SOME RESULTS ABOUT A POSITIVE DEFINITE MATRIX TIMES A NEGATIVE SEMIDEFINITE MATRIX

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Lemma 17 (definiteness and trace) If $A = (a_{ij})$ is a real symmetric positive definite matrix and $H = (b_{ij})$ is a real symmetric negative semidefinite matrix, then

$$\operatorname{tr}(AH) = \sum_{i,j} a_{ij} b_{ij} \leq 0.$$

I offer two proofs, the first of which being similar (I think — although I do not remember the exact details of the proof) to the one presented by Tyler in class. The second proof is similar to the one I presented on the board in class on Tuesday, October 22nd.

Proof. Since A is symmetric, there is an orthogonal matrix Q and diagonal matrix Λ of eigenvalues of A such that $A = Q^T \Lambda Q$. Note that the diagonal entries of Λ are all positive since Λ is positive definite. Hence, the square root of A given by $A^{1/2} = Q^T \Lambda^{1/2} Q$ exists.

Thus,

$$\operatorname{tr}(AH) = \operatorname{tr}\left(A^{1/2} A^{1/2} H\right).$$

By a property of the trace,

$$\begin{aligned}
&= \operatorname{tr}\left(A^{1/2} H A^{1/2}\right) \\
&= \sum_{i=1}^{n} \mathbf{e}_i^T A^{1/2} H A^{1/2} \mathbf{e}_i \\
&= \sum_{i=1}^{n} \left(\mathbf{e}_i^T A^{1/2}\right) H \left(A^{1/2} \mathbf{e}_i\right)
\end{aligned}$$

We have $\mathbf{e}_i^T A^{1/2} = \mathbf{e}_i^T \left(A^{1/2}\right)^T = \left(A^{1/2}\mathbf{e}_i\right)^T$ because

$$\left(A^{1/2}\right)^T = \left(Q^T \Lambda^{1/2} Q\right)^T = \left(\Lambda^{1/2} Q\right)^T Q = Q^T \left(\Lambda^{1/2}\right)^T Q = Q^T \Lambda^{1/2} Q = (A^{1/2}).$$

Therefore,

$$\operatorname{tr}(AH) = \sum_{i=1}^{n} \left(A^{1/2}\mathbf{e}_i\right)^T H \left(A^{1/2}\mathbf{e}_i\right)$$

Letting, $\mathbf{x}_i = A^{1/2}\mathbf{e}_i$,

$$= \sum_{i=1}^{n} \mathbf{x}_i^T H \mathbf{x}_i$$

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and since H is negative semidefinite, it follows that $\mathbf{x}_i^T H \mathbf{x}_i \leq 0$ for each $i \in \{1, \ldots, n\}$. Thus, the summation above is non-positive:

$$\operatorname{tr}(AH) \leq 0.$$

□

Proof. We have that $-H$ is a positive semidefinite matrix by Lemma 1. Also, A is a positive semidefinite matrix because $\mathbf{x}_i^T A \mathbf{x} < 0 \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. By Lemma 3, there exist matrices X and Y such that $A = X^T X$ and $-H = Y^T Y$.

Therefore,

$$\begin{aligned} -\operatorname{tr}(AH) &= \operatorname{tr}(A(-H)) \\ &= \operatorname{tr}\left(x^T X Y^T Y\right) \\ &= \operatorname{tr}\left(X Y^T Y X^T\right) \\ &= \operatorname{tr}\left(\left(Y X^T\right)^T \left(Y X^T\right)\right). \end{aligned}$$

Calling $C = Y X^T = (c_{ij})$,

$$\begin{aligned} -tr(AH) &= \operatorname{tr}\left(C^T C\right) \\ &= \sum_{i=1}^n \sum_{j=1}^n c_{ij}^2 \\ &\geq 0. \end{aligned}$$

Hence, $\operatorname{tr}(AH) \leq 0$. □

Lemma 1. If H is a $n \times n$ real symmetric negative semidefinite matrix, then $B \coloneqq -H$ is a positive semidefinite matrix.

Proof. Since H is negative semidefinite, for each $\mathbf{x} \in \mathbb{R}^n$, we have $\mathbf{x}^T H \mathbf{x} \leq 0$. Thus, $\mathbf{x}^T(-H)\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Hence, $B \coloneqq -H$ is positive semidefinite. $\square$

Lemma 2. In the cases below, if $A = (a_i j)$ and $B = (b_i j)$ are $n \times n$ real symmetric positive semidefinite matrices, then the diagonal entries of AB are non-negative. That is, $\mathbf{e}_i^T AB\mathbf{e}_i \geq 0$ for every $i \in \{1, \ldots, n\}$.

Note: These cases are by no means exhaustive (as far as I know), they are just cases I was able to come up with which make this lemma true.

Case 1. ($B = A$) *Note that if $B = A$, then Case 2 below automatically applies so this proof is not really necessary.*

Proof. Suppose for the sake of contradiction that there is a $k \in \{1, \ldots, n\}$ such that $\mathbf{e}_k^T AB\mathbf{e}_k < 0$.

