Parameteric mollification of C^1 planer curves and constant curvature invariance

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Abstract

Mollification is a remarkable procedure for approximating real valued functions with smooth C^∞ functions. We describe a geometric adaptation of this procedure for approximating certain planar curves. For C^1 curves with a weak curvature vector we give a mollification procedure leaving curves of constant curvature invariant in anology to mollifications of functions leaving solutions of Laplace's equation (harmonic functions) invariant.

Let Γ be a C^1 planar curve parameterized by arclength on an open interval $(-L, M)$ for some positive numbers L and M. Specifically, we begin with

$$\gamma = (\gamma_1, \gamma_2) \in C^1((-L, M) \to \mathbb{R}^2) \tag{1}$$

for which

$$\dot{\gamma} = \frac{d\gamma}{ds} \qquad \text{satisfies} \qquad \dot{\gamma}_1^2 + \dot{\gamma}_2^2 = 1. \tag{2}$$

We restrict attention further to curves admitting a weak curvature vector in the following sense: There exist functions $D_1\dot{\gamma}_1, D_1\dot{\gamma}_2 \in L^1_{loc}(-L, M)$ with

$$\int_{(-L,M)} \dot{\gamma}_j \, \frac{d\phi}{ds} = -\int_{(-L,M)} D_1\dot{\gamma}_j \; \phi, \qquad \phi \in C_c^\infty(-L, M), \; j = 1, 2. \tag{3}$$

Examples of such curves are the following:

Example 1 Given $a > 0$, consider $\gamma : (-\pi a/2, \pi a/2) \to \mathbb{R}^2$ by

$$\gamma(s) = \begin{cases} (a, 0) + a(-\cos(s/a), \sin(s/a)), & -\pi a/2 < s \le 0 \\ (-a, 0) + a(\cos(s/a), \sin(s/a)), & 0 \le s < \pi a/2. \end{cases} \tag{4}$$

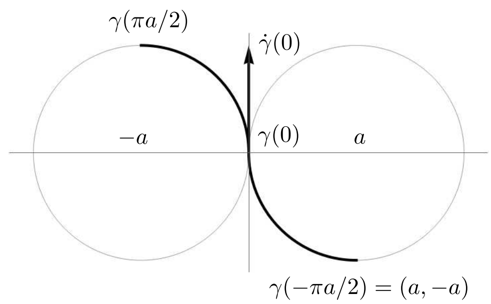

Figure 1: Example 1: A curve that follows two tangent circles of the same radius

Example 2 Given $a, c > 0$ with $c < a < 2c$ so that $b = 2c - a > 0$, consider $\gamma : (-\pi a/2, \pi b/2) \to \mathbb{R}^2$ by

$$\gamma(s) = \begin{cases} (c, 0) + a(-\cos(s/a), \sin(s/a), & -\pi a/2 < s \leq 0 \\ (-c, 0) + b(\cos(s/b), \sin(s/b)), & 0 \leq s < \pi b/2. \end{cases} \tag{5}$$

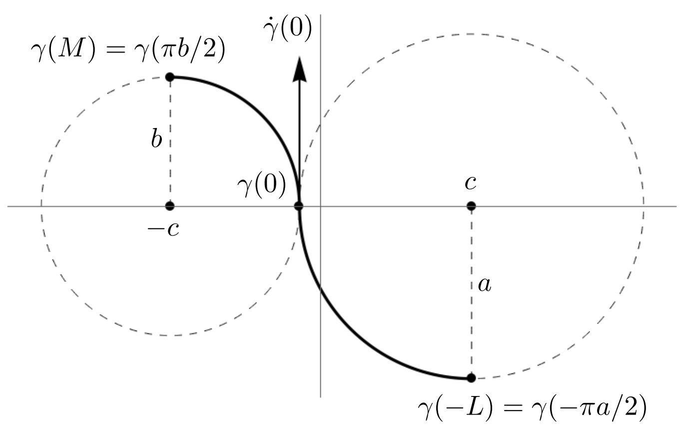

Figure 2: Example 2: A curve that follows two tangent circles of different radii

Given an open set $\Omega \subset \mathbb{R}^n$, the mollification procedure familiar from the theory of partial differential equations provides a remarkable way to associate with a function $u \in L^1_{loc}(\Omega)$ for each $\sigma > 0$ an approximating function in $C^\infty(\Omega_\sigma)$ where

$$\Omega_\sigma = \{\mathbf{x} \in \Omega : \operatorname{dist}(\mathbf{x}, \partial\Omega) > \sigma\}$$

is a slightly smaller domain. Specifically, we extend u to a real valued function on $\mathbb{R}^n$ (in any manner) and consider $\mu_\sigma * u : \Omega_\sigma \to \mathbb{R}$ by

$$\mu_\sigma * u(\mathbf{x}) = \int_{\xi \in \mathbb{R}^n} \mu_\sigma(\mathbf{x} - \xi)\ u(\xi) \tag{6}$$

where

$$\mu_\sigma(\mathbf{x}) = \frac{1}{\sigma^n I} \phi\left(\frac{\mathbf{x}}{\sigma}\right); \qquad I = \int_{\mathbb{R}^n} \phi \tag{7}$$

and $\phi \in C_c^\infty(\mathbb{R}^n)$ is given by

$$\phi(\mathbf{x}) = \begin{cases} e^{-1/(1-|\mathbf{x}|^2)}, & \mathbf{x} \in B_1(\mathbf{0}) \\ 0, & \mathbf{x} \in \mathbb{R}^n \backslash B_1(\mathbf{0}). \end{cases}$$

Here we have used the notation $B_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}$ for the ball of radius $r > 0$ centered at $\mathbf{p} \in \mathbb{R}^n$. Among the remarkable properties of the mollification $\mu_\sigma * u$, it may be recalled that mollification respects the partial differential equation of Laplace in the sense that if

$$\Delta u = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} = 0 \qquad \text{on} \qquad \Omega,$$

then

$$\mu_\sigma * u \equiv u\Big|_{\Omega_\sigma}.$$

See for example Theorem 6 in Section 2.2 of [Eva10].

The objective of this paper is to adapt the mollification procedure for application to certain planar curves in a manner that takes into account the local geometry of the curve and the curvature of the curve in particular. One may simply mollify the coordinate functions γ_1 and γ_2 of a curve or even any pair of L^1_{loc} functions defined on an interval $(-L, M) \subset \mathbb{R}$ to obtain a smooth function $\mu_\sigma * \gamma \in C^\infty((-L+\sigma, M-\sigma) \to \mathbb{R}^2)$. It is not surprising that applying this procedure to a parameterization of a circular arc $\gamma(s) = a(\cos(s/a), \sin(s/a))$ results in a parameterization of a circular arc of (smaller) radius

$$\tilde{a} = a \int_{-\sigma}^{\sigma} \mu_\sigma(t)\ \cos t\, dt < a \tag{8}$$

at least for σ small. There is in fact a unique value $\sigma_0 \doteq 4.997a$ of σ for which $\tilde{a}$ decreases to zero as a function of σ with

$$\lim_{\sigma \searrow 0} \tilde{a} = a \qquad \text{and} \qquad \lim_{\sigma \searrow \sigma_0} \tilde{a} = 0.$$

That the mollified curve is a (smaller) circle is to be expected because

$$\mu_\sigma * \gamma(t) = \int_{t-\sigma}^{t+\sigma} \mu_\sigma(t-s)\ \gamma(s)\, ds \tag{9}$$

is a weighted average, symmetric with respect to the arclength parameter s in the integration.

With the curve in Example 1 this simple pointwise mollification results in a C^∞ parameterization $\mu_\sigma * \gamma$ for $\sigma < \pi a/4$ consisting of portions of circles parameterized as

$$\begin{aligned} &\mu_\sigma * \gamma(t) = (a,0) + \tilde{a}(-\cos t, \sin t) \qquad \text{for} \qquad -\pi a/2 + \sigma \le t \le -\sigma \qquad \text{and} \\ &\mu_\sigma * \gamma(t) = (-a,0) + \tilde{a}(\cos t, \sin t) \qquad \text{for} \qquad \sigma \le t \le \pi a/2 - \sigma \end{aligned}$$

with a smooth C^∞ transition occurring for $-\sigma < t < \sigma$ with

$$\mu_\sigma * \gamma(-t) = -\mu_\sigma * \gamma(t)$$

by the symmetry of the original concatenated tangent circular arcs. In particular, the central point $\gamma(0) = (0,0) = \mu_\sigma * \gamma(0)$ is preserved.

Since mollification commutes with taking (weak) derivatives a tangent vector to the mollified curve is given by

$$(\mu_\sigma * \gamma)'(t) = \int_{t-\sigma}^{t+\sigma} \mu_\sigma(t-s)\ \dot{\gamma}(s)\, ds = \mu_\sigma * \dot{\gamma}(t). \tag{10}$$

Since the tangent indicatrix

$$\dot{\gamma} : (-\pi/2, \pi/2) \to \mathbb{S}^1 = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = 1\}$$

lies entirely in the closed second quadrant, the tangent vector $(\mu_\sigma * \gamma)'$ is biased toward the second quadrant during the transition with $(\mu_\sigma * \gamma_1)'(0) < 0 = \dot{\gamma}_1(0)$ in particular. A close up of the transition region corresponding to mollification parameter $\sigma = \pi a/5$ is shown in Figure 3.

Note finally that the invariance of harmonic functions under mollification results in the predictable property that a portion of a curve that is locally a straight line is invariant on some interior segment under this basic mollification. This is because harmonic functions in one dimension are affine.

The discussion so far illustrates the basic idea of how mollification incorporates **weighted averaging** to modify the pointwise values of a given function to determine a C^∞ approximation and how this basic idea may be applied to the coordinate

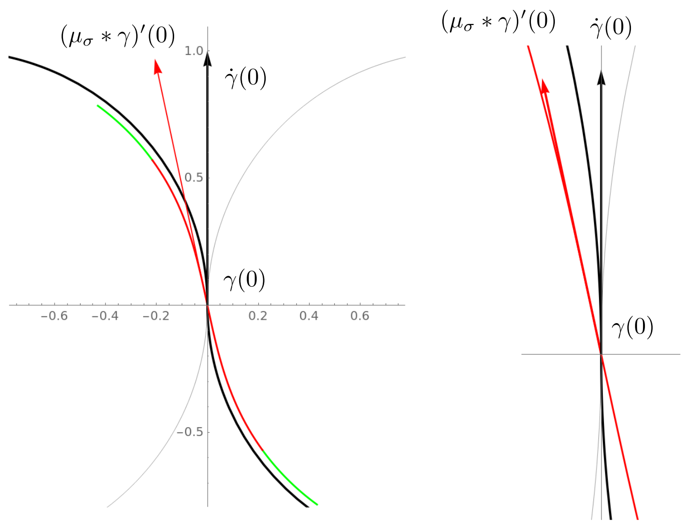

Figure 3: Example 1: Simple pointwise mollification of a curve along two tangent circles of radius 1 (left). Here the mollification parameter is $\sigma = \pi/5$. Circular arcs of smaller radius are shown in green; the transition interval is in red. On the right is a close up of essentially the same geometry though to display the tangent vectors accurately, the radius has been taken larger. In this case $a = 5$ and $\sigma = \pi$.

functions of a curve to essentially mollify position. In order to mollify the geometry of the curve rather than the merely pointwise positions of points on the curve, or to express what we even mean by mollifying geometry, we proceed to consider weighted averages of different quantities.

1 Tangent mollification

The new idea introduced here may be understood to consist of the following two steps:

(i) Given $\gamma(s_0)$ and an arclength $s \neq s_0$, say $s < s_0$, consider a **position** $\alpha(s_0)$ **obtained** by starting at $\gamma(s)$ and proceeding along a curve with "homogeneous geometry" in the direction of $\dot\gamma(s)$ for an arclength $s_0 - s$.

(ii) As the new mollification of γ take a **weighted average** $\nu(s_0)$ of the values $\alpha(s_0)$ with respect to the arclength s.

Step **(i)** contemplates a well-defined auxiliary curve determined by the pointwise geometric information at $\gamma(s)$; we give one example presently.

Take $\sigma > 0$ as usual. For each $s_0 \in (-L+\sigma, M-\sigma)$ and s with $s_0-\sigma < s < s_0+\sigma$ consider the straight line curve along the directed tangent to Γ at $\gamma(s)$. That is, consider $\beta : \mathbb{R} \to \mathbb{R}^2$ given by

$$\beta(t) = \gamma(s) + t\dot{\gamma}(s).$$

Take $\alpha(s_0) = \beta(s_0 - s) = \gamma(s) + (s_0 - s)\dot{\gamma}(s)$ and

$$\nu(s_0) = \int_{s_0-\sigma}^{s_0+\sigma} \mu_\sigma(s_0 - s)\alpha(s_0)\, ds = \int_{s_0-\sigma}^{s_0+\sigma} \mu_\sigma(s_0 - s)\beta(s_0 - s)\, ds.$$

In simpler notation somewhat obscuring the basic idea this new mollification is given by

$$\begin{aligned}\nu(t) &= \int_{t-\sigma}^{t+\sigma} \mu_\sigma(t-s)\gamma(s)\, ds + \int_{t-\sigma}^{t+\sigma} (t-s)\mu_\sigma(t-s)\dot{\gamma}(s)\, ds \\ &= \mu_\sigma * \gamma(t) + \int_{t-\sigma}^{t+\sigma} (t-s)\mu_\sigma(t-s)\dot{\gamma}(s)\, ds. \qquad (11)\end{aligned}$$

In this formula the new element appears as an addition to the basic mollification of γ considered in the previous discussion. Furthermore, when $\dot{\gamma}(s) = \mathbf{v}$ is a constant (unit) vector for $t - \sigma < s < t + \sigma$, then the addional term vanishes because

$$\int_{t-\sigma}^{t+\sigma} (t-s)\mu_\sigma(t-s)\, ds = \int_{-\sigma}^{\sigma} \eta\mu_\sigma(\eta)\, d\eta = 0$$

is the integral of an odd function, and $\nu(t) = \mu_\sigma * \gamma(t) = \gamma(t)$ as before.

On the other hand, mollification of a circular arc is now somewhat different. Again one obtains for small enough $\sigma > 0$ a circular arc parameterized by $\nu(t) = \tilde{a}(\cos t, \sin t)$ for $t \in \mathbb{R}$, but in this instance the radius

$$\begin{aligned}\tilde{a} &= a\int_{-\sigma}^{\sigma} \mu_\sigma(s)\cos(s/a)\, ds + \int_{-\sigma}^{\sigma} s\ \mu_\sigma(s)\ \sin(s/a)\, ds \\ &= \int_{-\sigma}^{\sigma} \mu_\sigma(s)[a\cos(s/a) + s\sin(s/a)]\, ds \qquad (12)\end{aligned}$$

satisfies $\tilde{a} > a$ at least for σ small enough. Figure 4 illustrates this inequality for $\sigma < \pi a/2$. Notice the points $\alpha(s_0) = \alpha(s_0; s)$ averaged to obtain $\nu(s_0)$ lie outside

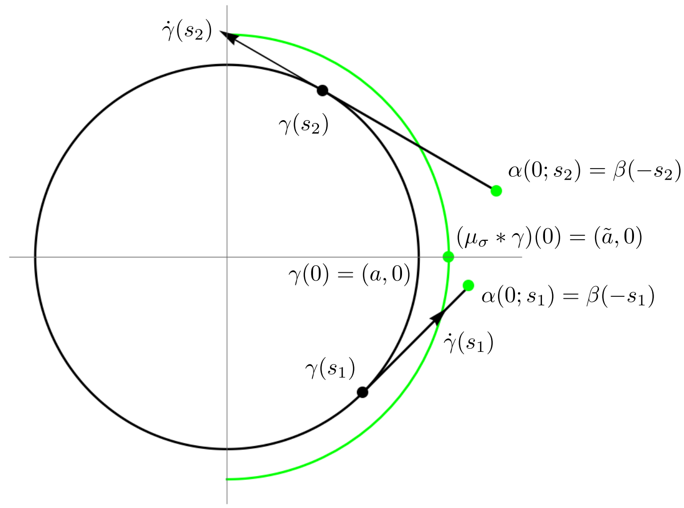

Figure 4: Tangent mollification of a circular curve. Here the particular circle has radius $a = 4/3$ as indicated by the length of the unit tangent vectors $\dot\gamma(s_1)$ and $\dot\gamma s_2$ and the mollification parameter is $\sigma = \pi/2$.

the domain enclosed by the circle and project to points $|\alpha(s_0)|\gamma(s_0)/|\gamma(s_0)|$ with $|\alpha(s_0)| > a$. Analytically, the inequality $\tilde a > a$ for the integral quantity in (12) is not entirely obvious. It is relatively straightforward however to show the integrand $a\cos(s/a) + s\sin(s/a)$ satisfies

$$a\cos(s/a) + s\sin(s/a) > a \qquad \text{for} \qquad 0 < s \le \frac{\pi a}{2}.$$

Thus, the inequality certainly holds for $\sigma \le \pi a/2$. In fact, the integral expression $\tilde a$ given in (12) increases with σ for $0 < \sigma < \sigma_{\max} \doteq 2.83a$ to a maximum value $\tilde a_{\max} \doteq 1.29\ a$ and then decreases to zero for $\sigma_{\max} < \sigma < \sigma_0 \doteq 6.30\ a$.

After this initial observation it should perhaps be noted that the new geometric tangent mollification $\nu : (-L + \sigma, M - \sigma) \to \mathbb{R}^2$ given in (11) satisfies

$$\nu \in C^\infty((-L + \sigma, M - \sigma) \to \mathbb{R}^2).$$

This is not quite for the usual reason that a standard mollification $\mu_\sigma * u$ or $\mu_\sigma * \gamma$ is smooth. The difference is explained in the proof of the following theorem stated under somewhat more general assumptions and allowing also for a space curve

$$\gamma : (-L, M) \to \mathbb{R}^n,$$

for some $n \in \{2, 3, 4, \ldots\}$.

Theorem 1 (regularity of tangent mollification) If $\gamma \in C^0((-L, M) \to \mathbb{R}^n)$ admits a weak tangent indicatrix $D_1\gamma : (-L, M) \to \mathbb{S}^{n-1}$ in the sense that there exist

$$D_1\gamma_1,\ D_1\gamma_2, \ldots, D_1\gamma_n \in L^1_{loc}(-L, M) \tag{13}$$

for which

$$\int_{(-L,M)} \gamma_j \,\frac{d\phi}{ds} = -\int_{(-L,M)} D_1\gamma_j\ \phi, \qquad \text{for all } \phi \in C_c^\infty(-L, M),\ j = 1, 2 \tag{14}$$

and

$$[D_1\gamma_1(s)]^2 + [D_2\gamma_2(s)]^2 + \cdots + [D_2\gamma_n(s)]^2 = 1 \qquad \text{for almost every } s \in (-L, M)$$

then for $\sigma > 0$ the tangent mollification $\nu : (-L + \sigma, M - \sigma) \to \mathbb{R}^n$ given by

$$\nu(t) = \mu_\sigma * \gamma(t) + \int_{s\in(-L,M)} (t - s)\mu_\sigma(t - s)(D_1\gamma_1(s), D_1\gamma_2(s), \ldots, D_2\gamma_n(s)) \tag{15}$$

satisfies $\nu \in C^\infty((-L + \sigma, M - \sigma) \to \mathbb{R}^n)$.

Note: In (3), (13), (14), and (15) we have used the notation $D_1\gamma$ and $D_1\dot\gamma$ to distinguish weak derivatives from the corresponding classical derivatives $\dot\gamma$ and $\ddot\gamma$. Since the existence of certain classical derivatives are assumed generally in the constructions presented here along with and in contrast to weak derivatives, we wil continue to maintain this notational distinction.

Proof of Theorem 1: The first term $\mu_\sigma * \gamma$ in (15) is C^∞ regular for the usual basic reason that the derivatives with respect to t fall only on the mollifier μ_σ in the integrand. Technically, one considers difference quotients and applies the dominated convergence theorem. See for example Theorem 7 in Appendix C of [Eva10] or Chapter 4 of [EG92].

