 2.$$

To see the last assertion note that $2 - a^2 > 0$, so the last inequality is equivalent to

$$4(1-a^2) < 4 - 4a^2 + a^4 \qquad \text{or} \qquad a^4 > 0.$$

This completes the analysis (or at least some analysis) of T_- which is a "convex down" curve connecting $(a, b) = (-1, \ln 2 - 1)$ to $(a, b) = (0, 0)$ as indicated in Figure 5.

Since f is even, a symmetric situation prevails for tangent lines to G at points $(t, \ln(1+t^2))$ with $t > 0$ and $t \neq 1$. Thus, it is natural to extend the sets/curves $T_\pm$ to include slope-intercept pairs (a, b) corresponding to these tangent lines:

$$T_+ = \left\{ \left(a, \ln\left(2\,\frac{1+\sqrt{1-a^2}}{a^2}\right) - 1 - \sqrt{1-a^2} \right) : 0 < |a| < 1 \right\}$$

$$T_- = \left\{ \left(a, \ln\left(2\,\frac{1-\sqrt{1-a^2}}{a^2} \right) - 1 + \sqrt{1-a^2} \right) : 0 < |a| < 1 \right\}$$

This extension is also indicated in Figure 5. It would be a good exercise to go through the analysis associated with "determining" these curves for $0 < a < 1$ independently and see if what we have done above for $-1 < a < 0$ can be streamlined. Of course, it is a little ambiguous in the statement of the problem what exactly it means to "determine" the set A. I am relatively satisfied to have checked the monotonicity and convexity of the curves $T_\pm$. Of course (also) these curves are not in A, but having "determined" them, we are in a pretty good position to complete the problem. Let's consider some slope-intercept pairs (a, b) which are actually in A.

First of all, if we take $(t, \ln(1+t^2)) \in T_+$ with $-\infty < t < -1$, then we know the tangent line determined by $\ell(x) = ax + b$ with $a = f'(t)$ and $b = b(t_+(a)) = b_+(f'(t))$ satisfies

$$\begin{cases} \ell(x) > f(x), & -\infty < x < t \\ \ell(t) = f(t), & \\ \ell(x) > f(x), & t < x < x_1 \\ \ell(x_1) = f(x_1), & \\ \ell(x) < f(x), & x_1 < x < \infty. \end{cases} \tag{16}$$

Furthermore, we know $\ell'(x_1) = a < f'(x_1)$ and $x_1 > -1$. Based on this information, we can show

$$\{(a, \beta) : \beta > b\} \subset A.$$

That is, the line $\{(x, ax + \beta) : x \in \mathbb{R}\}$ intersects the graph G in exactly one point. Let $\mu(x) = ax + \beta$. Then immediately we have

$$\mu(x) = ax + b + (\beta - b) > \ell(x) \geq f(x) \qquad \text{for} \qquad x \geq x_1.$$

Because $\lim_{x \nearrow +\infty} \mu(x) = -\infty$ we know by the intermediate value theorem that there is some $x_\beta > x_1$ with $\mu(x_\beta) = f(x_\beta)$. Thus, $(x_\beta, f(x_\beta))$ is one point of intersection of

$$L_\beta = \{(a, ax + \beta) : x \in \mathbb{R}\}$$

with G. Furthermore, we can take

$$x_\beta = \min\{x \in \mathbb{R} : \mu(x) = f(x)\}.$$

Then

$$\begin{cases} \mu(x) > f(x), & -\infty < x < x_\beta \\ \mu(x_\beta) = f(x_\beta), & \\ \mu(x) < f(x), & x_\beta < x < \infty. \end{cases}$$

To see the last inequality note that if $x_1 < x_\beta < 1$, then

$$\mu'(x_\beta) = a = \ell'(x_1) < f'(x_1) < f'(x_\beta).$$

Thus for $x_\beta < x \leq 1$ we have

$$\mu(x) = \mu(x_\beta) + \int_{x_\beta}^{x} a\, d\xi < f(x_\beta) + \int_{x_\beta}^{x} f'(\xi)\, d\xi = f(x).$$

In particular $\mu(1) < f(1)$. For $x > 1$, the function f is increasing while the affine function μ is decreasing. Consequently

$$\mu(x) = \mu(1) + \int_{x_\beta}^{x} a\, d\xi < \mu(1) < f(1) < f(x).$$

If $x_\beta \geq 1$, then the last string of inequalities can be modified to give

$$\mu(x) = \mu(x_\beta) + \int_{x_\beta}^{x} a\, d\xi < \mu(x_\beta) \leq f(x_\beta) < f(x) \qquad \text{for} \qquad x > x_\beta.$$

A similar argument applies to the points

$$\{(a, \beta) : \beta > b\}$$

when $(a, b) \in T_+$ with $0 < a < 1$ and $\beta > b$. These points are all in the set A.

