Solution to 2010 Putnam B1

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B1. Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

Solution:

The answer is **no**.

Suppose such a sequence existed. Then, note that

$$\left(\sum a_i^2\right)^2 = \left(\sum a_i^4\right) + 2 \cdot \sum_{i \neq j} a_i^2 a_j^2 \implies 4 = 4 + 2 \cdot \sum_{i \neq j} a_i^2 a_j^2 \implies \sum_{i \neq j} a_i^2 a_j^2 = 0.$$

Since a_i and a_j are real numbers, $a_i^2 a_j^2 \geq 0$, which means that all products in the sum must be equal to 0. If all a_i terms were equal to 0, then the sum in the original equation would equal 0, a contradiction. Else, suppose that any a_i were nonzero; WLOG, let $a_1 \neq 0$. This forces $a_2 = a_3 = \cdots = 0$; thus, $a_1^m = m$ must hold for all $m \in \mathbb{Z}^+$ for some constant real number a_1. Setting $m = 2$ gives $a_1^2 = 2 \implies a_1 = \pm\sqrt{2}$, but $m = 1$ gives $a_1 = 1 \neq \pm\sqrt{2}$, a contradiction.

$\square$