As mentioned above, this reasoning does not apply precisely the same way to the second term

$$\mathbf{T}(t) = \int_{s\in(-L,M)} (t - s)\mu_\sigma(t - s)(D_1\gamma_1(s), D_1\gamma_2(s), \ldots, D_1\gamma_n(s)).$$

Setting $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ so that for $j = 1, 2, \ldots, n$

$$T_j(t) = \int_{s\in(-L,M)} (t - s)\mu_\sigma(t - s)D_1\gamma_j(s),$$

one finds in this case

$$\frac{dT_j}{dt} = \int_{s\in(-L,M)} \mu_\sigma(t-s)\ D_1\gamma_j(s) + \int_{s\in(-L,M)} (t-s)\frac{d\mu_\sigma}{ds}(t-s)\ D_1\gamma_j(s).$$

The integral expressions on the right are easily seen to be continuous in t and since $\mu'_\sigma \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\mu'_\sigma) = \mathrm{supp}(\mu_\sigma)$, the expressions are again differentiable with respect to t with

$$\frac{d^kT_j}{dt^k} = k\int_{s\in(-L,M)} \frac{d^{k-1}\mu_\sigma}{ds^{k-1}}(t-s)\ D_1\gamma_j(s) + \int_{s\in(-L,M)} (t-s)\ \frac{d^k\mu_\sigma}{ds^k}(t-s)\ D_1\gamma_j(s)$$

and

$$\frac{d^k\nu}{dt^k}(t) = (\mu_\sigma * \gamma_j)^{(k)}(t) + k\int_{s\in(-L,M)} \frac{d^{k-1}\mu_\sigma}{ds^{k-1}}(t-s)\ D_1\gamma_j(s) + \int_{s\in(-L,M)} (t-s)\ \frac{d^k\mu_\sigma}{ds^k}(t-s)\ D_1\gamma_j(s). \tag{16}$$

for $k = 2, 3, \ldots$. □

A mollified function $\mu_\sigma * u$ approximates the function $u \in L^1_{loc}(\Omega)$ in various senses, and it is natural to ask if the tangent mollification ν converges to γ in the same ways and under the same or similar assumptions. Note first the following:

(i) If K is a compact subset of $(-L, M)$, then for each $j = 1, 2, \ldots, n$

$$\lim_{\sigma\searrow 0} \|\mu_\sigma * \gamma_j - \gamma_j\|_{L^1(K)} = 0.$$

(ii) For almost every $t \in (-L, M)$

$$\lim_{\sigma\searrow 0} \mu_\sigma * \gamma(t) = \gamma(t). \tag{17}$$

(iii) If $\gamma \in C^0((-L, M) \to \mathbb{R}^n)$, then $\mu_\sigma * \gamma$ converges uniformly to γ on every compact subset of $(-L, M)$ and (17) holds at every point $t \in (-L, M)$.

(iv) If $\gamma_j \in C^k(-L, M)$ for some $k \in \mathbb{N}$ and $j \in \{1, 2, \ldots, n\}$, then for each compact set K with $K \subset (-L, M)$

$$\lim_{\sigma \searrow 0} \|\mu_\sigma * \gamma_j - \gamma_j\|_{C^k(K)} = 0.$$

(v) If $\gamma_j \in W^{k,p}(-L, M)$ for some $k \in \mathbb{N}$ and $j \in \{1, 2, \ldots, n\}$, then for each open set U with $U \subset\subset (-L, M)$

$$\lim_{\sigma \searrow 0} \sum_{\ell=0}^{k} \left\| \frac{d^\ell \mu_\sigma * \gamma_j}{dt^k} - D_1^\ell \gamma_j \right\|_{L^p(U)} = 0.$$

Good references for these assertions are [EG92] and [Fol76].

We now seek to obtain some analogous results for the tangent mollification $\nu \in C^\infty((-L + \sigma, M - \sigma) \to \mathbb{R}^n)$. We begin with a result analogous to the pointwise convergence of **(iii)** above.

Theorem 2 (pointwise convergence of the tangent mollification) If

$$\gamma \in C^1((-L, M) \to \mathbb{R}^n),$$

then $\nu \in C^\infty((-L + \sigma, M - \sigma) \to \mathbb{R}^n)$ given by (15) converges pointwise uniformly to γ on every compact subset of $(-L, M)$ and pointwise at every point in particular.

Notice the "additional" regularity required here in the sense that standard pointwise mollification $\mu_\sigma * \gamma$ converges uniformly on compact subsets to γ whenever

$$\gamma \in C^0((-L, M) \to \mathbb{R}^n).$$

Proof of Theorem 2: Since the first term $\mu_\sigma * \gamma(t)$ in (15) converges to $\gamma(t)$ uniformly on compact subsets, the convergence claimed in the theorem follows if the remaining term $T(t)$ tends to zero. Also, in this case, the weak derivative $D_1\gamma_j = \dot{\gamma}_j$ is a classical derivative for $j = 1, 2, \ldots, n$. In fact, for $j = 1, 2, \ldots, n$

$$\begin{aligned} |T_j(t)| &= \left| \int_{s \in (-L,M)} (t-s)\mu_\sigma(t-s)\ \dot{\gamma}_j(s) \right| \\ &= \left| \int_{s \in (t-\sigma, t+\sigma)} (t-s)\mu_\sigma(t-s)\ \dot{\gamma}_j(s) \right| \\ &\le \sigma |\mu_\sigma * \dot{\gamma}_j(t)|. \end{aligned} \tag{18}$$

If K is a compact subset of $(-L, M)$, then under the assumptions here $\mu_\sigma * \dot{\gamma}_j$ converges uniformly on K to $\dot{\gamma}_j$. In particular for $t \in K$ there holds

$$|T_j(t)| \leq \sigma \, \|\mu_\sigma * \dot{\gamma}_j\|_{C^0(K)}$$

and $\|\mu_\sigma * \dot{\gamma}_j\|_{C^0(K)}$ remains bounded as σ tends to zero. $\square$

The estimate (18) does not depend on the existence of the classical derivative $\dot{\gamma}_j$ and can be written in the form

$$\begin{aligned} |T_j(t)| &\leq \left| \int_{s \in (-L,M)} (t-s)\mu_\sigma(t-s) \; D_1\gamma_j(s) \right| \\ &= \left| \int_{s \in (t-\sigma, t+\sigma)} (t-s)\mu_\sigma(t-s) \; D_1\gamma_j(s) \right| \\ &\leq \sigma |\mu_\sigma * D_1\gamma_j(t)|. \end{aligned} \tag{19}$$

under the initial assumptions on γ given in Theorem 1. As a result the various convergence results considered presently are basically obtained by simply showing the function $G \in C^\infty(-L+\sigma, M-\sigma)$ given by

$$G(t) = \mu_\sigma * D_1\gamma_j(t)$$

remains bounded in some norm uniformly in σ as σ tends to zero.

Pointwise almost everywhere convergence of the tangent mollification ν analogous to **(ii)** holds whenever ν is defined:

Theorem 3 (almost everywhere convergence of the tangent mollification) If $\gamma \in C^0((-L, M) \to \mathbb{R}^n)$ has a weak unit tangent indacatrix according to the conditions of Theorem 1, then $\nu = \nu \in C^\infty((-L+\sigma, M-\sigma) \to \mathbb{R}^n)$ given by (15) converges pointwise to γ at almost every point $t \in (-L, M)$.

Note: "Additional" regularity is also required here in the sense that γ is required, for example, to have weak first order derivatives while **(ii)** only requires

$$\gamma \in L^1_{loc}((-L, M) \to \mathbb{R}^n).$$

On the other hand, this additional regularity may be viewed as primarilly required in order to make sense of the tangent mollification ν itself rather than for the convergence.

Proof of Theorem 3: Let A_j denote the collection of Lebesgue points

$$A_j = \left\{ t \in (-L, M) : \lim_{r \searrow 0} \frac{1}{r} \int_{s \in (t-r, t+r)} |D_1\gamma_j(s) - D_1\gamma_j(t)| = 0 \right\}$$

of $D_1\gamma_j$ in $(-L, M)$. Taking $t \in A_j$ we have

$$\lim_{\sigma \searrow 0} \mu_\sigma * D_1\gamma_j(t) = D_1\gamma_j(t).$$

Therefore, $|\mu_\sigma * D_1\gamma_j(t)|$ remains bounded as σ tends to zero, and starting with (19) we have

$$\lim_{\sigma \searrow 0} \left| \int_{s \in (-L,M)} (t-s)\mu_\sigma(t-s)\ D_1\gamma_j(s) \right| \le \lim_{\sigma \searrow 0} \sigma |\mu_\sigma * D_1\gamma_j(t)| = 0. \qquad \square$$

If γ admits classical derivatives of higher orders, then a formula for the higher order derivatives of the tangent mollification ν is available which is somewhat simpler than the one given in (16). Specifically, if $\gamma \in C^1((-L, M) \to \mathbb{R}^n)$, then (15) can be written as

$$\begin{aligned} \nu(t) &= \mu_\sigma * \gamma(t) + \int_{s \in (-L,M)} (t-s)\mu_\sigma(t-s)\ \dot{\gamma}(s) \\ &= \mu_\sigma * \gamma(t) + \int_{\eta \in (-\sigma,\sigma)} \eta\ \mu_\sigma(\eta)\ \dot{\gamma}(t-\eta) \end{aligned}$$

in terms of the classical derivative $\dot{\gamma}$. If $\gamma \in C^{k+1}((-L, M) \to \mathbb{R}^n)$ for $k \ge 1$, then additional derivatives may fall on the classical derivative in the integrand of the second term to yield

$$\begin{aligned} \frac{d^k\nu}{dt^k}(t) &= (\mu_\sigma * \gamma)^{(k)}(t) + \int_{\eta \in (-\sigma,\sigma)} \eta\ \mu_\sigma(\eta)\ \frac{d^{k+1}\gamma}{ds^{k+1}}(t-\eta) \\ &= \mu_\sigma * \frac{d^k\gamma}{ds^k}(t) + \int_{\eta \in (-\sigma,\sigma)} \eta\ \mu_\sigma(\eta)\ \frac{d^{k+1}\gamma}{ds^{k+1}}(t-\eta) \\ &= \mu_\sigma * (\gamma - \mathbf{w})\,(t) + t\ \mu_\sigma * \frac{d^{k+1}\gamma}{ds^{k+1}}(t) \end{aligned} \qquad (20)$$

where $\mathbf{w} \in C^0((-L, M) \to \mathbb{R}^2)$ is given by

$$\mathbf{w}(s) = s\ \frac{d^{k+1}\gamma}{ds^{k+1}}(s).$$

Notice that given a compact set $K \subset (-L, M)$ one has in this case for $t \in K$ and $j = 1, 2, \ldots, n$ the estimate

$$\left| \frac{d^k T_j}{dt^k}(t) \right| = \left| \int_{\eta \in (-\sigma,\sigma)} \eta \; \mu_\sigma(\eta) \; \frac{d^{k+1}\gamma_j}{ds^{k+1}}(t-\eta) \right| \leq \sigma \left\| \frac{d^{k+1}\gamma_j}{ds^{k+1}} \right\|_{C^0(K)}. \tag{21}$$

Consequently, one obtains a simple convergence result for the higher order derivatives of the tangent mollification:

Theorem 4 (convergence of higher order derivatives; tangent mollification) If $\gamma \in C^{k+1}((-L, M) \to \mathbb{R}^n)$ for some $k \geq 1$, then the tangent mollification ν converges to γ as σ tends to zero in $C^k([-L + \delta, M - \delta] \to \mathbb{R}^n)$ for every $\delta > 0$.

If $\gamma \in L^p_{loc}((-L, M) \to \mathbb{R}^n)$ for some p with $1 \leq p < \infty$ in the sense that for each compact set K with $K \subset (-L, M)$

$$\sum_{j=1}^{n} \int_K |\gamma_j|^p < \infty,$$

then it is well known that for each $K \subset\subset (-L, M)$ and $j = 1, 2, \ldots, n$ there holds

$$\lim_{\sigma \searrow 0} \|\mu_\sigma * \gamma_j - \gamma_j\|_{L^p(K)} = 0.$$

Starting again from (19)

$$\begin{aligned} |T(t)| &= \left| \int_{s \in (-L,M)} (t-s)\mu_\sigma(t-s) \; D_1\gamma_j(s) \right| \\ &\leq \sigma \left| \int_{s \in (-\sigma,\sigma)} \mu_\sigma(s) \; D_1\gamma_j(t-s) \right| \\ &= \sigma \left| \int_{r \in (-1,1)} \mu_1(r) \; D_1\gamma_j(t - r\sigma) \right| \\ &\leq \sigma \int_{r \in (-1,1)} \mu_1(r) \; |D_1\gamma_j(t - r\sigma)|. \end{aligned} \tag{22}$$

Therefore, taking $K \subset\subset (-L, M)$ and $\delta > 0$ small enough so that

$$K \subset\subset (-L + 2\delta, M - 2\delta)$$

we find for $\sigma < \delta$

$$\begin{aligned}\int_K |T_j| &= \int_{t\in K}\left|\int_{s\in(-L,M)}(t-s)\mu_\sigma(t-s)\ D_1\gamma_j(s)\right|\\ &\le \sigma\int_{r\in(-1,1)}\int_{t\in K}\mu_1(r)\ |D_1\gamma_j(t-r\sigma)|\\ &\le \sigma\int_{r\in(-1,1)}\mu_1(r)\ \|D_1\gamma_j\|_{L^1(-L+\delta,M-\delta)}\\ &= \sigma\|D_1\gamma_j\|_{L^1(-L+\delta,M-\delta)}. \end{aligned}\tag{23}$$

We conclude the following under the hypothesis of Theorem 3:

Theorem 5 (local L^1 convergence of the tangent mollification) If $\gamma \in C^0((-L,M)\to\mathbb{R}^n)$ has a weak unit tangent indacatrix according to the conditions of Theorem 1, then $\nu = \nu \in C^\infty((-L+\sigma, M-\sigma)\to\mathbb{R}^n)$ given by (15) satisfies

$$\lim_{\sigma\searrow 0}\sum_{j=1}^n \|\nu_j - \gamma_j\|_{L^1(K)} = 0$$

for every compact set K with $K \subset (-L,M)$.

Proof: Again since $\|\nu_j-\gamma_j\|_{L^1(K)} \le \|\mu_\sigma * \gamma_j - \gamma_j\|_{L^1(K)} + \|T_j\|_{L^1(K)}$ it is enough to show the latter term tends to zero. Since $\delta > 0$ is fixed in the estimate (23) and $\|D_1\gamma_j\|_{L^1(-L+\delta,M-\delta)}$ remains bounded accordingly (as a fixed constant) we have immediately that

$$\lim_{\sigma\searrow 0}\|T_j\|_{L^1(K)} = 0. \qquad \square$$

Notice that in the inequalities leading to (23) we have essentially obtained a fixed bound for the quantity

$$\|\ \mu_\sigma * |D_1\gamma_j|\ \|_{L^1(K)}.$$

Theorem 5 then essentially follows from the observation that

$$\|T_j\|_{L^1(K)} \le \sigma\|\ \mu_\sigma * |D_1\gamma_j|\ \|_{L^1(K)}.$$

For convenience, let us denote by $H \in C^\infty(-L+\delta, M-\delta)$ the function with values

$$H(t) = (\mu_\sigma * |D_1\gamma_j|)(t)$$

and recall the basic pointwise estimate (22) according to which $|T_j(t)| \le \sigma H(t)$.

We follow the approach of [EG92] to find an appropriate bound to show L^p convergence of ν to γ for $1 < p < \infty$:

$$
\begin{aligned}
H(t) &= \int_{s\in(t-\sigma,t+\sigma)} \mu_\sigma(t-s)\ |D_1\gamma_j(s)| \\
&= \int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)\ |D_1\gamma_j(t-\eta)| \\
&= \int_{r\in(-1,1)} \mu_1(r)^{1-1/p}\ \mu_1(r)^{1/p}\ |D_1\gamma_j(t-\sigma r)| \\
&\le \int_{r\in(-1,1)} \mu_1(r)^{1-1/p}\ \mu_1(r)^{1/p}\ |D_1\gamma_j(t-\sigma r)| \\
&\le \left(\int_{r\in(-1,1)} \mu_1(r)\right)^{1-1/p} \left(\int_{r\in(-1,1)} \mu_1(r)\ |D_1\gamma_j(t-\sigma r)|^p\right)^{1/p} \\
&= \left(\int_{r\in(-1,1)} \mu_1(r)\ |D_1\gamma_j(t-\sigma r)|^p\right)^{1/p}.
\end{aligned}
$$

Thus, for any $K \subset\subset (-L+2\delta, M-2\delta)$ as above and $\sigma < \delta$

$$
\begin{aligned}
\|H\|^p_{L^p(K)} &\le \int_{t\in K}\int_{r\in(-1,1)} \mu_1(r)\ |D_1\gamma_j(t-\sigma r)|^p \\
&= \int_{r\in(-1,1)} \mu_1(r) \int_{t\in K} |D_1\gamma_j(t-\sigma r)|^p \\
&\le \int_{r\in(-1,1)} \mu_1(r)\ \|D_1\gamma_j\|^p_{L^p(-L+\delta,M-\delta)} \\
&= \|D_1\gamma_j\|^p_{L^p(-L+\delta,M-\delta)}.
\end{aligned}
$$

The quantity on the right is now a fixed non-negative constant independent of σ.

Theorem 6 (local L^p convergence of the tangent mollification) If $\gamma \in C^0((-L,M) \to \mathbb{R}^n)$ has a weak unit tangent indacatrix according to the conditions of Theorem 1 satisfying

$$D_1\gamma_j \in L^p_{loc}(-L,M)$$

for $j = 1, 2, \dots, n$ and some p with $1 < p < \infty$, then

$$\nu = \nu \in C^\infty((-L+\sigma, M-\sigma) \to \mathbb{R}^n)$$

given by (15) satisfies

$$\lim_{\sigma\searrow 0}\sum_{j=1}^{n}\|\nu_j-\gamma_j\|_{L^p(K)}=0$$

for every compact set K with $K\subset(-L,M)$.

Proof: As before fix $\delta>0$ with $K\subset\subset(-L+2\delta,M-2\delta)$ and consider $\sigma<\delta$. Since $\|\nu_j-\gamma_j\|_{L^p(K)}\le\|\mu_\sigma*\gamma_j-\gamma_j\|_{L^p(K)}+\|T_j\|_{L^p(K)}$ it is enough to show

$$\lim_{\sigma\searrow 0}\|T_j\|_{L^p(K)}=0. \tag{24}$$

In fact, the basic pointwise estimate (22) gives $|T_j(t)|\le\sigma H$, so

$$\|T_j\|_{L^p(K)}\le\sigma\|H\|_{L^p(K)}\le\sigma\|D_1\gamma_j\|_{L^p(-L+\delta,M-\delta)}.$$

Therefore, (24) holds. □

It perhaps remains to discuss the convergence of weak derivatives of higher order in $L^p(K)$ where K is a compact subset of $(-L,M)$ when γ admits higher order weak derivatives. We postpone discussion of this topic to another occasion.

Because tangent mollification has been formulated for curves with lower regularity than those in Example 1 and Example 2, there is a natural lower regularity example to consider.

Example 3 (tangent mollification of a curve with a corner) Consider the graph Γ of the function $u:\mathbb{R}\to\mathbb{R}$ with values $u(x)=a|x|$ for some constant $a>0$. This curve is parameterized by arclength with $\gamma:\mathbb{R}\to\mathbb{R}^2$ given by

$$\gamma(s)=\frac{1}{\sqrt{1+a^2}}\,(s,a|s|). \tag{25}$$

Since the curve is given as a graph one may consider three mollifcations. The first is obtained by mollifying the function u and considering the graph Γ_σ of $\mu_\sigma*u:\mathbb{R}\to\mathbb{R}$. The formula and properties of this graph are well-known:

$$\begin{aligned}\mu_\sigma*u(x)&=\int_{x-\sigma}^{x+\sigma}\mu_\sigma(\xi-x)u(\xi)\,d\xi\\&=\begin{cases}a|x|, & |x|\ge\sigma\\ a\Big[\Big(\int_{-\sigma}^{x}\mu_\sigma(\eta)\,d\eta-\int_{x}^{\sigma}\mu_\sigma(\eta)\,d\eta\Big)\;x\\ \qquad-\int_{-\sigma}^{x}\eta\;\mu_\sigma(\eta)\,d\eta+\int_{x}^{\sigma}\eta\;\mu_\sigma(\eta)\,d\eta\Big], & |x|\le\sigma.\end{cases}\end{aligned}$$

The function $\mu_\sigma * u$ in this case is positive, even, and convex for $|x| < \sigma$ as indicated on the left in Figure 5.

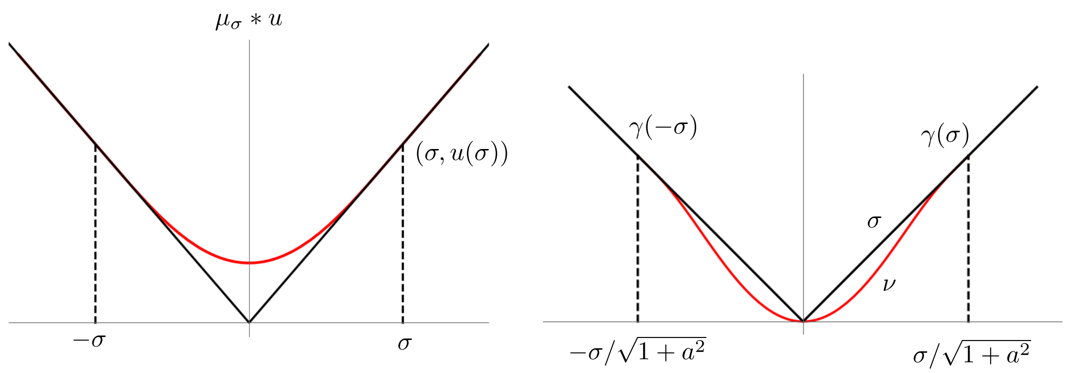

Figure 5: Mollifications of a curve with a corner. Mollification of the graph and basic coordinate mollification (left); tangent mollification (right).