Since A is positive semidefinite, it follows that $A = Q^T \Lambda Q$ for some orthogonal matrix Q and diagonal matrix Λ having the eigenvalues of A on its diagonal.

Thus,

$$\mathbf{e}_k^T Q^T \Lambda Q B \mathbf{e}_k < 0$$

Now consider the special case where $B = A$. Then $B = Q^T \Lambda Q$, and so

$$\begin{aligned} 0 &> \mathbf{e}_k^T Q^T \Lambda Q Q^T \Lambda Q \mathbf{e}_k \\ &= \mathbf{e}_k^T Q^T \Lambda \Lambda Q \mathbf{e}_k \end{aligned}$$

Letting $\mathbf{v}_k = \Lambda Q \mathbf{e}_k$, we have

$$0 > \mathbf{v}_k^T \mathbf{v}_k,$$

which is a contradiction because the squared norm of a vector in $\mathbb{R}^n$ cannot be negative.

Thus, it must be the case that $\mathbf{e}_i^T AB\mathbf{e}_i \geq 0$ for all $i \in \{1, \ldots, n\}$. $\square$

Case 2. ($\operatorname{sign}(a_{ij}) = \operatorname{sign}(b_{ij})$ for each $i, j \in \{1, \ldots, n\}$, where $\operatorname{sign}(x)$ is the signum function.)

Proof. For each $i \in \{1, \ldots, n\}$, we have

$$\mathbf{e}_i^T AB\mathbf{e}_i = (AB)_{ii} = \sum_{j=1}^{n} a_{ij} b_{ji} = \sum_{j=1}^{n} a_{ij} b_{ij}.$$

Since we assumed $\operatorname{sign}(a_{ij}) = \operatorname{sign}(b_{ij})$, it follows that $a_{ij} b_{ij} \geq 0$. Hence $\sum_{j=1}^{n} a_{ij} b_{ij} \geq 0$. That is, each diagonal entry of the product AB is non-negative. $\square$

Non-Fact. (Exercise 3.2) If $A = (a_{ij})$ is an $n \times n$ real symmetric positive definite matrix and $H = (h_{ij})$ is an $n \times n$ real symmetric negative semidefinite matrix, then the diagonal entries of AH are non-positive.

Note: There are certain conditions we can impose on A and H that make this statement true. See below.

Counterexample. Consider the 2×2 matrices

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix} \qquad \text{and} \qquad H = \begin{pmatrix} -1 & 3 \\ 3 & -10 \end{pmatrix}.$$

The eigenvalues of A are $\frac{1}{2}\left(11 \pm \sqrt{117}\right) > \frac{1}{2}\left(11 - \sqrt{121}\right) = 0$, Hence, A is positive definite. The eigenvalues of H are $\frac{1}{2}\left(-11 \pm \sqrt{117}\right) < \frac{1}{2}\left(-11 + \sqrt{121}\right) = 0 \leq 0$. Hence, H is a negative semidefinite matrix.

However, we have that

$$AH = \begin{pmatrix} 8 & -27 \\ 27 & -91 \end{pmatrix}$$

so that the entry $(AH)_{11}$ is positive. Hence the statement is not true in general.

Cases where the statement is true. We have that A is a positive semidefinite matrix since A is a positive definite matrix. This is clear because for all $\mathbf{x} \in \mathbb{R}^n$, we have $\mathbf{x}^T A \mathbf{x} > 0 \geq 0$. We also have that $-H$ is a positive semidefinite matrix by Lemma 1.

For the first case, where $A = -H$, we can apply Case 1 of Lemma 2 to see that $\mathbf{e}_i^T A(-H)\mathbf{e}_i \geq 0$. Hence, $\mathbf{e}_i^T AH\mathbf{e}_i \leq 0$.

For the second case, where $\text{sign}(a_{ij}) = -\text{sign}(h_{ij})$ for all $i, j \in \{1, \ldots, n\}$, we have that $\text{sign}\,(a_{ij}) = \text{sign}(-h_{ij})$. Hence, we can apply Case 2 of Lemma 2 to see that $\mathbf{e}_i^T A(-H)\mathbf{e}_i \geq 0$. That is, $\mathbf{e}_i^T AH\mathbf{e}_i \leq 0$.

Lemma 3. If A is a $n \times n$ real symmetric positive semidefinite matrix, then there is a matrix X such that $A = X^T X$.

Proof. Writing $A = Q^T \Lambda Q$ as before, the square root $\Lambda^{1/2}$ exists because the eigenvalues of A are non-negative. Thus, $A = \left(Q^T \Lambda^{1/2}\right)\left(\Lambda^{1/2} Q\right) = \left(\Lambda^{1/2} Q\right)^T \left(\Lambda^{1/2} Q\right)$. Calling $X = \Lambda^{1/2} Q$, we have $A = X^T X$. □