Next consider $(a, b_-) \in T_-$ with $-1 < a < 0$ and $a = f'(t_-)$. In this case we have shown

$$\begin{cases} \ell_-(x) > f(x), & -\infty < x < x_2 \\ \ell_-(x_2) = f(x_2), & \\ \ell_-(x) < f(x), & x_2 < x < t_- \\ \ell_-(t_-) = f(t_-), & \\ \ell_-(x) < f(x), & t_- < x < \infty \end{cases} \tag{17}$$

with $\ell_-'(x_2) = a < f'(x_2)$ and $x_2 < t_-$. Arguments similar to those used in the case $a = f'(t)$ with $t < -1$ give

$$\{(a, \beta) : \beta < b_-\} \subset A.$$

Symmetrically we have also

$$\{(a, \beta) : \beta < b_-\} \subset A$$

when $0 < a < 1$ and $(a, b_-) \in T_-$.

Considering and comparing (16) and (17) we see that the points (a, β) with $b_- < \beta < b$ are not in A. In fact, if $\mu(x) = ax + \beta$ for such a point, then

$$\begin{cases} \mu(x_2) > f(x_2), \\ \mu(t) < f(t), \\ \mu(t_-) > f(t_-), \\ \mu(x_1) < f(x_1). \end{cases}$$

It follows from the intermediate value theorem that $L_\mu = \{(a, \mu(x)) = (a, ax + \beta) : x \in \mathbb{R}\}$ intersects the graph G at some point $(x_+, f(x_*))$ with $x_2 < x_* < t$ and at another point $(x_{**}, f(x_{**}))$ with $t < x_{**} < t_-$. In fact, there must be at least three points of intersection of the line L_μ with G. See Figure 2. I believe there should be exactly three. In any case, $(a, \beta) \notin A$ for $b_- < \beta < b$.

Overall, we conclude the set of points directly below and including T_+ and directly above and including T_- is not in A. This set may be seen in Figure 5.

The points (a, b) with $a = 0$ and $|a| \geq 1$ should be considered separately.

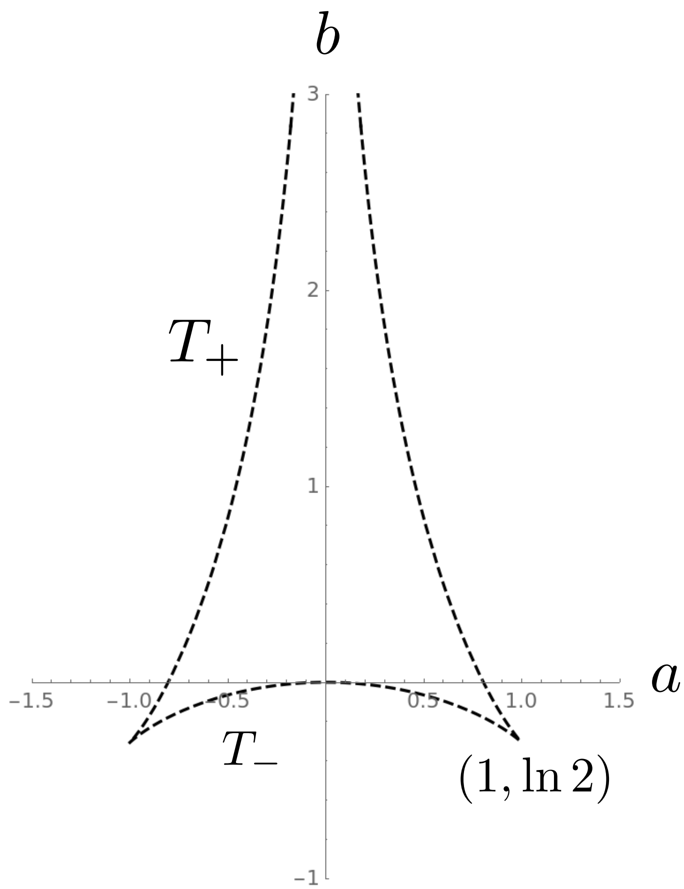

Figure 5: The curves T_+ and T_- determined by tangent lines at the points $(t, \ln(1+t^2))$ to the graph of f where $f(x) = \ln(1+x^2)$ and $t \neq \pm 1$. Each such line intersects the graph G in exactly two points, so these curves are not in the set A of slope-intercept pairs (a, b) corresponding to lines that intersect the graph exactly once.