The basic coordinate mollification is given by

$$\mu_\sigma * \gamma(t) = \int_{t-\sigma}^{t+\sigma} \mu_\sigma(s-t)\gamma(s)\,ds$$
$$= \begin{cases} \dfrac{1}{\sqrt{1+a^2}}(t, a|t|) & |t| \geq \sigma \\ \dfrac{1}{\sqrt{1+a^2}}\Bigg(t, \quad \Bigg[\Bigg(\displaystyle\int_{-\sigma}^{t} \mu_\sigma(\eta)\,d\eta - \int_t^\sigma \mu_\sigma(\eta)\,d\eta\Bigg)\, t \\ \qquad\qquad - \int_{-\sigma}^t \eta\, \mu_\sigma(\eta)\,d\eta + \int_t^\sigma \eta\, \mu_\sigma(\eta)\,d\eta\Bigg]\Bigg), & |t| \leq \sigma. \end{cases}$$

Comparison of the formulas for $\mu_\sigma * u$ and $\mu_\sigma * \gamma$ reveals that these mollifications are giving essentially the same curve/graph though the geometric significance of the mollification parameter σ is distinct relative to a scaling. Specifically the graph of $\mu_\sigma * u$ and the curve parameterized by $\mu_\tau * \gamma$ are identical if

$$\sigma = \frac{\tau}{\sqrt{1+a^2}}.$$

It may also be remarked that the equivalence of the graph mollification of a curve with a corner and the coordinate mollification depends also on the fact that I have chosen coordinates so that the function u is even. To see this, consider a case when $a < 1$ and the coordinates are rotated so that the same curve with an angle at the origin is given by the graph of a function $u : \mathbb{R} \to \mathbb{R}$ with

$$u(x) = x \ \tan\psi \ \chi_{[0,\infty)}(x) = \begin{cases} 0, & x \le 0 \\ x \ \tan\psi, & x \ge 0 \end{cases}$$

and $\psi = 2\tan^{-1} a$ satisfying $0 < \psi < \pi/2$. In this case, coordinate mollification will produce geometrically the same result as illustrated on the left in Figure 5. The function $\mu_\sigma * u$ however has a graph with transition nontrivially modifying the value of u for $-\sigma < x < \sigma$ and corresponding to the nonsymmetric interval of arclengths $-L < s < M$ with $-L = -\sigma < 0 < \sigma \sec\psi = M$. Thus, one sees the mollification of this graph with an angle is not the same as the coordinate mollification.

The tangent mollification is quite different from the graph mollification and the coordinate mollification considered above. One does have again $\nu(t) \equiv (t, a|t|)/\sqrt{1+a^2}$ for $|t| \ge \sigma$ simply because one is averaging a constant value $\alpha(t;s) \equiv \gamma(t)$ for each t in these intervals. For $|t| < \sigma$, say $-\sigma < t < 0$, we find

$$\alpha(t;s) = \begin{cases} \gamma(t), & t-\sigma \le s < 0 \\ (\gamma_1(t), -\gamma_2(t)), & 0 < s \le t+\sigma \end{cases}$$

as indicated in Figure 6. When $0 < t < \sigma$ one finds in contrast

$$\alpha(t;s) = \begin{cases} (\gamma_1(t), -\gamma_2(t)), & t-\sigma \le s < 0 \\ \gamma(t), & 0 < s \le t+\sigma \end{cases}$$

The consequent averaging gives

$$\begin{aligned} \nu(t) &= \frac{1}{\sqrt{1+a^2}} \left(\int_{t-\sigma}^{t+\sigma} \mu_\sigma(t-s) \ s\,ds, \right. \\ &\qquad\qquad \left. \int_{t-\sigma}^{0} \mu_\sigma(t-s) \ (-at)\,ds + \int_0^{t+\sigma} \mu_\sigma(t-s) \ at\,ds \right) \\ &= \frac{1}{\sqrt{1+a^2}} \left(\quad t, \quad a \left(\int_{-\sigma}^{t} \mu_\sigma(\eta)\,d\eta - \int_t^{\sigma} \mu_\sigma(\eta)\,d\eta \right) \ t \quad \right). \end{aligned}$$

Notice that for $t < 0$ one has

$$-1 < \int_{-\sigma}^{t} \mu_\sigma(\eta)\,d\eta - \int_t^{\sigma} \mu_\sigma(\eta)\,d\eta < 0.$$

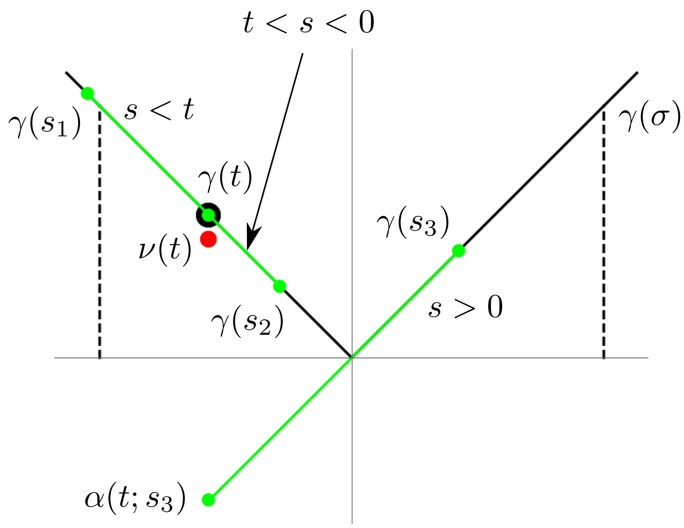

Figure 6: Tangent mollification of a curve with a corner: Averaged values. Consider a point $\gamma(t)$ with t fixed as indicated in the illustration. A tangent mollified point $\nu(t)$ is obtained as a weighted average with respect to $s \in (t-\sigma, t+\sigma)$ of points $\alpha(t;s)$. Given s_1 with $t-\sigma < s_1 < t$, we associate with the point $\gamma(s_1)$ the point $\alpha(t;s_1) = \gamma(s_1) + (t-s_1)\dot{\gamma}(s_1) \equiv \gamma(t)$. Likewise $\alpha(t;s_2) = \gamma(s_2) + (t-s_2)\dot{\gamma}(s_2) \equiv \gamma(t)$ when $t < s_2 < 0$. For $0 < s_3 < t+\sigma$ the point $\alpha(t,s_3) = \gamma(s_3) + (t-s_3)\dot{\gamma}(s_2) \neq \gamma(t)$.

Consequently

$$0 < \nu_2(t) < \frac{a}{\sqrt{1+a^2}}\,|t| = |\nu_1(t)|$$

for $-\sigma < t < 0$. On the other hand, for $t > 0$ there holds

$$0 < \int_{-\sigma}^{t} \mu_\sigma(\eta)\,d\eta - \int_{t}^{\sigma} \mu_\sigma(\eta)\,d\eta < 1$$

so that

$$0 < \nu_2(t) < \frac{a}{\sqrt{1+a^2}}\,t = \nu_1(t)$$

for $0 < t < \sigma$. Remarkably then $\nu(0) = (0,0)$ as indicated on the right in Figure 5.

It would be interesting to compute the tangent $\nu' : (-\sigma,\sigma) \to \mathbb{R}^2$ and curvature vector associated with $\nu \in C^\infty((-\sigma,\sigma) \to \mathbb{R}^2)$ in this example. As indicated on the right in Figure 5 there are certainly points of vanishing curvature. We leave these calculations and the associated analysis to the ambitious reader.

Let us finally examine briefly the application of tangent mollification to the curve of Example 1. As mentioned above the tangent mollification parameterizes circles of larger radius with

$$\nu(t) = (\pm 1, 0) + \tilde{a}(\mp \cos t, \sin t) \qquad \text{for} \qquad \sigma \leq |t| \leq \frac{\pi a}{2} - \sigma$$

with

$$\tilde{a} = \int_{-\sigma}^{\sigma} \mu_\sigma(s)[a\cos(s/a) + s\sin(s/a)]\, ds > a$$

as indicated in the close up on the left in Figure 7.

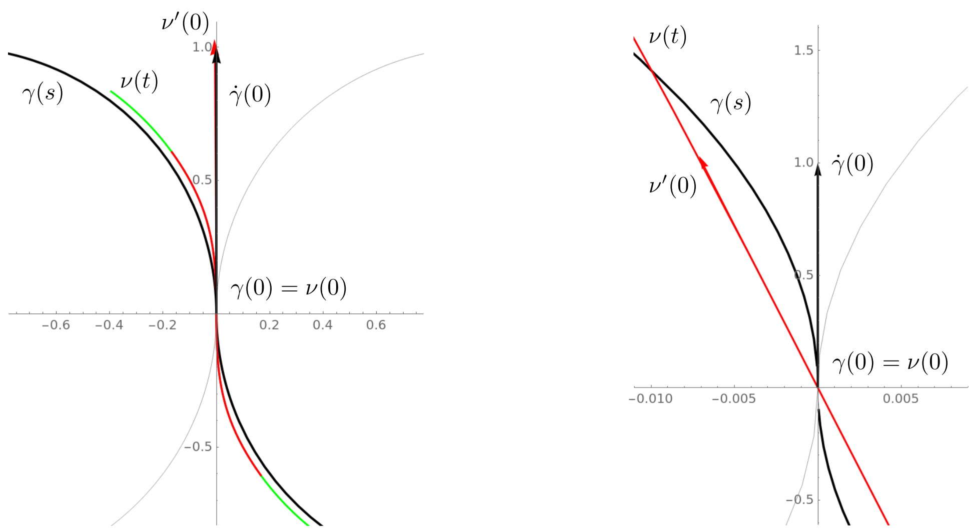

Figure 7: Tangent mollification of the two tangent circular arcs from Example 1. In the close up on the left we have taken $a = 1$ and mollification parameter $\sigma = \pi/5$. The axes are shown at the same scale, and the detail near $\gamma(0) = \nu(0) = (0, 0)$ is not clearly visible. In the close up on the right we have taken the circular radius $a = 100$, mollification parameter $\sigma = 20\pi$, and the horizontal axis is scaled by a factor of approximately 100 relative to the vertical axis.

In order to understand the C^∞ transition occurring for $-\sigma < t < \sigma$ it is useful to consider a result applying to a somewhat more general class of curves.

Lemma 1 If $\gamma \in C^1((-L, L) \to \mathbb{R}^n)$ satisfies $\gamma(-s) = -\gamma(s)$, then $\nu(-t) = -\nu(t)$ for $-\sigma < t < \sigma$ and

$$\frac{d\nu}{dt}(0) = 2\int_0^\sigma [2\mu_\sigma(\eta) + \eta\mu'(\eta)]\ \dot\gamma(\eta)\, d\eta. \tag{26}$$

Proof: These assertions follow directly from computation(s): Notice that differentiating the relation $\gamma(-s) = -\gamma(s)$ one obtains $\dot\gamma(-s) = \dot\gamma(-s)$. Therefore,

$$\begin{aligned}
\nu(-t) &= \int_{s\in(t-\sigma,t+\sigma)} \mu_\sigma(-t-s)\ \gamma(s) + \int_{s\in(t-\sigma,t+\sigma)} (-t-s)\mu_\sigma(-t-s)\ \dot\gamma(s)\\
&= \int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)\ \gamma(-t-\eta) + \int_{\eta\in(-\sigma,\sigma)} \eta\ \mu_\sigma(\eta)\ \dot\gamma(-t-\eta)\\
&= -\int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)\ \gamma(t+\eta) + \int_{\eta\in(-\sigma,\sigma)} \eta\ \mu_\sigma(\eta)\ \dot\gamma(t+\eta)\\
&= -\int_{s\in(t-\sigma,t+\sigma)} \mu_\sigma(s-t)\ \gamma(s) + \int_{s\in(t-\sigma,t+\sigma)} (s-t)\mu_\sigma(s-t)\ \dot\gamma(s)\\
&= -\int_{s\in(t-\sigma,t+\sigma)} \mu_\sigma(t-s)\ \gamma(s) - \int_{s\in(t-\sigma,t+\sigma)} (t-s)\mu_\sigma(t-s)\ \dot\gamma(s)\\
&= -\nu(t).
\end{aligned}$$

For the derivative

$$\begin{aligned}
\nu'(t) &= 2\int_{s\in(t-\sigma,t+\sigma)} \mu_\sigma(t-s)\ \dot\gamma(s) + \int_{s\in(t-\sigma,t+\sigma)} (t-s)\mu'_\sigma(t-s)\ \dot\gamma(s)\\
&= 2\int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)\ \dot\gamma(t-\eta) + \int_{\eta\in(-\sigma,\sigma)} \eta\ \mu'_\sigma(\eta)\ \dot\gamma(t-\eta).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nu'(0) &= 2\int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)\ \dot\gamma(-\eta) + \int_{\eta\in(-\sigma,\sigma)} \eta\ \mu'_\sigma(\eta)\ \dot\gamma(-\eta)\\
&= \int_{\eta\in(-\sigma,\sigma)} [2\mu_\sigma(\eta) + \eta\ \mu'_\sigma(\eta)]\dot\gamma(\eta)\\
&= \int_{-\sigma}^0 [2\mu_\sigma(\eta) + \eta\ \mu'_\sigma(\eta)]\dot\gamma(\eta)\, d\eta + \int_0^\sigma [2\mu_\sigma(\eta) + \eta\ \mu'_\sigma(\eta)]\dot\gamma(\eta)\, d\eta\\
&= 2\int_0^\sigma [2\mu_\sigma(\eta) + \eta\ \mu'_\sigma(\eta)]\dot\gamma(\eta)\, d\eta. \qquad \square
\end{aligned}$$

Lemma 1 applies to the curve of Example 1 and yields immediately that $\nu(0) = (0,0) = \gamma(0)$. In cases like this one where $\gamma \in C^2((0,L) \to \mathbb{R}^n)$ the formula for $\nu'(0)$ may be further manipulated:

$$\begin{aligned} \nu'(0) &= 4\int_0^\sigma \mu_\sigma(s)\ \dot\gamma(s)\,ds + 2\int_0^\sigma s\ \mu_\sigma'(s)\ \dot\gamma(s)\,ds \\ &= 4\int_0^\sigma \mu_\sigma(s)\ \dot\gamma(s)\,ds - 2\int_0^\sigma \mu_\sigma(s)\ \frac{d}{ds}[s\ \dot\gamma(s)]\,ds \\ &= 2\int_0^\sigma \mu_\sigma(s)[\dot\gamma(s) - s\ \ddot\gamma(s)]\,ds. \end{aligned} \tag{27}$$

Referring back to (4) for the formula for γ in Example 1 we find

$$\begin{aligned} \dot\gamma(s) &= \left(-\sin\left(\frac{s}{a}\right), \cos\left(\frac{s}{a}\right)\right) \\ \ddot\gamma(s) &= -\frac{1}{a}\left(\cos\left(\frac{s}{a}\right), \sin\left(\frac{s}{a}\right)\right) \end{aligned}$$

for $0 \le s < L = \pi a/2$. Taking the components separately

$$\dot\gamma_1(s) - s\ \ddot\gamma_1(s) = \frac{s}{a}\ \cos\left(\frac{s}{a}\right) - \sin\left(\frac{s}{a}\right) \le 0 \tag{28}$$

for $0 \le s < \pi a/2$ with equality only for $s = 0$. To see this note

$$\frac{d}{ds}[\dot\gamma_1(s) - s\ \ddot\gamma_1(s)] = -\frac{s}{a^2}\ \sin\left(\frac{s}{a}\right).$$

For the second component

$$\dot\gamma_2(s) - s\ \ddot\gamma_2(s) = \frac{s}{a}\ \sin\left(\frac{s}{a}\right) + \cos\left(\frac{s}{a}\right) > 0.$$

Since $\mu_\sigma \ge 0$, this latter inequality means the vector $\nu'(0)$ given in (27) points upward, which is perhaps to be expected. Taking account of the inequality (28) for the first component of $\nu'(0)$ however gives that $\nu'(0)$ points into the second quadrant, that is, this tangent is biased toward the second quadrant like $(\mu_\sigma * \gamma)'(0)$. This possibly unexpected behavior means first of all that the tangent mollification curve intersects the concatenated circular arcs at $\gamma(0) = \nu(0) = (0,0)$ transversely and necessarily intersects the curve Γ at (at least) two additional points in order to join smoothly with the exterior circular arcs determined by $\nu(t)$ for $|t| > \sigma$ (shown in green on the left in Figure 7). This transverse crossing is illustrated on the right in Figure 7 where it appears that in fact there are exactly three intersection points of the tangent mollification curve with the original concatenated circles Γ. We have not verified this detailed behavior with an explicit computation and/or estimates, but the numerical indication is relatively persuasive. Another numerical calculation confirming the same conclusion is indicated/illustrated in Figure tex1plotB.

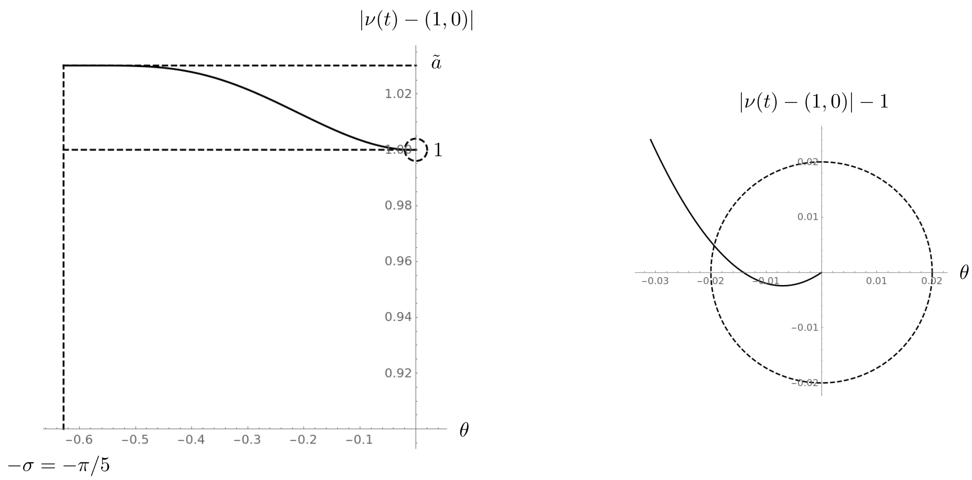

Figure 8: Tangent mollification of the two tangent circular arcs from Example 1. On the left we have again taken $a = 1$ and mollification parameter $\sigma = \pi/5$. Taking the right center $(1, 0)$ of the unit radius circular arc determined by Γ for $-\pi/2 < s \leq 0$, each point $\nu(t)$ for $t \leq 0$ determines a polar radius $|\nu(t) - (1, 0)|$ and for $t < 0$ a (reverse oriented) polar angle $\theta = \arctan(\nu_2(t)/\nu_1(t))$ (measured from the negative x_2-axis) with $\lim_{t\nearrow 0} \theta(t) = 0$. This polar angle is thus comparable to the arclength parameter s for $-\pi/2 < s \leq 0$ along the original circular arc, and we have plotted the new polar radius as a function of the polar angle. The plot on the right shows a close up of the circled region near the point $(\theta, |\nu - (1, 0)|) = (0, 1)$ with some normalizations making the detail visible. Aside from the vertical translation associated with the quantity $|\nu - (1, 0)| - 1$, the vertical axis has been scaled by 100.

Of additional significant interest is the behavior of the curvature

$$\vec{k} = \frac{1}{|\nu'(t)|} \frac{d}{dt} \left(\frac{\nu'(t)}{|\nu'(t)|} \right) = \frac{1}{|\nu'(t)|^4} \left(|\nu'(t)|^2 \, \nu''(t) - (\nu'(t) \cdot \nu''(t)) \, \nu'(t) \right) \tag{29}$$

of the tangent mollification throughout the transition interval $-\sigma < t < \sigma$. The direct computation of the curvature vector in (29) is again complicated, and we rely on a numerical calculation which is relatively persuasive and easily obtained. Figure 9 shows the curvature vector plotted on the interval $(-\pi/2+\sigma, \pi/2-\sigma)$ with $\sigma = \pi a/5$. The illustration indicates that the modulus $|\vec{k}|$ of the curvature vector is nowhere greater than $1/\tilde{a}$ and decreases monotonically to zero as a function of t for $-\pi/2+\sigma < t < 0$ and then increases symmetrically for $0 < t < \pi/2-\sigma$. The symmetry follows from the relation $\nu(-t) = -\nu(t)$ from Lemma 1.

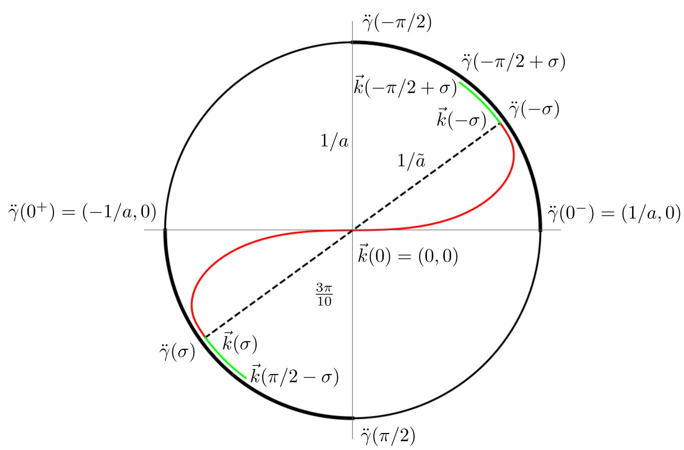

Figure 9: Tangent mollification of the two tangent circular arcs from Example 1. Curvature vectors $\vec{k}$ associated with ν and $\ddot{\gamma}$. The path of $\vec{k}$ consists of two circular arcs of radius $1/\tilde{a} < 1/a$ indicated in green along with a transition curve plotted in red. The transition of the curvature vector from $\vec{k}(-\sigma) = (a/\tilde{a})\ddot{\gamma}(-\sigma)$ to $\vec{k}(\sigma) = (a/\tilde{a})\ddot{\gamma}(\sigma)$ mollifies the jump discontinuity in curvature present in Γ. Here $\sigma = \pi a/5$

The parameterization $\vec{k} \in C^\infty((-\pi/2+\sigma, \pi/2-\sigma) \to \mathbb{R}^2)$ of

$$K_\nu = \left\{\vec{k}(t) : -\frac{\pi}{2} + \sigma \le t \le \frac{\pi}{2} - \sigma\right\}$$

is singular with respect to arclength in the sense that

$$\frac{d\vec{k}}{dt}(0) = (0, 0).$$

This follows from the fact that the unit tangent vector $\mathbf{T}(t) = \nu'(t)/|\nu'(t)|$ differentiated in (29) satisfies $\mathbf{T}(-t) = \mathbf{T}(t)$. Notice however, that the path K_ν traced by the curvature vector and illustrated in Figure 9 appears to be geometrically nonsingular with well-defined unit tangent $\mathbf{u} = \mathbf{u}(0)$ at $(0,0)$. In fact, one might at first be inclined to assume $\mathbf{u}(0) = (-1,0)$ at $\vec{k}(0) = (0,0)$. The nonsingularity with respect to intrinsic arclength appears to be still indicated numerically also at smaller scales. The unit tangent vector $\mathbf{u}(0)$ if it exists, however, appears to be of the form $\mathbf{u}(0) = (-\sqrt{1-\epsilon^2}, -\epsilon)$ for some $\epsilon > 0$; see Figure 10.

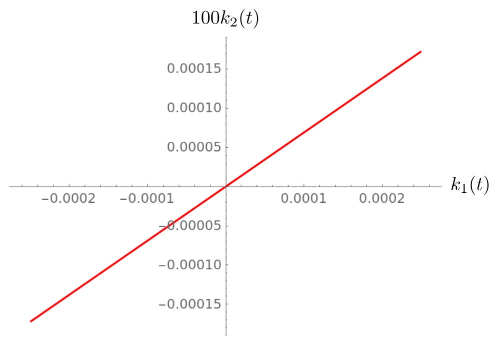

Figure 10: Tangent mollification of the two tangent circular arcs from Example 1. Plot of $\{(k_1(t), 100k_2(t)) : |t| \leq 0.0001\}$ where $\vec{k} = (k_1, k_2)$. Here the values indicated on the axes and relevant for the parameter t are determined by the choice $a = 1$ and $\sigma = \pi/5$, but gometrically the set K_ν is independent of the radius a.

2 Mollification of curvature

Generalizing the idea of the previous section to the second order condition of homogeneity, that is constant curvature, is in principle straightforward. Given $s_0 \in (-L+\sigma, M-\sigma)$ and s with $s_0 - \sigma < s < s_0 + \sigma$, one considers the circle $\partial B_r(\mathbf{p})$ passing through $\gamma(s)$ and tangent to the curve Γ at $\gamma(s)$ with

$$r = \frac{1}{|D_1\dot{\gamma}(s)|} \qquad \text{and} \qquad \mathbf{p} = \gamma(s) + \frac{D_1\dot{\gamma}(s)}{|D_1\dot{\gamma}(s)|^2}$$

the radius and center of curvature respectively. Parameterizing this circle by arclength t from $\gamma(s)$ one obtains a function $\beta \in C^\infty(\mathbb{R} \to \mathbb{R}^2)$ satisfying $\beta(0) = \gamma(s)$ and $\dot{\beta}(0) = \dot{\gamma}(s)$. One then writes $\alpha(s_0; s) = \beta(s_0 - s)$ and considers the point $\beta(s_0 - s)$

in comparison with $\gamma(s_0)$ so that averaging with respect to a mollifier gives $\nu \in C^\infty((-L+\sigma, M-\sigma) \to \mathbb{R}^2$ by

$$\nu(s_0) = \int_{s\in(s_0-\sigma,s_0+\sigma)} \mu_\sigma(s_0-s)\beta(s_0-s)$$

which we call the **curvature mollification** of Γ. Of course, the regularity of ν should be checked based on various assumptions about the curve Γ. The question of regularity will be considered in detail below.

In practice perhaps the most natural, convenient, and practical way to carry out this mollification procedure involves the introduction of a global inclination angle along Γ. It is not surprising, and it may even be considered well-known, that a global inclination angle $\psi : (-L, M) \to \mathbb{R}$ exists satisfying

$$(\cos\psi, \sin\psi) = \dot{\gamma}. \tag{30}$$

In the case when $\gamma \in C^2((-L, M) \to \mathbb{R}^2)$ the inclination angle can be used to define the curvature of the curve γ. A discussion of this topic along with a proof and related formulas may be found in [McC24]. The inclination angle as described here is not unique, but in this instance we can take any convenient initial angle ψ_0 satisfying

$$(\cos\psi_0, \sin\psi_0) = \dot{\gamma}(0);$$

there is then a unique inclination angle along Γ satisfying $\psi(0) = \psi_0$. In addition to (30) we write

$$\dot{\gamma}^\perp = (-\sin\psi, \cos\psi).$$

We may then write

$$\beta(t) = \gamma(s) + \frac{D_1\dot{\gamma}(s)}{|D_1\dot{\gamma}(s)|^2} + \frac{1}{|D_1\dot{\gamma}(s)|}(\cos\theta(t), \sin\theta(t))$$

where

$$\begin{aligned}\theta(t) &= \psi(s) - \frac{D_1\dot{\gamma}(s)\cdot\dot{\gamma}(s)^\perp}{|D_1\dot{\gamma}(s)|}\left(\frac{\pi}{2} - |D_1\dot{\gamma}(s)|\ t\right)\\ &= \psi(s) + [D_1\dot{\gamma}(s)\cdot\dot{\gamma}(s)^\perp]\left(t - \frac{\pi}{2|D_1\dot{\gamma}(s)|}\right).\end{aligned}$$

Finally,

$$\alpha(s_0; s) = \beta(s_0 - s) = \gamma(s) + \frac{D_1\dot{\gamma}(s)}{|D_1\dot{\gamma}(s)|^2} + \frac{1}{|D_1\dot{\gamma}(s)|}(\cos\theta(s_0-s), \sin\theta(s_0-s))$$

and we can write for $-L+\sigma \le t \le M-\sigma$

$$\nu(t) = \mu_\sigma * \gamma(t) + \int_{s\in(t-\sigma,t+\sigma)} \mu_\sigma(t-s)\frac{D_1\dot\gamma(s)}{|D_1\dot\gamma(s)|^2}$$
$$+ \int_{s\in(t-\sigma,t+\sigma)} \frac{\mu_\sigma(t-s)}{|D_1\dot\gamma(s)|}\left(\cos\left[\psi(s) + [D_1\dot\gamma(s)\cdot\dot\gamma(s)^\perp]\left(t-s-\frac{\pi}{2|D_1\dot\gamma(s)|}\right)\right],\right.$$
$$\left.\sin\left[\psi(s) + [D_1\dot\gamma(s)\cdot\dot\gamma(s)^\perp]\left(t-s-\frac{\pi}{2|D_1\dot\gamma(s)|}\right)\right]\right)$$

For convenience we denote the argument of the trigonometric functions in the last integral by

$$\Theta(t;s) = \psi(s) + [D_1\dot\gamma(s)\cdot\dot\gamma(s)^\perp]\left(t-s-\frac{\pi}{2|D_1\dot\gamma(s)|}\right)$$

so that the curvature mollification takes the form

$$\nu(t) = \mu_\sigma * \gamma(t) + \int_{s\in(t-\sigma,t+\sigma)} \frac{\mu_\sigma(t-s)}{|D_1\dot\gamma(s)|}\left[\frac{D_1\dot\gamma(s)}{|D_1\dot\gamma(s)|} + \Big(\cos\Theta(t;s), \sin\Theta(t;s)\Big)\right] \tag{31}$$

The geometric connection of the weak curvature vector $D_1\dot\gamma$ to the inclination angle ψ can be strengthened with consequences for the expression (31) and for the angle $\Theta(t;s)$ in particular. For each $s_0 \in (-L,M)$ there is some $j\in\{1,2\}$, some $\delta>0$, and some smooth function $f\in C^\infty[s_0-\delta, s_0+\delta]$ such that

$$\psi(s) = f\circ\dot\gamma_j(s) \qquad \text{for} \qquad s_0-\delta < s < s_0+\delta.$$

The function f is some local inverse of a trigonometric function; see [McC24]. By the chain rule for weak derivatives (Lemma 7.5 in [GT83]) we have that

$$\psi_0 = \psi\big|_{(s_0-\delta,s_0+\delta)}$$

has a weak derivative $D_1\psi_0 \in L^1_{loc}(s_0-\delta, s_0+\delta)$ given by $D_1\psi_0(s) = f'\circ\dot\gamma_j(s)\ D_1\dot\gamma_j(s)$.

It is fairly straightforward to show that setting

$$D_1\psi(s_0) = D_1\psi_0(s_0) \qquad \text{for} \qquad s_0\in(-L,M) \tag{32}$$

one obtains a well-defined weak derivative $D_1\psi \in L^1_{loc}(-L,M)$ for the inclination angle $\psi\in C^0(-L,M)$. Applying the chain rule again to $\dot\gamma = (\cos\psi, \sin\psi)$ we have

$$D_1\dot\gamma_1 = -\sin\psi\ D_1\psi$$
$$D_1\dot\gamma_2 = \cos\psi\ D_1\psi,$$

that is, the weak curvature vector satisfies

$$D_1\dot{\gamma} = D_1\psi\ (-\sin\psi, \cos\psi) = D_1\psi\ \dot{\gamma}^{\perp}.$$

Consequently, $|D_1\dot{\gamma}| = |D_1\psi|$ and $D_1\dot{\gamma}\cdot\dot{\gamma}^{\perp} = D_1\psi$. In particular,

$$\frac{D_1\dot{\gamma}\cdot\dot{\gamma}^{\perp}}{|D_1\dot{\gamma}|} = \frac{D_1\psi}{|D_1\psi|} \in \{\pm 1\}.$$

Returning to (31) the angle $\Theta(t;s)$ now takes the form

$$\Theta(t;s) = \psi(s) + (t-s)D_1\psi(s) - \frac{D_1\psi(s)}{|D_1\psi(s)|}\ \frac{\pi}{2}$$

so that

$$\cos\Theta(t;s) = \frac{D_1\psi(s)}{|D_1\psi(s)|}\sin[\psi(s) + (t-s)D_1\psi(s)], \tag{33}$$

$$\sin\Theta(t;s) = -\frac{D_1\psi(s)}{|D_1\psi(s)|}\cos[\psi(s) + (t-s)D_1\psi(s)] \tag{34}$$

and the curvature mollification (31) can also be written as

$$\begin{aligned}\nu(t) &= \mu_\sigma * \gamma(t)\\ &\quad + \int_{s\in(-L,M)} \mu_\sigma(t-s)\frac{1}{D_1\psi(s)}\Bigg[\Bigg(\sin[\psi(s)+(t-s)D_1\psi(s)] - \sin\psi(s),\\ &\qquad\qquad \cos\psi(s) - \cos[\psi(s)+(t-s)D_1\psi(s)]\Bigg)\Bigg]\end{aligned} \tag{35}$$

Having formally obtained the two equivalent forms of the curvature mollification (31) and (35), we now impose two additional restrictions:

(RC) We assume the original curve $\Gamma = \{\gamma(s) : -L < s < M\}$ has a **locally integrable radius of curvature**, that is, the function $\rho : (-L, M) \to \mathbb{R}$ given by

$$\rho(s) = \frac{1}{|D_1\dot{\gamma}(s)|} = \frac{1}{|D_1\psi(s)|}$$

satisfies $\rho \in L^1_{loc}(-L, M)$.

(MCB) We assume Γ has a **locally bounded weak curvature modulus**, that is, the function $\kappa : (-L, M) \to \mathbb{R}$ give by $\kappa(s) = |D_1\dot{\gamma}(s)|$ satisfies $\kappa \in L^\infty_{loc}(-L, M)$.

The form of the curvature mollification given in (31) suggests the first restriction **(RC)** on the radius of curvature. The origin of the second restriction **(MCB)** on the absolute value of the curvature will become clear in the proof of the following result.

Theorem 7 (regularity of curvature mollification) If $\gamma \in C^1((-L, M) \to \mathbb{R}^2)$ admits a weak curvature vector $D_1\dot{\gamma} : (-L, M) \to \mathbb{S}^1$ in the sense described by (1), (2), and (3) in the introduction and in addition satisfies condition **(RC)** of having a locally integrable radius of curvature $\rho = 1/|D_1\dot{\gamma}|$ and condition **(MCB)** of having a locally bounded weak curvature modulus $\kappa = |D_1\dot{\gamma}|$ then, letting $\psi \in C^0(-L, M)$ denote the inclination of Γ so that

$$\dot{\gamma}(s) = (\cos\psi(s), \sin\psi(s)) \qquad \text{for} \qquad s \in (-L, M),$$

one has that for $\sigma > 0$ the curvature mollification $\nu : (-L+\sigma, M-\sigma) \to \mathbb{R}^2$ given by

$$\begin{aligned} \nu(t) &= \mu_\sigma * \gamma(t) \\ &\quad + \int_{s\in(-L,M)} \mu_\sigma(t-s)\ \rho(s) \left[\rho(s)\ D_1\dot{\gamma}(s) + (\cos\Theta(t;s), \sin\Theta(t;s))\right] \end{aligned} \tag{36}$$

where

$$\Theta(t;s) = \psi(s) + (t-s)D_1\psi(s) - \frac{D_1\psi(s)}{|D_1\psi(s)|}\ \frac{\pi}{2} \tag{37}$$

satisfies $\nu \in C^\infty((-L+\sigma, M-\sigma) \to \mathbb{R}^2)$.

Note that in (37) we are using the implication that ψ is weakly differentiable with $D_1\psi \in L^1_{loc}(-L, M)$; see (32).

Proof of Theorem 7: As with tangent mollification the curvature mollification takes the form of an additional term or terms added to the coordinate mollification $\mu_\sigma * \gamma$, and we consider these as separate components setting

$$S_j(t) = \int_{s\in(-L,M)} \mu_\sigma(t-s)\ \rho(s)^2\ D_1\dot{\gamma}_j(s) \qquad \text{for } j = 1, 2, \tag{38}$$

$$T_1(t) = \int_{s\in(-L,M)} \mu_\sigma(t-s)\ \rho(s)\ \cos\Theta(t;s), \qquad \text{and}$$

$$T_2(t) = \int_{s\in(-L,M)} \mu_\sigma(t-s)\ \rho(s)\ \sin\Theta(t;s). \tag{39}$$

Notice that

$$|\rho(s)^2\ D_1\dot{\gamma}_j(s)| \le \rho(s) \left| \frac{D_1\dot{\gamma}(s)}{|D_1\dot{\gamma}(s)|} \right| = \rho(s).$$

Thus, $S_j = \mu_\sigma * g$ is the mollification of the locally integrable function $g = \rho^2 D_1\dot{\gamma}_j$, and $S_j \in C^\infty(-L+\sigma, M-\sigma)$ for $j = 1, 2$ with the usual formula

$$\frac{d^k S_j}{dt^k}(t) = \int_{s\in(-L,M)} \frac{d^k\mu_\sigma}{ds^k}(t-s)\ \rho(s)^2\ D_1\dot{\gamma}_j(s) \qquad \text{for } j = 1, 2$$

or

$$\frac{d^k S_j}{dt^k}(t) = \int_{s\in(-L,M)} \mu_\sigma(t-s) \frac{d^k}{ds^k} \left(\frac{\ddot{\gamma}_j}{|\ddot{\gamma}|^2} \right)(s) = \mu_\sigma * \frac{d^k}{ds^k} \left(\rho^2\ \ddot{\gamma} \right)(t) \tag{40}$$

in case $\gamma \in C^{k+2}((-L, M) \to \mathbb{R}^2)$.

Consider next the regularity of T_1. The function Θ is an affine function of t with

$$\frac{\partial\Theta}{\partial t}(t; s) = D_1\psi(s)$$

independent of t. Thus for t fixed, $h_1 : (-L, M) \to \mathbb{R}$ by

$$h_1(s) = -\rho(s)\ \sin\Theta(t; s)\ \frac{\partial\Theta}{\partial t}(t; s)$$

satisfies

$$|h_1(s)| \le \rho(s)\ |D_1\psi(s)| = \rho(s)\ |D_1\dot{\gamma}(s)| = 1$$

and hence $h_1 \in L^\infty(-L, M)$. It follows that

$$\begin{aligned}
\frac{dT_1}{dt}(t) &= \int_{s\in(-L,M)} \frac{d\mu_\sigma}{ds}(t-s)\ \rho(s)\ \cos\Theta(t; s) \\
&\qquad - \int_{s\in(-L,M)} \mu_\sigma(t-s)\ \rho(s)\ \sin\Theta(t; s)\ D_1\psi(s) \\
&= \int_{s\in(-L,M)} \frac{d\mu_\sigma}{ds}(t-s)\ \rho(s)\ \cos\Theta(t; s) \\
&\qquad - \int_{s\in(-L,M)} \mu_\sigma(t-s)\ \rho(s)\ \sin\Theta(t; s)\ \frac{\partial\Theta}{\partial t}(t; s).
\end{aligned}$$

The factor

$$\rho(s)\ \sin\Theta(t; s)\ D_1\psi(s) = \rho(s)\ \sin\Theta(t; s)\ \frac{\partial\Theta}{\partial t}(t; s)$$

in the last integrand has

$$\left|\frac{\partial}{\partial t}\left(\rho(s)\ \sin\Theta(t;s)\ D_1\psi(s)\right)\right| = \left|\rho(s)\ \cos\Theta(t;s)\left[D_1\psi(s)\right]^2\right|$$
$$\leq |D_1\psi(s)|.$$

Since $D_1\dot\gamma_j \in L^1_{loc}(-L,M)$ for $j=1,2$ and $|D_1\psi(s)| = |D_1\dot\gamma| \leq \sqrt{2}(|D_1\dot\gamma_1| + |D_1\dot\gamma_2|)$, we have for fixed t that $h_2 \in L^1_{loc}(-L,M)$ where

$$h_2(s) = -\rho(s)\ \cos\Theta(t;s)\left[D_1\psi(s)\right]^2,$$

and

$$\begin{aligned}\frac{d^2T_1}{dt^2}(t) = &\int_{s\in(-L,M)} \frac{d^2\mu_\sigma}{ds^2}(t-s)\ \rho(s)\ \cos\Theta(t;s)\\ &- 2\int_{s\in(-L,M)} \frac{d\mu_\sigma}{ds}(t-s)\ \rho(s)\ \sin\Theta(t;s)\ \frac{\partial\Theta}{\partial t}(t;s)\\ &- \int_{s\in(-L,M)} \mu_\sigma(t-s)\ \rho(s)\ \cos\Theta(t;s)\left[\frac{\partial\Theta}{\partial t}(t;s)\right]^2.\end{aligned}$$

The higher order derivatives follow the same pattern except that for each $j = 3,4,5,\ldots$ we need an estimate on

$$\begin{aligned}&\left|\frac{\partial}{\partial t}\left(\rho(s)\ \left(\frac{1+(-1)^j}{2}\sin\Theta(t;s) + \frac{1-(-1)^j}{2}\cos\Theta(t;s)\right)\left[D_1\psi(s)\right]^{j-1}\right)\right|\\ &\qquad= \left|\rho(s)\ \left(\frac{1+(-1)^j}{2}\cos\Theta(t;s) - \frac{1-(-1)^j}{2}\sin\Theta(t;s)\right)\left[D_1\psi(s)\right]^j\right|\\ &\qquad\leq |D_1\psi(s)|^{j-1}\\ &\qquad= |D_1\dot\gamma(s)|^{j-1}.\end{aligned} \tag{41}$$

The appearance of the power in (41) is the reason for assumption **(MCB)**. With the value in (41) bounded by a constant, we conclude that for fixed t we have $h_j \in L^\infty_{loc}(-L,M) \subset L^1_{loc}(-L,M)$ where

$$h_j(s) = \rho(s)\ \left(\frac{1+(-1)^j}{2}\cos\Theta(t;s) - \frac{1-(-1)^j}{2}\sin\Theta(t;s)\right)\left(\frac{\partial\Theta}{\partial t}(t;s)\right)^j,$$

and by induction

$$\frac{d^k T_1}{dt^k}(t) = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) \int_{s\in(-L,M)} \frac{d^{k-j}\mu_\sigma}{ds^{k-j}}(t-s)\rho(s) \; \frac{d^j}{d\Theta^j}[\cos\Theta](t;s) \; \left[\frac{\partial\Theta}{\partial t}(t;s)\right]^j. \quad (42)$$

Similarly,

$$\frac{d^k T_2}{dt^k}(t) = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) \int_{s\in(-L,M)} \frac{d^{k-j}\mu_\sigma}{ds^{k-j}}(t-s)\rho(s) \; \frac{d^j}{d\Theta^j}[\sin\Theta](t;s) \; \left[\frac{\partial\Theta}{\partial t}(t;s)\right]^j. \quad (43)$$

Since $\nu_j(t) = \mu_\sigma * \gamma_j(t) + S_j(t) + T_j(t)$, this completes the proof that ν is infinitely differentiable. $\Box$

In retrospect it is not entirely surprising the curvature mollification

$$\nu \in C^2((-L, M) \to \mathbb{R}^2)$$

may not have additional regularity in situations where there are points of arbitrarily large curvature on Γ, though this is not a problem with either the pointwise mollification of γ or the tangent mollification for that matter. In any case, $\gamma \in C^2((-L, M) \to \mathbb{R}^2)$ is definitely adequate to ensure the curvature mollification satisfies $\nu \in C^\infty((-L, M) \to \mathbb{R}^2)$.

Perhaps it is somewhat more surprising, or troubling, that the condition **(RC)** on the radius of curvature is required. One might suspect points of zero curvature (infinite radius of curvature) could or should be included with a pointwise transition to something like the averaged values of the tangent mollification. The current construction, however, definitely uses integrals involving the radius of curvature and center of curvature in a fundamental way, and it is not clear how to construct such a natural generalization for curves with vanishing curvature. Thus, curvature mollification as presented so far applies essentially to curves with curvature bounded away from zero, which is a standard assumption in the structure theory for curves in differential geometry. In the last section we do offer an approach to obtaining such a generalization but without proofs of regularity or convergence properties.

Before we attempt to address questions of convergence for the curvature mollification, we consider a simple explicit example giving one of the key properties of the construction.

2.1 Curvature mollification of a circular arc

Consider the circular arc $\Gamma = \{a(\cos(s/a), \sin(s/a)) : s \in \mathbb{R}\}$ parameterized by arclength as above. Here $\rho(s) \equiv a$ and $D_1\dot{\gamma}(s) = \ddot{\gamma}(s) = -(\cos(s/a), \sin(s/a))/a$. We may take $\psi(s) = s/a + \pi/2$ so that

$$\Theta(t; s) = \frac{s}{a} + \frac{\pi}{2} + \frac{1}{a}\ \left[\cos^2\left(\frac{s}{a}\right) + \sin^2\left(\frac{s}{a}\right)\right]\left(t - s - \frac{a\pi}{2}\right) = \frac{t}{a}.$$

Consequently,

$$\begin{aligned}\beta(t-s) &= \gamma(s) + \rho(s)^2\ddot{\gamma}(s) + \rho(s)(\cos\Theta(t;s), \sin\Theta(t;s)) \\ &= a(\cos(s/a), \sin(s/a)) - a(\cos(s/a), \sin(s/a)) + a(\cos(t/a), \sin(t/a)) \\ &\equiv \gamma(t). \end{aligned} \tag{44}$$

Thus, we have for the average $\nu : \mathbb{R} \to \mathbb{R}^2$

$$\nu(t) = \int_{s\in\mathbb{R}} \mu_\sigma(t-s)\beta(t-s) = \int_{s\in\mathbb{R}} \mu_\sigma(t-s)\gamma(t) \equiv \gamma(t), \tag{45}$$

and the circle is invariant under curvature mollification.

Conceptually the result of the computations (44) and (45) is essentialy obvious if we have obtained the correct formula for the curvature mollification. At each point $\gamma(t)$ one averages values obtained by moving a particular arclength s along the curve, finding the tangent circle at the point $\gamma(t+s)$ of the appropriate curvature, parameterizing this circle by arclength, and then following that circle in the reverse direction an arclength s. If the original curve is a circle, this procedure leads one precisely back to the point $\gamma(t)$, and the average of such values is of course also $\gamma(t)$. When the curve is not a circle, however, the construction leads to something rather more interesting.

2.2 Convergence

We now examine convergence for curvature mollification. Since the formula (36) has the form $\nu = \mu_\sigma * \gamma + \mathbf{S} + \mathbf{T}$ and the pointwise mollification $\mu_\sigma * \gamma$ converges to γ in various senses, the proofs of convergence, like those for tangent mollification, primarily consist of showing the additional quantity $\mathbf{S} + \mathbf{T} = (S_1, S_2) + (T_1, T_2)$ tends to zero.

Here we will use more explicitly the geometric significance of the weak curvature relation $D_1\dot{\gamma} = D_1\psi(-\sin\psi, \cos\psi)$ associated with the form (35). Accordingly we write $S_1(t) + T_1(t)$ as

$$(S_1 + T_1)(t) = \int_{s\in(-L,M)} \mu_\sigma(t-s)\ \rho(s) \left[\rho(s)\ D_1\dot{\gamma}_1(s) + \cos\Theta(t;s)\right]$$
$$= \int_{s\in(-L,M)} \mu_\sigma(t-s)\ \frac{1}{D_1\psi(s)} \left[\sin[\psi(s) + (t-s)D_1\psi(s)] - \sin\psi(s)\right].$$

In this way we obtain a pointwise estimate

$$|(S_1 + T_1)(t)| \leq \int_{s\in(-L,M)} |t-s|\mu_\sigma(t-s) < \sigma. \tag{46}$$

Similarly, we find

$$|(S_2 + T_2)(t)| < \sigma \qquad \text{for} \qquad -L + \sigma < t < M - \sigma. \tag{47}$$

We conclude the following:

Theorem 8 (pointwise convergence of the curvature mollification) If

$$\gamma \in C^1((-L, M) \to \mathbb{R}^n)$$

satisfies the conditions of Theorem 7 then $\nu \in C^\infty((-L+\sigma, M-\sigma) \to \mathbb{R}^n)$ given by (36) converges pointwise uniformly to γ on every compact subset of $(-L, M)$ and pointwise at every point in particular as σ tends to zero.

Since we have assumed no additional regularity in Theorem 8, that is in addition to the regularity required to construct the curvature mollification, there is no result concerning almost everywhere pointwise convergence for curvature mollification analogous to Theorem 3 for tangent mollification.

Notice finally the comparison of the main zero order pointwise estimates (46) and (46) for the curvature mollification with the corresponding estimates (22) for the tangent mollification. Though the curvature mollification has nominally a much more complicated structure and formula, these pointwise estimates for the curvature mollification are surprisingly simple.

2.3 Higher order derivatives for curvature mollification

In situations when γ enjoys higher regularity, the derivatives of ν can also be written in a "mollification form" analogous to $\mu_\sigma * \gamma$ and the expression in (20) for the tangent mollification excluding derivatives of the mollifier μ_σ. This has already been accomplished for the component functions S_1 and S_2 in (40). We proceed to find a similar form for T_1 and T_2.

If $\gamma \in C^{k+2}((-L, M) \to \mathbb{R}^2)$ for some $k \in \{1, 2, 3, \ldots\}$, then we write $\dot{\gamma}$, $\ddot{\gamma}$, ..., $\gamma^{(k+2)}$ for the classical derivatives and note that the reasoning leading to (32) applies to show the inclination angle ψ has classical derivatives $\dot{\psi}$, ..., $\psi^{(k+1)}$. Furthermore, the condition **(MCB)** that the modulus of curvature $\kappa = |D_1\dot{\gamma}| = |\ddot{\gamma}|$ is locally bounded yields for each set $K \subset\subset (-L, M)$ an inequality

$$|\dot{\psi}(s)| = |\ddot{\gamma}(s)| \leq \|\ddot{\gamma}\|_{C^0(K)} < \infty$$

holding for every $s \in K$. We strengthen the assumption **(RC)** that the radius of curvature is locally integrable to the following:

(RCB) The curve $\Gamma = \{\gamma(s) : -L < s < M\}$ has a **locally bounded radius of curvature**, that is, the function $\rho : (-L, M) \to \mathbb{R}$ given by

$$\rho(s) = \frac{1}{|\ddot{\gamma}(s)|} = \frac{1}{|\dot{\psi}(s)|}$$

satisfies $\rho \in L^\infty_{loc}(-L, M)$.

Again, this assumption implies for each compact set $K \subset\subset (-L, M)$ a pointwise inequality

$$0 < \frac{1}{\|\ 1/|\ddot{\gamma}|\ \|_{C^0(K)}} \leq \kappa(s) = |\dot{\psi}(s)| = |\ddot{\gamma}(s)| \qquad \text{for} \qquad s \in K.$$

With these comments in mind and the central role played by the classical derivatives of the inclination angle in particular, we may start with the form (35) for the curvature mollification ν and write it as

$$\begin{aligned}\nu(t) = \mu_\sigma * \gamma(t) \\ + \int_{s\in(-L,M)} \mu_\sigma(t-s)\frac{1}{\dot{\psi}(s)} \bigg[\bigg(& \sin[\psi(s) + (t-s)\dot{\psi}(s)] - \sin\psi(s), \\ & \cos\psi(s) - \cos[\psi(s) + (t-s)\dot{\psi}(s)]\bigg)\bigg]\end{aligned}$$

with

$$S_1(t) = -\int_{s\in(-L,M)} \mu_\sigma(t-s)\ \frac{1}{\dot\psi(s)} \sin\psi(s)$$
$$S_2(t) = \int_{s\in(-L,M)} \mu_\sigma(t-s)\ \frac{1}{\dot\psi(s)} \cos\psi(s)$$
$$T_1(t) = \int_{s\in(-L,M)} \mu_\sigma(t-s)\ \frac{1}{\dot\psi(s)} \sin[\psi(s)+(t-s)\dot\psi(s)], \qquad \text{and}$$
$$T_2(t) = -\int_{s\in(-L,M)} \mu_\sigma(t-s)\ \frac{1}{\dot\psi(s)} \cos[\psi(s)+(t-s)\dot\psi(s)].$$

Here we have taken account of the expressions (33) and (34) as well as the additional regularity of ψ. As noted above, the components S_1 and S_2 take the form of a simple mollification, and we focus on the components T_1 and T_2.

Changing variables in $\mathbf{T}$, we can write

$$\nu(t) = \mu_\sigma * \gamma(t) + \mu_\sigma * (\rho^2\ddot\gamma)(t) + \int_{\eta\in(-L,M)} \mu_\sigma(\eta)\rho(t-\eta)^2\mathbf{w}(t-\eta,\eta)$$

where $\mathbf{w}\in C^k((-L,M)^2\to\mathbb{R}^2)$ with

$$\begin{aligned}\rho(s)^2\mathbf{w}(s,\eta) &= -\frac{1}{\dot\psi(s)}\Big(-\sin[\psi(s)+\eta\ \dot\psi(s)],\ \cos[\psi(s)+\eta\ \dot\psi(s)]\Big)\\ &= -\rho(s)^2\dot\psi(s)\Big(-\sin[\psi(s)+\eta\ \dot\psi(s)],\ \cos[\psi(s)+\eta\ \dot\psi(s)]\Big).\end{aligned} \tag{48}$$

In this form we can differentiate with the derivatives falling on factors multiplied by the mollifier $\mu_\sigma = \mu_\sigma(\eta)$ under the integral rather than on the mollifier to obtain

$$\begin{aligned}\frac{d^k\nu}{dt^k}(t) &= \mu_\sigma * \frac{d^k}{ds^k}(\gamma+\rho^2\ddot\gamma)(t) + \int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)\frac{\partial^k}{\partial s^k}(\rho^2\mathbf{w})(t-\eta,\eta)\\ &= \mu_\sigma * \frac{d^k}{ds^k}(\gamma+\rho^2\ddot\gamma)(t) + \int_{s\in(-L,M)} \mu_\sigma(t-s)\frac{\partial^k}{\partial s^k}(\rho^2\mathbf{w})(s,t-s).\end{aligned} \tag{49}$$

Notice the factors involving

$$\frac{d^k}{ds^k}(\rho^2\ddot\gamma) \qquad \text{and} \qquad \frac{\partial^k}{\partial s^k}(\rho^2\mathbf{w}) \tag{50}$$

in the mollification and the last integrand in (49). These derivatives involve derivatives of the radius of curvature $\rho(s) = 1/|\ddot{\gamma}(s)| = 1/|\dot{\psi}(s)|$, and for the integrals to be well-defined one needs powers of ρ to be locally integrable. This is the reason for the assumption **(RCB)**. Notice further the components of $\mathbf{w} = \mathbf{w}(s, \eta)$ in (48) are products involving compositions of simple trigonometric functions.

We now apply the generalized Leibniz rule and the Faà di Bruno formula for derivatives of compositions to obtain explicit expressions for the derivatives of $\rho^2 \mathbf{w} = \rho^2(w_1, w_2)$ appearing in (49) and (50). Suppressing the dependence on s in $\rho = \rho(s)$ and $\psi^{(k)} = \psi^{(k)}(s)$ for $k = 0, 1, 2, 3, \ldots$, we find

$$\frac{\partial^k}{\partial s^k}(\rho^2 w_1)(s, \eta) = \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{d^{k-\ell}}{ds^{k-\ell}}(\rho^2 \dot{\psi})\ \frac{d^\ell}{ds^\ell} \sin(\psi + \eta\dot{\psi}) \tag{51}$$

$$\frac{\partial^k}{\partial s^k}(\rho^2 w_2)(s, \eta) = -\sum_{\ell=0}^{k} \binom{k}{\ell} \frac{d^{k-\ell}}{ds^{k-\ell}}(\rho^2 \dot{\psi})\ \frac{d^\ell}{ds^\ell} \cos(\psi + \eta\dot{\psi}) \tag{52}$$

with

$$\frac{d^\ell}{ds^\ell} \sin(\psi + \eta\dot{\psi}) = \sum_{m=1}^{\ell} B_{\ell,m}(\zeta)\ \frac{d^m}{d\theta^m} \sin\theta\Big|_{\theta=\psi+\eta\dot{\psi}} \tag{53}$$

$$\frac{d^\ell}{ds^\ell} \cos(\psi + \eta\dot{\psi}) = \sum_{m=1}^{\ell} B_{\ell,m}(\zeta)\ \frac{d^m}{d\theta^m} \cos\theta\Big|_{\theta=\psi+\eta\dot{\psi}} \tag{54}$$

for $\ell \geq 1$ where $B_{\ell,m} : \mathbb{R}^{\ell-m+1} \to \mathbb{R}$ is the (multivariable) Bell polynomial discussed below and

$$\zeta = \left(\dot{\psi} + \eta\ddot{\psi}, \ddot{\psi} + \eta\psi^{(3)}, \ldots, \psi^{(\ell-m+1)} + \eta\psi^{(\ell-m+2)}\right) \in \mathbb{R}^{\ell-m+1}.$$

We apply the same expansion to the derivatives of $\rho^2\ddot{\gamma} = \rho^2\dot{\psi}(-\sin\psi, \cos\psi)$ appearing in (49) and (50):

$$\frac{d^k}{ds^k}(\rho^2\dot{\psi}\sin\psi) = \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{d^{k-\ell}}{ds^{k-\ell}}(\rho^2 \dot{\psi})\ \frac{d^\ell}{ds^\ell} \sin(\psi) \tag{55}$$

$$\frac{d^k}{ds^k}(\rho^2\dot{\psi}\cos\psi) = \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{d^{k-\ell}}{ds^{k-\ell}}(\rho^2 \dot{\psi})\ \frac{d^\ell}{ds^\ell} \cos(\psi) \tag{56}$$

with

$$\frac{d^\ell}{ds^\ell}\sin(\psi) = \sum_{m=1}^{\ell} B_{\ell,m}(\zeta_0)\ \frac{d^m}{d\theta^m}\sin\theta\Big|_{\theta=\psi} \tag{57}$$

$$\frac{d^\ell}{ds^\ell}\cos(\psi) = \sum_{m=1}^{\ell} B_{\ell,m}(\zeta_0)\ \frac{d^m}{d\theta^m}\cos\theta\Big|_{\theta=\psi} \tag{58}$$

and

$$\zeta_0 = \left(\dot\psi, \ddot\psi, \ldots, \psi^{(\ell-m+1)}\right) \in \mathbb{R}^{\ell-m+1}.$$

To complete these expansion formulas, for each $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ and each $\ell, m \in \mathbb{N} = \{1, 2, 3, \ldots\}$ with $m \le \ell \le k$ consider the multiindex set

$$\Gamma_{\ell,m} = \left\{\beta \in \mathbb{N}_0^{\ell-m+1} : |\beta| = \sum_{j-1}^{\ell-m+1} \beta_j = m \quad \text{and} \quad \sum_{j-1}^{\ell-m+1} j\beta_j = \ell\right\}.$$

The Bell polynomials referenced above are then given by

$$B_{\ell,m}(\mathbf{z}) = \ell! \sum_{\beta\in\Gamma_{\ell,m}} \frac{1}{\beta!\ (m!\mathbf{1})^\beta}\ \mathbf{z}^\beta$$

where

$$\left\{\begin{array}{l} \beta! = \beta_1!\beta_2!\cdots\beta_{\ell-m+1}! \\ \mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^{\ell-m+1}, \qquad \text{and} \\ \mathbf{z}^\beta = z_1^{\beta_1} z_2^{\beta_2} \cdots z_{\ell-,+1}^{\beta_{\ell-m+1}} \end{array}\right.$$

for a multiindex $\beta \in \mathbb{N}_0^{\ell-m+1}$ and $\mathbf{z} = (z_1, z_2, \ldots, z_{\ell-m+1}) \in \mathbb{R}^{\ell-m+1}$.

As in the proof of Theorem 8 it is crucial to consider the cancellation occurring in the sum $\mathbf{S} + \mathbf{T}$ rather than in either $\mathbf{S}$ or $\mathbf{T}$ separately; see for example (46). For the higher order derivatives the situation is of course rather more complicated. We start with (49) from which we have

$$\begin{aligned} \left|\frac{d^k}{dt^k}(\mathbf{S}+\mathbf{T})(t)\right| &= \left|\int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta) \left[\frac{d^k}{ds^k}(\rho^2\ddot\gamma)(t-\eta) - \frac{\partial^k}{\partial s^k}(\rho^2\mathbf{w})(t-\eta,\eta)\right]\right| \\ &\le \int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta) \left|\frac{d^k}{ds^k}(\rho^2\ddot\gamma)(t-\eta) - \frac{\partial^k}{\partial s^k}(\rho^2\mathbf{w})(t-\eta,\eta)\right| \end{aligned} \tag{59}$$

Using (51) and (55) we estimate (at least the second component in) the second factor in the integrand in (59) where the suppressed arguments are $s = t - \eta$:

$$
\begin{aligned}
&\left| \frac{d^k(-\rho^2\dot{\psi}\sin\psi)}{ds^k}(t-\eta) - \frac{\partial^k w_1}{\partial s^k}(t-\eta,\eta) \right| \\
&\qquad = \left| \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{d^{k-\ell}}{ds^{k-\ell}}(\rho^2\dot{\psi}) \left(\frac{d^\ell}{ds^\ell}\sin(\psi+\eta\dot{\psi}) - \frac{d^\ell}{ds^\ell}\sin(\psi) \right) \right| \\
&\qquad \leq \sum_{\ell=0}^{k} \binom{k}{\ell} \left| \frac{d^{k-\ell}}{ds^{k-\ell}}(\rho^2\dot{\psi}) \right| \, \left| \frac{d^\ell}{ds^\ell}\sin(\psi+\eta\dot{\psi}) - \frac{d^\ell}{ds^\ell}\sin(\psi) \right|. \qquad (60)
\end{aligned}
$$

Under the current assumptions **(RCB)** and **(MCB)** the nonnegative functions $f_{k\ell} \in C^0(-L, M)$ with values

$$
f_{k\ell}(s) = \left| \frac{d^{k-\ell}}{ds^{k-\ell}}(\rho^2\dot{\psi}) \right| = \left| \frac{d^{k-\ell}}{ds^{k-\ell}} \left(\frac{1}{\dot{\psi}} \right) \right|
$$

are locally bounded, and we can also use (53) and (57) for $1 \leq \ell \leq k$ to obtain

$$
\begin{aligned}
&\left| \frac{d^\ell}{ds^\ell}\sin(\psi+\eta\dot{\psi}) - \frac{d^\ell}{ds^\ell}\sin(\psi) \right| \\
&\qquad = \left| \sum_{m=1}^{\ell} \left[B_{\ell,m}(\zeta) \, \frac{d^m}{d\theta^m}\sin\theta\Big|_{\theta=\psi+\eta\dot{\psi}} - B_{\ell,m}(\zeta_0) \, \frac{d^m}{d\theta^m}\sin\theta\Big|_{\theta=\psi} \right] \right| \\
&\qquad \leq \sum_{m=1}^{\ell} |B_{\ell,m}(\zeta)| \, \left| \frac{d^m}{d\theta^m}\sin\theta\Big|_{\theta=\psi+\eta\dot{\psi}} - \frac{d^m}{d\theta^m}\sin\theta\Big|_{\theta=\psi} \right| \\
&\qquad\qquad + \sum_{m=1}^{\ell} |B_{\ell,m}(\zeta) - B_{\ell,m}(\zeta_0)| \, \left| \frac{d^m}{d\theta^m}\sin\theta\Big|_{\theta=\psi} \right| \\
&\qquad = \sum_{m=1}^{\ell} |B_{\ell,m}(\zeta)| \, \left| \frac{d^m}{d\theta^m}\sin\theta\Big|_{\theta=\psi_*} \right| \, |\eta| \, |\dot{\psi}| \\
&\qquad\qquad + \sum_{m=1}^{\ell} |B_{\ell,m}(\zeta) - B_{\ell,m}(\zeta_0)| \, \left| \frac{d^m}{d\theta^m}\sin\theta\Big|_{\theta=\psi} \right| \qquad (61)
\end{aligned}
$$

where ψ_* is some value between $\psi(s)$ and $\psi(s)+\eta\dot{\psi}(s)$. Continuing with the terms in the first summation of the expression (61) we have for $-\sigma < \eta < \sigma$ as in the integrand of (59)

$$|B_{\ell,m}(\zeta)| \ \left| \frac{d^m}{d\theta^m} \sin\theta \Big|_{\theta=\psi_*} \right| \ |\eta| \ |\dot\psi| \le \sigma \ \left| \dot\psi \ B_{\ell,m}(\zeta) \right|,$$

and again the nonnegative functions $g_{\ell,m} \in C^0((-L, M) \times \mathbb{R})$ with values

$$g_{\ell,m}(s,\eta) = \left| \dot\psi(s) \ B_{\ell,m}(\zeta) \right|$$

where

$$\zeta = \zeta(s,\eta) = \left(\dot\psi(s) + \eta\ddot\psi(s), \ddot\psi(s) + \eta\psi^{(3)}(s), \ldots, \psi^{(\ell-m+1)}(s) + \eta\psi^{(\ell-m+2)}(s) \right)$$

are locally bounded.

The terms in the second summation of the expression (61) are evidently bounded by

$$\begin{aligned} |B_{\ell,m}(\zeta) - B_{\ell,m}(\zeta_0)| &= \left| \sum_{j=1}^{\ell-m+1} \left(\int_0^1 \frac{\partial B_{\ell,m}}{\partial z_j}(\zeta_*)\, dt \right) \ \eta \ \psi^{(j+1)} \right| \\ &\le \sigma \left| \sum_{j=1}^{\ell-m+1} \psi^{(j+1)} \left(\int_0^1 \frac{\partial B_{\ell,m}}{\partial z_j}(\zeta_*)\, dt \right) \right| \end{aligned}$$

where

$$\begin{aligned} \zeta_* &= \zeta_0 + t(\zeta - \zeta_0) \\ &= \left(\dot\psi + t\eta\ddot\psi, \ddot\psi + t\eta\psi^{(3)}, \ldots, \psi^{(\ell-m+1)} + t\eta\psi^{(\ell-m+2)} \right). \end{aligned}$$

Here we have used a multivariable mean value theorem.

Assuming without loss of generality that $|\eta| \le \sigma < 1$, we observe that given a compact set $K \subset (-L, M)$ all possible values of

$$\zeta_* = \zeta_*(s,\eta) = \left(\dot\psi(s) + t\eta\ddot\psi(s), \ddot\psi(s) + t\eta\psi^{(3)}(s), \ldots, \psi^{(\ell-m+1)}(s) + t\eta\psi^{(\ell-m+2)}(s) \right)$$

for all $s \in K$, all $|\eta| \le \sigma$, and all $0 \le t \le 1$ fall into some compact subset $K_{\ell,m}$ of $\mathbb{R}^{\ell-m+1}$. Consequently, given $\delta > 0$ and $t \in (-L+\delta, M-\delta)$, there is a fixed constant $C_\delta > 0$ independent of σ (and of ℓ and of m) such that

$$|B_{\ell,m}(\zeta(t-\eta,\eta) - B_{\ell,m}(\zeta_0(t-\eta))| \le C_\delta \ \sigma \tag{62}$$

for $\sigma < \delta/2$. Taking $K = (-L+\delta/2, M-\delta/2)$ and returning to (61) we can estimate

$$\left|\frac{d^\ell}{ds^\ell}\sin(\psi+\eta\dot{\psi}) - \frac{d^\ell}{ds^\ell}\sin(\psi)\right| \le \sigma\left(\max_{\ell,m}\|g_{l,m}\|_{C^0(K\times[-\delta/2,\delta/2])} + \ell C_\delta\right). \tag{63}$$

We are now in a position to state and prove a result analogous to Theorem 4 for the curvature mollification:

Theorem 9 (convergence of higher order derivatives; curvature mollification) If $\gamma \in C^{k+2}((-L,M)\to\mathbb{R}^2)$ for some $k\ge 1$ satisfies **(RCB)** and **(MCB)**, then the curvature mollification ν converges to γ as σ tends to zero in $C^k([-L+\delta, M-\delta]\to\mathbb{R}^2)$ for every $\delta > 0$.

Proof: Recalling the estimates associated with (59) and (60) we start with $\delta > 0$ fixed and take $t \in (-L+\delta, M-\delta)$ and $0<\sigma<\delta/2$. Then if $|\eta|<\sigma$ we know $t-\eta \in K = (-L+\delta/2, M-\delta/2)$ and

$$\begin{aligned}
\left|\frac{d^k}{dt^k}(S_1+T_1)(t)\right| &= \left|\int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)\left[\frac{d^k(-\rho^2\dot{\psi}\sin\psi)}{ds^k}(t-\eta) - \frac{\partial^k}{\partial s^k}(\rho^2 w_1)(t-\eta,\eta)\right]\right| \\
&\le \int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)\left|\frac{d^k(-\rho^2\dot{\psi}\sin\psi)}{ds^k}(t-\eta) - \frac{\partial^k}{\partial s^k}(\rho^2 w_1)(t-\eta,\eta)\right| \\
&\le \int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)\sum_{\ell=0}^{k}\binom{k}{\ell}\left|\frac{d^{k-\ell}}{ds^{k-\ell}}(\rho^2\dot{\psi})\right| \\
&\qquad\qquad \left|\frac{d^\ell}{ds^\ell}\sin(\psi+\eta\dot{\psi}) - \frac{d^\ell}{ds^\ell}\sin(\psi)\right| \qquad (64) \\
&\le k(k!)\max_{k,\ell}\left\|\frac{d^{k-\ell}}{ds^{k-\ell}}(\rho^2\dot{\psi})\right\|_{C^0(K)} \\
&\qquad\qquad \int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)\left|\frac{d^\ell}{ds^\ell}\sin(\psi+\eta\dot{\psi}) - \frac{d^\ell}{ds^\ell}\sin(\psi)\right| \\
&\le k(k!)\max_{k,\ell}\left\|\frac{d^{k-\ell}}{ds^{k-\ell}}(\rho^2\dot{\psi})\right\|_{C^0(K)}\left(\max_{\ell,m}\|g_{l,m}\|_{C^0(K\times[-\delta/2,\delta/2])} + kC_\delta\right)\sigma.
\end{aligned}$$

The estimate (64) follows from (60). The last estimate uses (59). Since the constant

$$k(k!)\max_{k,\ell}\left\|\frac{d^{k-\ell}}{ds^{k-\ell}}(\rho^2\dot{\psi})\right\|_{C^0(K)}\left(\max_{\ell,m}\|g_{l,m}\|_{C^0(K\times[-\delta/2,\delta/2])} + kC_\delta\right)$$

is bounded independent of σ and t we conclude

$$\left|\frac{d^k}{dt^k}(S_1 + T_1)(t)\right|$$

tends to zero uniformly for $t \in (-L+\delta, M-\delta)$ as σ tends to zero. Similar estimates apply to the second component value

$$\left|\frac{d^k}{dt^k}(S_2 + T_2)(t)\right|$$

so that, at length,

$$\lim_{\sigma \searrow 0} \|\mathbf{S} + \mathbf{T}\|_{C^k([-L+\delta, L-\delta] \to \mathbb{R}^2)} = 0. \qquad \square$$

2.4 L^p convergence

Starting again with the surprisingly simple pointwise estimates (46) and (47) the proof of results analogous to Theorem 5 and Theorem 6 is also correspondingly simple.

Theorem 10 (local L^p convergence of the curvature mollification) If $\gamma \in C^1((-L, M) \to \mathbb{R}^2)$ admits a weak curvature vector $D_1\dot{\gamma} : (-L, M) \to \mathbb{R}^2$ satisfying the conditions of Theorem 7 and

$$D_1\dot{\gamma}_j \in L^p_{loc}(-L, M)$$

for $j = 1, 2, \dots, n$ and some p with $1 \le p < \infty$, then $\nu = \nu \in C^\infty((-L+\sigma, M-\sigma) \to \mathbb{R}^2)$ given by (36) satisfies

$$\lim_{\sigma \searrow 0} \sum_{j=1}^{2} \|\nu_j - \gamma_j\|_{L^p(K)} = 0$$

for every compact set K with $K \subset (-L, M)$.

Proof: Fix $\delta > 0$ with $K \subset\subset (-L + 2\delta, M - 2\delta)$ and consider $\sigma < \delta$ and $j \in \{1, 2\}$. Then

$$\begin{aligned}\|\nu_j - \gamma_j\|_{L^p(K)} &= \|\mu_\sigma * \gamma_j - \gamma_j + S_j + T_j\|_{L^p(K)} \\ &\le \|\mu_\sigma * \gamma_j - \gamma_j\|_{L^p(K)} + \|S_j + T_j\|_{L^p(K)}\end{aligned}$$

where S_j and T_j are given in (38-39) and

$$\lim_{\sigma \searrow 0} \|\mu_\sigma * \gamma_j - \gamma_j\|_{L^p(K)} = 0.$$

Thus, it is enough to show

$$\lim_{\sigma \searrow 0} \|S_j + T_j\|_{L^p(K)} = 0. \tag{65}$$

We have the surprisingly simple pointwise estimate

$$|(S_j + T_j)(t)| \le \sigma.$$

Therefore,

$$\|S_j + T_j\|_{L^p(K)} \le \sigma(L + M),$$

and

$$\lim_{\sigma \searrow 0} \|S_j + T_j\|_{L^p(K)} = 0. \qquad \square$$

2.5 Examples of curvature mollification

Example 1: Starting with γ having values given in (4) the inclination angle $\psi \in C^0(-\pi a/2, \pi a/2)$ may be assumed to have values

$$\psi(s) = \begin{cases} \pi/2 - s/a, & -\pi a/2 < s \le 0 \\ \pi/2 + s/a, & 0 \le s < \pi a/2. \end{cases} \tag{66}$$

Formula (49) for the higher order derivatives does not apply in this case because Γ is not a C^2 curve. It is still useful, especially for computational purposes, to have a formula for the higher order derivatives of ν without derivatives on the mollifier in the integration. Accordingly, we consider the general case of a curve satisfying the conditions of Theorem 7 and the additional condition that for some arclength $s_0 \in (-L, M)$ the parameterization γ satisfies

$$\gamma\Big|_{(-L,s_0]} \in C^\infty(-L, s_0] \qquad \text{and} \qquad \gamma\Big|_{[s_0,M)} \in C^\infty[s_0, M).$$

Since we still have $\gamma \in C^1(-L, M)$, the first derivative of the basic pointwise mollification satisfies

$$\frac{d}{dt}(\mu_\sigma * \gamma)(t) = \mu_\sigma * \dot{\gamma}(t)$$

as usual, though higher order derivatives will require more care. The integrands in the additional terms **S** and **T**, however, already lack enough regularity to give a formula differing from (49) in some cases. Let us assume for simplicity that $t-\sigma < s_0 < t+\sigma$. Then

$$
\begin{aligned}
\frac{d\mathbf{S}}{dt}(t) &= \frac{d}{dt}\left[\mu_\sigma * (\rho^2 D_1\dot\gamma)\right](t) \\
&= \int_{s\in(-L,s_0)} \frac{d\mu_\sigma}{ds}(t-s)(\rho^2\ddot\gamma)(s) + \int_{s\in(s_0,M)} \frac{d\mu_\sigma}{ds}(t-s)(\rho^2\ddot\gamma)(s) \qquad (67)\\
&= \int_{\eta\in(t-s_0,\sigma)} \frac{d\mu_\sigma}{ds}(\eta)(\rho^2\ddot\gamma)(t-\eta) + \int_{\eta\in(-\sigma,t-s_0)} \frac{d\mu_\sigma}{ds}(\eta)(\rho^2\ddot\gamma)(t-\eta) \qquad (68)\\
&= -\mu_\sigma(t-s_0)(\rho^2\ddot\gamma)(s_0^-) + \int_{\eta\in(t-s_0,\sigma)} \mu_\sigma(\eta)\frac{d}{ds}(\rho^2\ddot\gamma)(t-\eta) \\
&\qquad + \mu_\sigma(t-s_0)(\rho^2\ddot\gamma)(s_0^+) + \int_{\eta\in(-\sigma,t-s_0)} \mu_\sigma(\eta)\frac{d}{ds}(\rho^2\ddot\gamma)(t-\eta) \qquad (69)\\
&= \mu_\sigma * \frac{d}{ds}(\rho^2 D_1\dot\gamma)(t) + \mu_\sigma(t-s_0)\left((\rho^2\ddot\gamma)(s_0^+) - (\rho^2\ddot\gamma)(s_0^-)\right). \qquad (70)
\end{aligned}
$$

Instead of integrating by parts in (68) we may change variables directly in the mollification integral $\mu_\sigma * (\rho^2\ddot\gamma)$ and replace (67) and (68) with

$$
\frac{d}{dt}\left(\int_{\eta\in(t-s_0,\sigma)} \mu_\sigma(\eta)(\rho^2\ddot\gamma)(t-\eta) + \int_{\eta\in(-\sigma,t-s_0)} \mu_\sigma(\eta)(\rho^2\ddot\gamma)(t-\eta)\right).
$$

Then the derivatives under the integral fall directly on the factor $\rho^2\ddot\gamma$, but there are also t dependencies in the limits of integration which must be differentiated and these differentiations lead to the additional "jump" terms and the same formula given in (70). The derivative of $\rho^2\ddot\gamma$ should not be understood as any kind of weak derivative on the entire interval $(-L, M)$ but simply as the values of an L^1_{loc} function determined pointwise classically almost everywhere, namely on $(-L, M)\backslash\{s_0\}$. Thus, the function

$$
\frac{d}{ds}(\rho^2 D_1\dot\gamma)
$$

might be considered a "strong derivative" in the sense of the strong solutions of Chapter 9 of [GT83]. In this sense, the final result may be also written as

$$
\frac{d\mathbf{S}}{dt}(t) = \mu_\sigma * \frac{d}{ds}(\rho^2\ddot\gamma)(t) + \mu_\sigma(t-s_0)\left((\rho^2\ddot\gamma)(s_0^+) - (\rho^2\ddot\gamma)(s_0^-)\right)
$$

and compared to the value

$$\frac{d\mathbf{S}}{dt}(t) = \mu_\sigma * \frac{d}{ds}(\rho^2\ddot{\gamma})(t) \tag{71}$$

expressed in (49) and valid in the smooth case. Notice than that when either $s_0 \notin (t-\sigma, t+\sigma)$ or $(\rho^2\ddot{\gamma})(s_0^+) = (\rho^2\ddot{\gamma})(s_0^-)$, the "jump" terms vanish. Thus, (70) is a direct generalization of (71).

We next turn to the differentiation of

$$\mathbf{T}(t) = \int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)(\rho^2\mathbf{w})(t-\eta,\eta).$$

The same approach applied to $\mathbf{S}$ above yields

$$\begin{aligned}
\frac{d\mathbf{T}}{dt}(t) &= \frac{d}{dt}\int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta)(\rho^2\mathbf{w})(t-\eta,\eta) \\
&= \int_{\eta\in(-\sigma,t-s_0)} \mu_\sigma(\eta)\frac{\partial}{\partial s}(\rho^2\mathbf{w})(t-\eta,\eta) + \mu_\sigma(t-s_0)(\rho^2\mathbf{w})(s_0^+,t-s_0) \\
&\qquad\qquad + \int_{\eta\in(t-s_0,\sigma)} \mu_\sigma(\eta)\frac{\partial}{\partial s}(\rho^2\mathbf{w})(t-\eta,\eta) - \mu_\sigma(t-s_0)(\rho^2\mathbf{w})(s_0^-,t-s_0) \\
&= \int_{s\in(-L,M)} \mu_\sigma(t-s)\frac{\partial}{\partial s}(\rho^2\mathbf{w})(s,t-s) \\
&\qquad\qquad + \mu_\sigma(t-s_0)\Big((\rho^2\mathbf{w})(s_0^+,t-s_0) - (\rho^2\mathbf{w})(s_0^-,t-s_0)\Big).
\end{aligned} \tag{72}$$

Again the value

$$\frac{\partial}{\partial s}(\rho^2\mathbf{w})(s,t-s)$$

should be understood as a kind of "strong derivative" defined classically except for $s = s_0$, and (72) generalizes directly the formula

$$\frac{d\mathbf{T}}{dt}(t) = \int_{s\in(-L,M)} \mu_\sigma(t-s)\frac{\partial}{\partial s}(\rho^2\mathbf{w})(s,t-s)$$

valid when $\gamma \in C^3(-L,M)$ and appearing in (49).

A similar approach may be applied to the second derivative of the basic pointwise mollification:

$$\begin{aligned}\frac{d^2}{dt^2}(\mu_\sigma * \gamma)(t) &= \frac{d}{dt}\int_{\eta\in(-\sigma,t-s_0)} \mu_\sigma(\eta)\dot{\gamma}(t-\eta) + \frac{d}{dt}\int_{\eta\in(t-s_0,\sigma)} \mu_\sigma(\eta)\dot{\gamma}(t-\eta) \\ &= \int_{\eta\in(-\sigma,\sigma)} \mu_\sigma(\eta) D_1\dot{\gamma}(t-\eta) + \mu_\sigma(t-s_0)\left(\dot{\gamma}(s_0^+) - \dot{\gamma}(s_0^-)\right) \\ &= \mu_\sigma * D_1\dot{\gamma}(t).\end{aligned}$$

Here we have used the fact that the weak curvature vector $D_1\dot{\gamma}$ agrees with the "strong" curvature vector $\ddot{\gamma}$ at all points except $s = s_0$ where the latter is not defined. We have also used the assumption $\gamma \in C^1((-L, M) \to \mathbb{R}^2)$ according to which the "jump" terms vanish.

Starting with (69) we find

$$\begin{aligned}\frac{d^2\mathbf{S}}{dt^2}(t) = &\frac{d\mu_\sigma}{ds}(t-s_0)\left((\rho^2\ddot{\gamma})(s_0^+) - \rho^2\ddot{\gamma})(s_0^-)\right) \\ &+ \frac{d}{dt}\int_{\eta\in(t-s_0,\sigma)} \mu_\sigma(\eta)\frac{d}{ds}(\rho^2\ddot{\gamma})(t-\eta) \\ &+ \frac{d}{dt}\int_{\eta\in(-\sigma,t-s_0)} \mu_\sigma(\eta)\frac{d}{ds}(\rho^2\ddot{\gamma})(t-\eta) \\ = &\frac{d\mu_\sigma}{ds}(t-s_0)\left((\rho^2\ddot{\gamma})(s_0^+) - \rho^2\ddot{\gamma})(s_0^-)\right) + \mu_\sigma * \frac{d^2}{ds^2}(\rho^2 D_1\dot{\gamma})(t) \\ &+ \mu_\sigma(t-s_0)\left(\frac{d}{ds}(\rho^2\ddot{\gamma})(s_0^+) - \frac{d}{ds}(\rho^2\ddot{\gamma})(s_0^-)\right) \\ = &\mu_\sigma * \frac{d^2}{ds^2}(\rho^2 D_1\dot{\gamma})(t) + \sum_{j=0}^{1}\frac{d^{1-j}\mu_\sigma}{ds^{1-j}}(t-s_0)\left(\frac{d^j}{ds^j}(\rho^2\ddot{\gamma})(s_0^+) - \frac{d^j}{ds^j}(\rho^2\ddot{\gamma})(s_0^-)\right).\end{aligned} \tag{73}$$

Here the generalization of

$$\frac{d^2\mathbf{S}}{dt^2}(t) = \mu_\sigma * \frac{d^2}{ds^2}(\rho^2 D_1\dot{\gamma})(t)$$

which applies when $\gamma \in C^4((-L, M) \to \mathbb{R}^2$ and features in (49) involves a sum of "jump" terms involving lower order derivatives and pointwise values of derivatives of the mollifier.

Similarly,

$$\begin{aligned}
\frac{d^2\mathbf{T}}{dt^2}(t) = &\int_{\eta\in(-\sigma,t-s_0)} \mu_\sigma(\eta)\frac{\partial^2}{\partial s^2}(\rho^2\mathbf{w})(t-\eta,\eta) \\
&\quad + \sum_{j=0}^{1} \frac{d^{1-j}\mu_\sigma}{ds^{1-j}}(t-s_0)\left(\frac{\partial^j}{\partial s^j}(\rho^2\mathbf{w})(s_0^+,t-s_0) - \frac{\partial^j}{\partial s^j}(\rho^2\mathbf{w})(s_0^-,t-s_0)\right) \\
&\qquad + \mu_\sigma(t-s_0)\left(\frac{\partial}{\partial\eta}(\rho^2\mathbf{w})(s_0^+,t-s_0) - \frac{\partial}{\partial\eta}(\rho^2\mathbf{w})(s_0^-,t-s_0)\right) \\
= &\int_{s\in(-L,M)} \mu_\sigma(t-s)\frac{\partial^2}{\partial s^2}(\rho^2\mathbf{w})(s,t-s) \\
&\quad + \sum_{j=0}^{1} \frac{d^{1-j}\mu_\sigma}{ds^{1-j}}(t-s_0)\left(\frac{\partial^j}{\partial s^j}(\rho^2\mathbf{w})(s_0^+,t-s_0) - \frac{\partial^j}{\partial s^j}(\rho^2\mathbf{w})(s_0^-,t-s_0)\right) \\
&\qquad + \mu_\sigma(t-s_0)\left(\frac{\partial}{\partial\eta}(\rho^2\mathbf{w})(s_0^+,t-s_0) - \frac{\partial}{\partial\eta}(\rho^2\mathbf{w})(s_0^-,t-s_0)\right).
\end{aligned}$$

Adding the results obtained so far we have formulas for the velocity and acceleration vectors associated with the curvature mollification applicable to the curve of Example 1:

$$\begin{aligned}
\nu'(t) = \mu_\sigma * &\left(\dot\gamma + \frac{d}{ds}(\rho^2 D_1\dot\gamma)\right)(t) + \int_{s\in(-L,M)} \mu_\sigma(t-s)\frac{\partial}{\partial s}(\rho^2\mathbf{w})(s,t-s) \\
&\quad + \mu_\sigma(t-s_0)\left((\rho^2\ddot\gamma)(s_0^+) - (\rho^2\ddot\gamma)(s_0^-)\right) \\
&\qquad + \mu_\sigma(t-s_0)\left((\rho^2\mathbf{w})(s_0^+,t-s_0) - (\rho^2\mathbf{w})(s_0^-,t-s_0)\right),
\end{aligned}$$

and

$$\nu''(t) = \mu_\sigma * \left(D_1\dot{\gamma} + \frac{d^2}{ds^2}(\rho^2 D_1\dot{\gamma}) \right)(t) + \int_{s\in(-L,M)} \mu_\sigma(t-s)\frac{\partial^2}{\partial s^2}(\rho^2\mathbf{w})(s,t-s)$$
$$+ \sum_{j=0}^{1} \frac{d^{1-j}\mu_\sigma}{ds^{1-j}}(t-s_0)\left(\frac{d^j}{ds^j}(\rho^2\ddot{\gamma})(s_0^+) - \frac{d^j}{ds^j}(\rho^2\ddot{\gamma})(s_0^-) \right)$$
$$+ \sum_{j=0}^{1} \frac{d^{1-j}\mu_\sigma}{ds^{1-j}}(t-s_0)\left(\frac{\partial^j}{\partial s^j}(\rho^2\mathbf{w})(s_0^+,t-s_0) - \frac{\partial^j}{\partial s^j}(\rho^2\mathbf{w})(s_0^-,t-s_0) \right)$$
$$+ \mu_\sigma(t-s_0)\left(\frac{\partial}{\partial\eta}(\rho^2\mathbf{w})(s_0^+,t-s_0) - \frac{\partial}{\partial\eta}(\rho^2\mathbf{w})(s_0^-,t-s_0) \right).$$

From these we can construct, for example, the tangent indacatrix and curvature vector associated with the curvature mollification in Example 1. In particular, these formulas are particularly useful for numerical calculation.

The result of induction giving the full generalization of (49) for $k \geq 3$ is now clear:

$$\frac{d^k\nu}{dt^k}(t) = \mu_\sigma * \frac{d^k}{ds^k}(\gamma + \rho^2 D_1\dot{\gamma})(t) + \int_{s\in(-L,M)} \mu_\sigma(t-s)\frac{\partial^k}{\partial s^k}(\rho^2\mathbf{w})(s,t-s)$$
$$+ \sum_{j=3}^{k} \frac{d^{k-j}\mu_\sigma}{ds^{k-j}}(t-s_0)\left(\frac{d^{j-1}\gamma}{ds^{j-1}}(s_0^+) - \frac{d^{j-1}\gamma}{ds^{j-1}}(s_0^-) \right) \quad (74)$$
$$+ \sum_{j=0}^{k-1} \frac{d^{k-j}\mu_\sigma}{ds^{k-j}}(t-s_0)\left(\frac{d^j}{ds^j}(\rho^2\ddot{\gamma})(s_0^+) - \frac{d^j}{ds^j}(\rho^2\ddot{\gamma})(s_0^-) \right)$$
$$+ \sum_{j=0}^{k} \frac{d^{k-j}\mu_\sigma}{ds^{k-j}}(t-s_0)\left(\frac{\partial^j}{\partial s^j}(\rho^2\mathbf{w})(s_0^+,t-s_0) - \frac{\partial^j}{\partial s^j}(\rho^2\mathbf{w})(s_0^-,t-s_0) \right)$$
$$+ \sum_{j=2}^{k} \frac{d^{k-j}\mu_\sigma}{ds^{k-j}}(t-s_0)\left(\frac{\partial^{j-1}}{\partial\eta^{j-1}}(\rho^2\mathbf{w})(s_0^+,t-s_0) - \frac{\partial^{j-1}}{\partial\eta^{j-1}}(\rho^2\mathbf{w})(s_0^-,t-s_0) \right).$$

The only significant new terms appearing for $k \geq 3$ are in the summation in (74) which is a linear combination of "jumps" in the higher derivatives of γ with coefficients given by pointwise values of complementary derivatives of the mollifier. These accumulate of course due to the possible singularity of γ at $s = s_0$ for derivatives of order greater than one, starting with the weak curvature vector, but do not appear in the velocity and acceleration vectors of the curvature mollification ν, so play no role in the tangent indacatrix $\nu'/|\nu'|$ or curvature vector $\vec{k} = \nu''/|\nu'|^2 - (\nu' \cdot \nu'')\nu/|\nu'|^4$.

The values of ν, ν', ν'', and $\vec{k}$ associated with Example 1 may also be calculated directly in a much simpler form as follows:

$$\begin{aligned}\nu(t) &= a\int_{t-\sigma}^{0} \mu_\sigma(t-s)\left(1-\cos\left(\frac{t}{a}\right), \sin\left(\frac{t}{a}\right)\right) ds \\ &\qquad + a\int_{0}^{t+\sigma} \mu_\sigma(t-s)\left(-1+\cos\left(\frac{t}{a}\right), \sin\left(\frac{t}{a}\right)\right) ds \\ &= a\left(\left[\int_{t-\sigma}^{0} \mu_\sigma(t-s)\,ds - \int_{0}^{t+\sigma} \mu_\sigma(t-s)\,ds\right]\left(1-\cos\left(\frac{t}{a}\right)\right), \sin\left(\frac{t}{a}\right)\right). \\ &= a\left(\left[\int_{t}^{\sigma} \mu_\sigma(\eta)\,d\eta - \int_{-\sigma}^{t} \mu_\sigma(\eta)\,d\eta\right]\left(1-\cos\left(\frac{t}{a}\right)\right), \sin\left(\frac{t}{a}\right)\right).\end{aligned}$$

$$\begin{aligned}\nu'(t) = \Bigg(&\left[\int_{t}^{\sigma} \mu_\sigma(\eta)\,d\eta - \int_{-\sigma}^{t} \mu_\sigma(\eta)\,d\eta\right]\sin\left(\frac{t}{a}\right) - 2a\mu_\sigma(t)\left(1-\cos\left(\frac{t}{a}\right)\right), \\ &\cos\left(\frac{t}{a}\right)\Bigg).\end{aligned}$$

$$\begin{aligned}\nu''(t) = \frac{1}{a}\Bigg(&\left[\int_{t}^{\sigma} \mu_\sigma(\eta)\,d\eta - \int_{-\sigma}^{t} \mu_\sigma(\eta)\,d\eta\right]\cos\left(\frac{t}{a}\right) - 4a\mu_\sigma(t)\sin\left(\frac{t}{a}\right) \\ &- 2a\mu'_\sigma(t)\left(1-\cos\left(\frac{t}{a}\right)\right), -\sin\left(\frac{t}{a}\right)\Bigg).\end{aligned}$$

These values are relatively easy to approximate computationally as illustrated in Figure 11 and Figure 12 for the case $a = 1$. The only delicate point is the apparent singularity in the mollifier μ_σ near the boundary of the support at $s = \pm\sigma$.

Continuing with the assumption $a = 1$, one sees immediately from the formulas that $\nu(0) = \gamma(0)$ and $\nu'(0) = \gamma'(0) = (0, 1)$. It may be observed also that $\nu = (\nu_1, \nu_2)$ for $-\sigma \le s, t \le 0$ has first coordinate

$$\begin{aligned}\nu_1(t) &= \left[\int_{t-\sigma}^{0} \mu_\sigma(t-s)\,ds - \int_{0}^{t+\sigma} \mu_\sigma(t-s)\,ds\right](1-\cos t) \\ &= \left[\int_{t}^{\sigma} \mu_\sigma(\eta)\,d\eta - \int_{-\sigma}^{t} \mu_\sigma(\eta)\,d\eta\right](1-\cos t)\end{aligned}$$

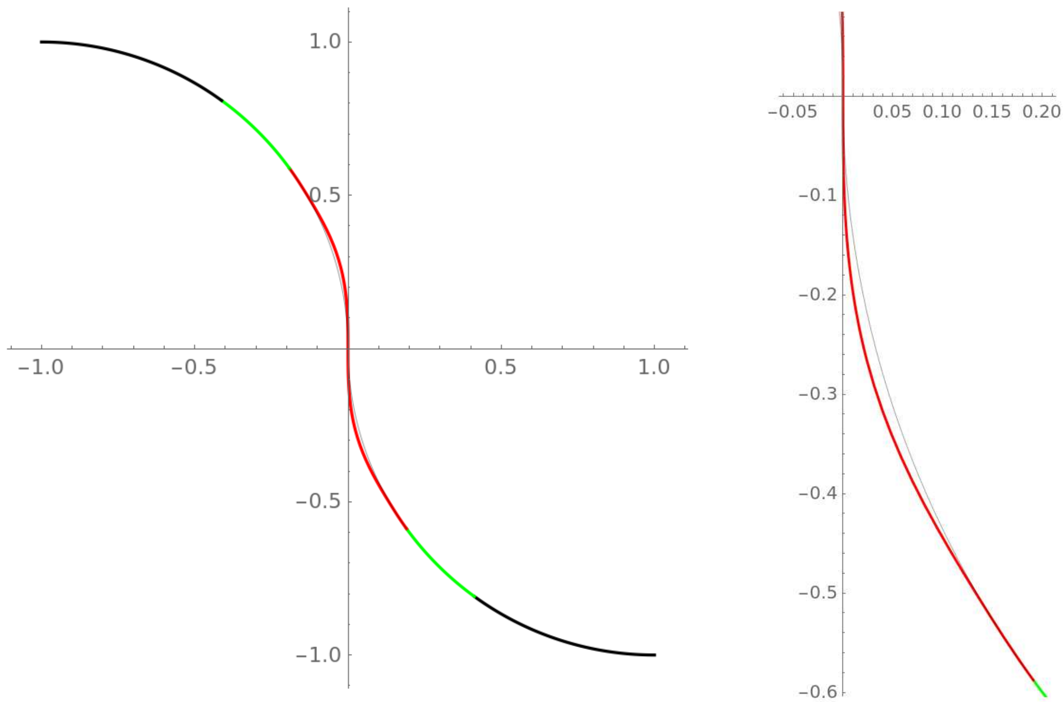

Figure 11: The curvature mollification of two concatenated circles of the same radius as given in Example 1. Here the radius of the circular arcs is $a = 1$ and the mollification parameter is $\sigma = \pi/5$. The green arcs are portions of the mollified curve agreeing identically, both geometrically and in parameterization, with the circular arcs of Example 1 for $-\pi/2 + \sigma \leq s = t \leq -\sigma$ and $\sigma < s = t \leq \pi/2 - \sigma$. The thin black circular arcs of Example 1 are shown in comparison to the red arc of the curvature mollification corresponding to $-\sigma < s, t < \sigma$ though t is not an arclength parameter for the curvature mollification on this interval.

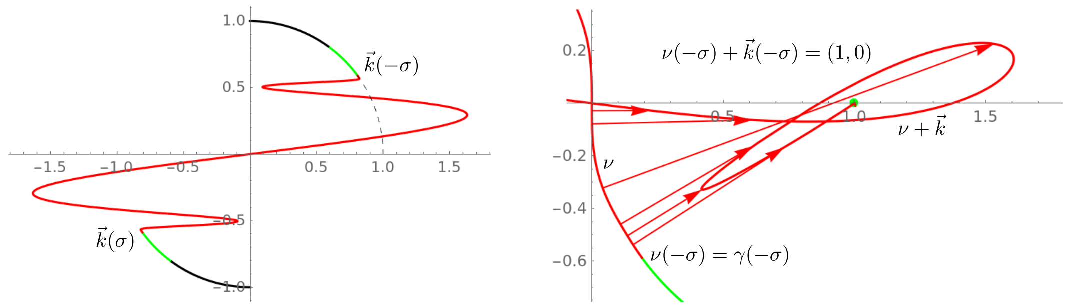

Figure 12: Example 1: Curvature associated with curvature mollification. On the left is shown the entire plot of the curvature vector $\vec{k}$ associated with the curvature mollification ν of Example 1. On the right a portion of the curvature mollification is shown featuring the image of the parameter interval $-\sigma < t < 0$ along with the evolute given by $\nu + \vec{k}$. Certain curvature vectors $\vec{k}(t)$ are also indicated in the tangent space of $\mathbb{R}^2$ at $\nu(t)$. Notice the evolute of the circular arc $\{\nu(t) : -\pi/2 + \sigma < t < -\sigma\}$ indicated in green is the single center of curvature $\{(1, 0)\}$ where the nonsingular evolute of ν emerges.

satisfying $0 < \nu_1 < 1 - \cos t$ and second component $\nu_2 \equiv \sin t$. From this one can see the point ν lies outside the circle $\{(x, y) : (x-1)^2 + y^2 = 1\}$ as well as in the fourth quadrant. Consequently, the polar radius of the curvature mollification with respect to the center $(1, 0)$ of the initial circular arc of Example 1 for $-\sigma \le s, t \le 0$, gibven by

$$q(t) = |\nu(t) - (1, 0)|$$

satisfies $q(-\sigma) = q(0) = 1$ and $q(t) > 1$ for $-\sigma < t < 0$. In particular, there are points $\nu(t)$ for $t < 0$ close to both $q(0)$ and $q(-\sigma)$ with numerical curvature $|\vec{k}| < 1$ and points $\nu(t)$ for some $t < 0$ with $|\vec{k}| > 1$. This observation is verifiable both analytically and numerically; an illustration is given in Figure 12.

It is usual to assume a planar curve is given by a "regular" parameterization in differential geometry in several senses by requiring

1. The coordinate functions are differentiable (at least two or three times).

2. The tangent vector does not vanish (thus allowing reparameterization by arclength).

3. The curvature vector does not vanish (so that among other things an orientation is induced).

The parameterization of the curvature mollification in this case is interesting because, while the original example curve γ is apriori parameterized by arclength and satisfies the other two usual regularity conditions piecewise with smooth non-vanishing curvature except at the singular point $\gamma(0) = (0,0)$, the curvature mollification ν is infinitely differentiable and regular in the second sense, but has vanishing curvature at $\nu(0) = (0,0)$. The fact that $\vec{k}(0) = (0,0)$ also follows directly from the formulas because, as one easily checks, $\nu''(0) = (0,0)$.

Example 2: Starting with the curve Γ illustrated in Figure 2 for which γ has values

$$\gamma(s) = \begin{cases} (c,0) + a(-\cos(s/a), \sin(s/a), & -\pi a/2 < s \le 0 \\ (-c,0) + b(\cos(s/b), \sin(s/b)), & 0 \le s < \pi b/2, \end{cases}$$

the curvature mollification $\nu : [-\pi a/2 + \sigma, \pi b/2 - \sigma] \to \mathbb{R}^2$ satisfies

$$\nu(t) \equiv \gamma(t) \qquad \text{for } -\pi a/2 + \sigma \le t \le -\sigma \text{ and } \sigma < t < \pi b/2.$$

For t satisfying $-\sigma < t < \sigma$ one finds

$$\begin{aligned} \nu(t) &= \int_{t-\sigma}^{0} \mu_\sigma(t-s) \left[(c,0) + a\left(-\cos\frac{t}{a}, \sin\frac{t}{a} \right) \right] ds \\ &\qquad + \int_0^{t+\sigma} \mu_\sigma(t-s) \left[(-c,0) + b\left(\cos\frac{t}{b}, \sin\frac{t}{b} \right) \right] ds \\ &= (1-\lambda)\left(c - a\cos\frac{t}{a}, a\sin\frac{t}{a} \right) + \lambda\left(-c + b\cos\frac{t}{b}, b\sin\frac{t}{b} \right) \end{aligned}$$

where

$$\lambda = \lambda(t,\sigma) = \int_0^{t+\sigma} \mu_\sigma(t-s)\, ds = \int_{-\sigma}^{t} \mu(\eta)\, d\eta.$$

Noting that

$$\frac{\partial \lambda}{\partial t} = \mu_\sigma(t),$$

we see

$$\begin{aligned} \nu'(t) &= (1-\lambda)\left(\sin\frac{t}{a}, \cos\frac{t}{a} \right) + \lambda\left(-\sin\frac{t}{b}, \cos\frac{t}{b} \right) \\ &\qquad - \mu_\sigma(t)\left(2c - a\cos\frac{t}{a} - b\cos\frac{t}{b}, a\sin\frac{t}{a} - b - \sin\frac{t}{b} \right) \end{aligned} \tag{75}$$

and

$$\nu''(t) = \frac{1-\lambda}{a}\left(\cos\frac{t}{a}, -\sin\frac{t}{a}\right) - \frac{\lambda}{b}\left(\cos\frac{t}{b}, \sin\frac{t}{b}\right)$$
$$- 2\mu_\sigma(t)\left(\sin\frac{t}{a} + \sin\frac{t}{b}, \cos\frac{t}{a} - \cos\frac{t}{b}\right)$$
$$- \mu_\sigma'(t)\left(2c - a\cos\frac{t}{a} - b\cos\frac{t}{b}, a\sin\frac{t}{a} - b\sin\frac{t}{b}\right). \quad (76)$$

With the help of these explicit formulas it is relatively easy to examine the curvature mollification of the curve in Example 2 numerically and to some extent analytically. In particular, we can consider the curvature vector $\vec{k}(t) = \nu''(t)/|\nu'(t)|^2 - (\nu'(t)\cdot\nu''(t))\nu'(t)/|\nu'(t)|^4$. See Figure 13-Figure 14.

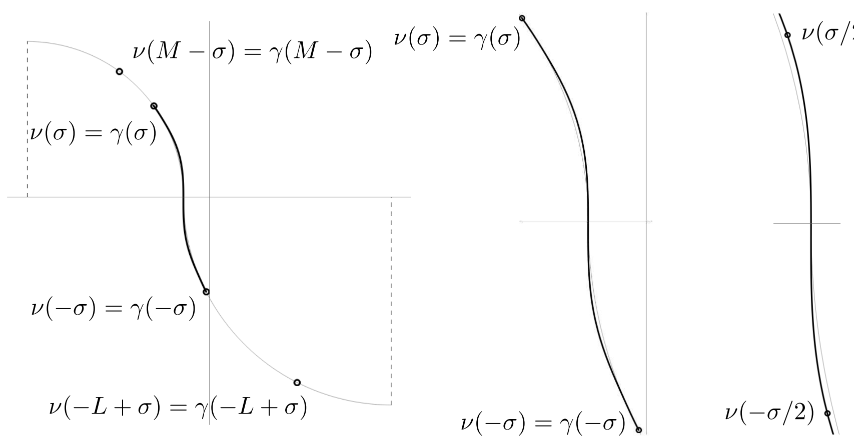

Figure 13: Curvature mollification for Example 2. Here we have chosen σ so the angle subtended on the smaller circular arc at $\gamma(\sigma)$ is $\pi/5$. The curvature mollification appears to follow the original curve Γ very closely, though the curvature plot in Figure 14 shows the curvature is actually fluctuating substantially. The two plots on the right give successively closer views of the curves near the common point $\nu(0) = \gamma(0)$ indicating the distinction between the image of ν and the image of γ.

Since $a+b=2c$, it is clear from the formula (75) for $\nu'(0)$ that $\nu'(0) = (0,1)$. We also see from (76) that

$$\begin{aligned}
\nu''(0) &= \left(\frac{1-\lambda}{a} - \frac{\lambda}{b}, 0\right) \\
&= \left(\frac{1}{a} - \left(\frac{1}{a} + \frac{1}{b}\right)\lambda, 0\right) \\
&= \left(\frac{1}{a} - \left(\frac{1}{a} + \frac{1}{b}\right)\int_0^\sigma \mu_\sigma(s)\, ds, 0\right) \\
&= \left(\frac{1}{a} - \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right), 0\right) \\
&= \left(-\frac{1}{2}\left(\frac{1}{b} - \frac{1}{a}\right), 0\right)
\end{aligned}$$

with the first coordinate satisfying

$$-\frac{1}{b} < -\frac{1}{2}\left(\frac{1}{b} - \frac{1}{a}\right) < 0$$

when $b < a$. In the special case $a = b$, we have Example 1 and $\vec{k}(0) = (0,0)$, and the numerical calculation suggests

$$\lim_{t \to 0} \frac{k_2(t)}{t} = 0$$

where $\vec{k} = (k_1, k_2)$. In fact, ...

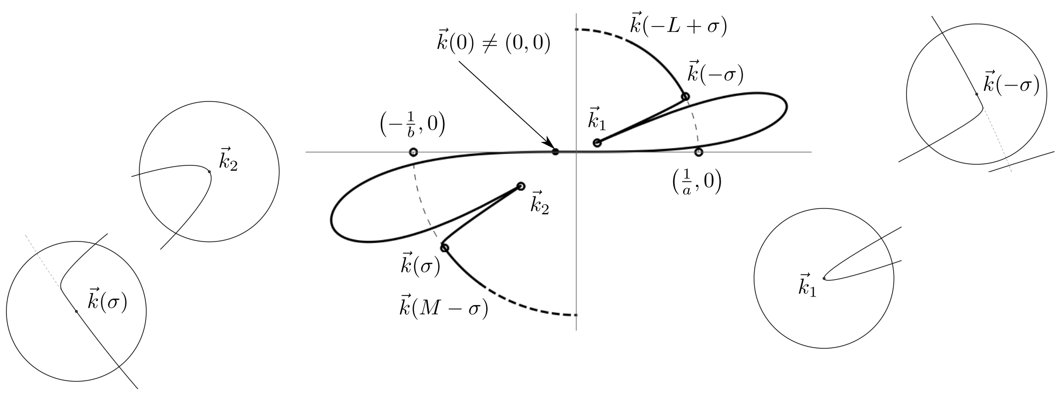

Figure 14: Example 2: Curvature associated with curvature mollification. In the middle is shown the plot of the curvature vector $\vec{k}$ associated with the curvature mollification ν of Example 2. The curvature vector associated with the original curve Γ is shown dashed. The values lie along a circle for $-\pi a/2 < s < 0$ and $0 < s < \pi b/2$; there is a jump discontinuity in the curvature of Γ at $s = 0$ from the circled point $(1/a, 0)$ to the circled point $(-1/b, 0)$. Four additional points of nominal interest are circled and close-up images are given on the left and right of the main plot. The points $\vec{k}(-\sigma)$ and $\vec{k}(\sigma)$ are the points at which the curvature of the curvature mollification first deviates from the circular curvature image of Γ. The points $\vec{k}_1$ and $\vec{k}_2$ are near points appearing to be singular in the initial plot. The close-up images at $\vec{k}_1$ and $\vec{k}_2$ illustrate the fact that the curvature vector depends smoothly on the parameter t.

3 Generalizations

It should be emphasized that the construction of the curvature mollification above depends essentially on the inclination angle ψ and is therefore, as it stands, a construction only for planar curves. Furthermore, I do not know a natural generalization of the inclination angle to curves of lower regularity than $\gamma \in C^1((-L, M) \to \mathbb{R}^2)$, so at least on the face of it, one is somewhat restricted in lowering the regularity assumed above. On the other hand, there is an interesting example to which both tangent mollification and curvature mollification apply.

Example 4 Consider the concatenated circles of the same radius parameterized by $\gamma : (-\pi a/2, \pi a/2) \to \mathbb{R}^2$ with

$$\gamma(s) = \begin{cases} (a, 0) + a(-\cos(s/a), -\sin(s/a)), & -\pi a/2 < s \leq 0 \\ (-a, 0) + a(\cos(s/a), \sin(s/a)), & 0 \leq s < \pi a/2. \end{cases} \tag{77}$$

This is a parameterization by arclength, and there is only one isolated point for which the values averaged in the tangent and curvature mollifications cannot be constructed. Consequently, the integrals defining those geometric mollifications are well-defined. On the other hand, the natural class into which this parameterization falls and with respect to which one can prove some kind of convergence is not entirely clear. The mollifications in this case are easily seen to be singular at the singular point of the original curve by elementary considerations. These remarks are illustrated in Figure 15 and Figure 16.

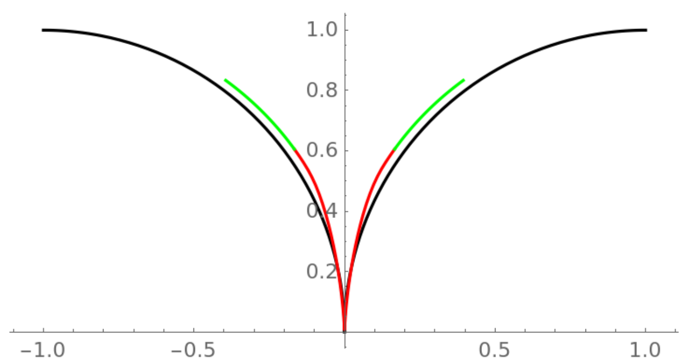

Figure 15: Tangent mollification of a cusp determined by concatenated circles as given by Example 4. The mollification itself preserves a cusp at $\nu(0) = (0, 0)$.

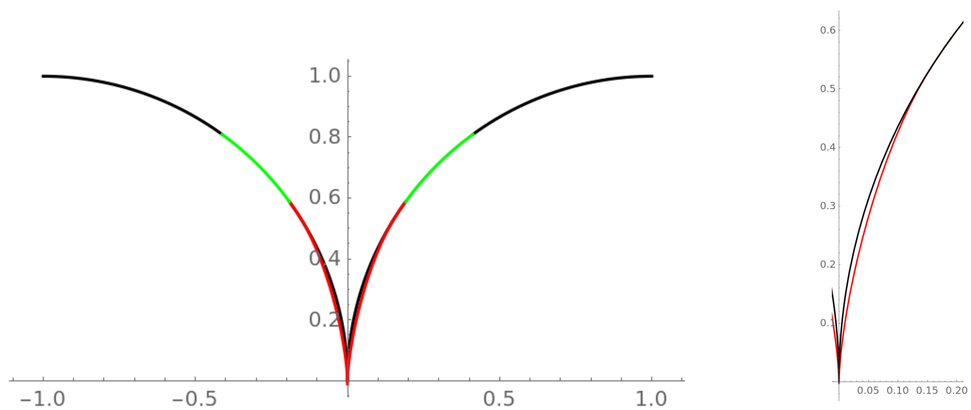

Figure 16: Curvature mollification of a cusp determined by concatenated circles as given by Example 4. This mollification also preserves a cusp at $\nu(0) = (0, 0)$ as indicated in the close-up on the right.

We finally return to the observation that the constructions of tangent mollification and curvature mollification are fundamentally distinct as presented above. In principle, tangent mollification is defined for a larger class of curves, and curvature mollification is not well-defined on curves of vanishing curvature in particular. On the other hand, tangent mollification does not give the same result or represent any kind of special case of curvature mollification with respect to curves where both are defined. Nevertheless, it is natural to look for a kind of blending of the two geometric mollifications which is well-defined on curves with vanishing curvature, reduces to tangent mollification on straight lines, and gives curvature mollification on circular arcs. This is possible using a variation on the Frenet formula(s) for planar curves. We note however the the basic structure theorem for planar curves (essentially that a curve is determined up to rigid motion using the Frenet formula by the curvature function $k = k(s)$) does require non-vanishing curvature, and we are not using the Frenet formula proper below.

Assume $\gamma \in C^2((-L, M) \to \mathbb{R}^2$ is an arclength parameterization. As outlined above such a parameterization determines an inclination angle $\psi \in C^1(-L, M)$ normalized without loss of generality by an initial condition $\psi(0) = \psi_0$. Recall the structure theorem for planar curves; see for example [dC76].

Theorem 11 Given $k \in C^2((-L, M) \to (0, \infty))$, $\mathbf{p}_0 \in \mathbb{R}^2$, and $\mathbf{v}_0 \in \mathbb{S}^1$, there exists a unique $\gamma \in C^2((-L, M) \to \mathbb{R}^2)$ satisfying

$$\begin{cases} \gamma(0) = \mathbf{p}_0, & \dot{\gamma}(0) = \mathbf{v}_0 \\ \ddot{\gamma} = k\mathbf{N}, & \dot{\mathbf{N}} = -\kappa\dot{\gamma} \\ |\dot{\gamma}| = |\mathbf{N}| = 1. \end{cases}$$

When we have an apriori inclination angle, an alternative formulation suitable to our geometric mollifications is possible.

Theorem 12 For each fixed $s \in (-L, M)$ the system of equations

$$\begin{cases} \frac{d\alpha}{d\eta} = \mathbf{v}, & \alpha(0) = \gamma(s) \\ \\ \frac{d\mathbf{v}}{d\eta} = \dot{\psi}(s)\ \mathbf{N}, & \mathbf{v}(0) = \dot{\gamma}(s) \\ \\ \frac{dN}{d\eta} = -\dot{\psi}(s)\ \mathbf{v}, & \mathbf{N}(0) = (-\sin\psi(s), \cos\psi(s)) \\ \\ |\mathbf{v}| = |\mathbf{N}| = 1 \end{cases} \tag{78}$$

has a unique solution $\alpha \in C^\infty(\mathbb{R} \to \mathbb{R}^2)$.

If $\ddot{\gamma}(s) \neq 0$ then $\dot{\psi}(s) \neq 0$, and the unique solution of (78) parameterizes a circle and is given explicitly by

$$\alpha(\eta) = \gamma(s) + \frac{1}{\dot{\psi}(s)}(-\sin\psi(s), \cos\psi(s)) - \frac{1}{\dot{\psi}(s)}(-\sin[\psi(s)+\eta\dot{\psi}(s)], \cos[\psi(s)+\eta\dot{\psi}(s)]).$$

If $\ddot{\gamma}(s) = 0$, the unique solution of (78) parameterizes a straight line and is given explicitly by

$$\alpha(\eta) = \gamma(s) + \eta(\cos\psi(s)), \sin\psi(s)).$$

Our construction blending tangent mollification and curvature mollification is given by

$$\nu(t) = \int_{s\in(-L,M)} \mu_\sigma(t-s)\alpha(s, t-s)$$

where $\alpha = \alpha(\eta) = \alpha(s, \eta)$ is the unique solution of (78). Again I will not state a regularity theorem or consider the convergence of this kind of geometric mollification, though in this case there are natural hypotheses leading to such theorems. Certainly the construction applies to the curve with a corner of Example 3 and agrees identically with the tangent mollification in this case as illustrated on the right in Figure 5. Also, this kind of geometric mollification leaves some interior portion of a long enough circular arc invariant. One final example suggests itself for application of this "blended" geometric mollification.

Example 5 Given $a > 0$ consider $\gamma : (-\pi/2, \infty) \to \mathbb{R}^2$ by

$$\gamma(s) = \begin{cases} (0, a) + a(\sin(s/a), -\cos(s/a)), & -\pi a/2 < s \leq 0 \\ (s, 0), & s \geq 0. \end{cases}$$

A third general construction we offer without proof involves adapting the "blended" geometric mollification just described to curves in a higher dimensional ambient $\mathbb{R}^n$. We can take as a starting point what we will call the **Frenet equations** by which we mean the collection of conditions

$$\begin{cases} \dot{\gamma} = \mathbf{v}, & \gamma(0) = \mathbf{p}_0 \\ \dot{\mathbf{v}} = k\mathbf{N}, & \mathbf{v}(0) = \mathbf{v}_0 \\ \dot{\mathbf{N}} = -k\mathbf{v} - \tau\mathbf{B}, & \mathbf{N}(0) = \mathbf{N}_0 \\ \dot{\mathbf{B}} = \tau\mathbf{N}, & \mathbf{B}(0) = \mathbf{B}_0 \\ \mathbf{B} = \mathbf{v} \times \mathbf{N} \\ |\mathbf{v}| = |\mathbf{N}| = 1. \end{cases} \tag{79}$$

In order to more or less clearly set the context we assume here the following:

1. $\gamma \in C^2((-L, M) \to \mathbb{R}^n)$ is either a given curve or a curve to be constructed according to a structure theorem.

2. $\mathbf{p}_0 \in \mathbb{R}^n$ and $\mathbf{v}_0, \mathbf{N}_0, \mathbf{B}_0 \in \mathbb{N}^{n-1}$ are fixed and given and $\{\mathbf{v}_0, \mathbf{N}_0, \mathbf{B}_0\}$ is an orthonormal set.

3. $k, \tau \in C^0(-L, M)$.

If $n = 3$ and $k > 0$, the structure theorem for space curves [dC76], or what do Carmo calls the *fundamental theorem of the local theory of curves,* asserts (79) has a unique solution $\gamma \in C^2((-L, M) \to \mathbb{R}^3)$. If $n = 2$ and one takes $\tau \equiv 0$ and $\mathbf{B} \equiv \mathbf{0} \in \mathbb{R}^2$, then a similar assertion holds as mentioned above. In certain constructions below, we will use a variant of (79), namely (80) in Theorem 13 below, in order to construct a geometric mollification of a given curve. It should be noted that in the context of the ideas below the relevant variant of (79) is not properly the Frenet equations as considered in the context of the structure theorem(s). To emphasize this point we note the condition $k > 0$ is included in the structure theorem for the simple reason that uniqueness fails without it. The condition $k > 0$ also makes certain considerations more straightforward, e.g., the definition of the principal normal vector, and is usually presented as an aspect of the regularity of the curve. This condition, however, is set aside in our variant (80) of (79). This allows the consideration of more general curves but also entails some technical complication(s).

When $n > 3$ some additional assumption is required to make sense of the product in the relation $\mathbf{B} = \mathbf{v} \times \mathbf{N}$. Though this situation is not our primary focus, one possibility is the following: We assume a curve $\Gamma \subset \mathbb{R}^n$ is parameterized by a function $\gamma \in C^1((-L, M) \to \mathbb{R}^n)$ satisfying $|\dot{\gamma}| = 1$ and endowed with a differentiable frame field, denoted symbolically by $\mathbf{F}$, so that $\{\dot{\gamma}(s), \mathbf{N}(s), \mathbf{F}(s)\}$ is an orthonormal collection of vectors in $T_{\gamma(s)}\mathbb{R}^n$ satisfying the equations

$$\begin{cases} \ddot{\gamma} = k\mathbf{N} \\ \dot{\mathbf{N}} = -k\dot{\gamma} - \tau\mathbf{F} \\ \dot{\mathbf{F}} = \tau\mathbf{N}. \end{cases}$$

In this case the ordered orthonormal basis $\{\dot{\gamma}(s), \mathbf{N}(s), \mathbf{F}(s)\}$ spans a three-dimensional subsapce $V_{\gamma(s)}$ of $\mathbb{R}^n$ and defines a cross product

$$\begin{aligned} (x_1\dot{\gamma}(s) + x_2\mathbf{N}(s) + x_3\mathbf{F}(s)) \ \times_{\mathbf{F}} \ & (y_1\dot{\gamma}(s) + y_2\mathbf{N}(s) + y_3\mathbf{F}(s)) \\ &= (x_2y_3 - x_3y_2)\dot{\gamma}(s) - (x_1y_3 - x_3y_1)\mathbf{N}(s) + (x_1y_2 - x_2y_1)\mathbf{F}(s) \end{aligned}$$

on $V_{\gamma(s)}$. For $n \geq 3$ we take the relation $B = \mathbf{v} \times \mathbf{N}$ of (79) to mean $\mathbf{B} = \mathbf{F}$ and

$$\mathbf{F} = \mathbf{v} \times_{\mathbf{F}} \mathbf{N}.$$

This interpretation/approach is also possible when $n = 3$, but one should be aware that the frame basis $\{\dot{\gamma}(s), \mathbf{N}(s), \mathbf{F}(s)\}$ may induce a cross-product on $\mathbb{R}^3$ different from the standard cross product.

Putting the construction below in the context of the mollification above, it may be said that we are introducing essentially four different kinds of mollification to which we can give the following names:

1. positional mollification,
2. tangent mollification,
3. curvature mollification, and

4. helical mollification.

Positional mollification, though applicable to parametric curves componentwise, is not properly a geometric mollification in the sense that the remaining three are. The first two kinds of mollification are moreoever essentially different from the last two. Curvature mollification as presented above is our primary subject of interest but will be extended to a more general setting below and will also be presented as a special case of **helical mollification** which is the main subject of the discussion below. Helical mollification, moreover, does admit certain relations to positional mollification and tangent mollification, and we will point out some of these relations. Finally, our primary interest is restricted to the mollification of planar curves, i.e., the case $n = 2$, and specifically Example 1-Example 4 and certain other examples with $n = 2$ arising more or less incidentally.

Theorem 13 Let $\gamma \in C^1((-L, M) \to \mathbb{R}^n)$ parameterize a curve

$$\Gamma = \{\gamma(s) : -L < s < M\}$$

with $|\dot{\gamma}| \equiv 1$. Given $s_0 \in (-L, M)$ and an orthonormal frame $\{\dot{\gamma}(s_0), \mathbf{N}_0, \mathbf{B}_0\}$ and **constants** $k, \tau \in \mathbb{R}$, the relations

$$\begin{cases} \dot{\alpha} = \mathbf{v}, & \alpha(0) = \gamma(s_0) \\ \dot{\mathbf{v}} = k\mathbf{N}, & \mathbf{v}(0) = \dot{\gamma}(s_0) \\ \dot{\mathbf{N}} = -k\mathbf{v} - \tau\mathbf{B}, & \mathbf{N}(0) = \mathbf{N}_0 \\ \dot{\mathbf{B}} = \tau\mathbf{N}, & \mathbf{B}(0) = \mathbf{B}_0 \\ \mathbf{B} = \mathbf{v} \times \mathbf{N} & \\ |\mathbf{v}| = |\mathbf{N}| = 1 & \end{cases} \tag{80}$$

admit a unique solution $\alpha \in C^\infty(\mathbb{R} \to \mathbb{R}^n)$.

We say the solution α from Theorem 13 parameterizes an adapted helix at $\gamma(s_0)$. Such a solution depends on many parameters, but we will primarily be interested in dependence on the arclength parameter η with respect to which the derivatives $\dot{\alpha}$, $\dot{\mathbf{v}}$, $\dot{\mathbf{N}}$ and $\dot{\mathbf{B}}$ are computed and the starting arclength parameter s_0 in relation to the given curve Γ. Accordingly, we write $\alpha = \alpha(s_0, \eta)$ with

$$\dot{\alpha}(s_0, \eta) = \frac{\partial}{\partial \eta}\alpha(s_0, \eta).$$

For $n = 2$ and $n = 3$, the proof of Theorem 13 is essentially a special case of the proof of the structure theorem associated with (79). Furthermore, since our primary interest is in the case $n = 2$, we omit the details in all cases and simply give formulas for the unique solution(s) in various cases. These will be useful for defining helical mollification.

If $k = 0$, then the unique solution satisfies

$$\alpha(s_0, \eta) = \gamma(s_0) + \eta\ \dot{\gamma}(s_0)$$

and parameterizes the tangent line to Γ at $\gamma(0)$. If $k \neq 0$ and $\tau = 0$, then the solution of Theorem 13 satisfies

$$\alpha(s_0, \eta) = \gamma(s_0) + \frac{1}{k}\ \mathbf{N}_0 - \frac{1}{k}\left(-\sin(k\eta)\ \dot{\gamma}(s_0) + \cos(k\eta)\ \mathbf{N}_0\right)$$

and parameterizes the tangent circle to Γ with center $\gamma(s_0) + \mathbf{N}_0/k$. Notice there is no requirement that $k > 0$ here. If $\kappa, \tau \neq 0$ (and $n \geq 3$)

$$\begin{aligned}\alpha(s_0, \eta) = \frac{1}{k^2 + \tau^2} \Bigg[& \left(\frac{k^2}{\sqrt{k^2+\tau^2}} \sin\left(\sqrt{k^2+\tau^2}\ \eta\right) + \tau^2\ \eta\right) \dot{\gamma}(s_0) \\ & + k\left(1 - \cos\left(\sqrt{k^2+\tau^2}\ \eta\right)\right) \mathbf{N}_0 \\ & + \kappa\tau \left(\frac{1}{\sqrt{k^2+\tau^2}} \sin\left(\sqrt{k^2+\tau^2}\ \eta\right) - \eta\right) \mathbf{B}_0 \Bigg].\end{aligned}$$

In this case α parameterizes a helix in a three-dimensional subspace of curvature $|k|$ and torsion $\pm\tau$ determined by the interaction of the constants k and τ with the choice of frame vectors $\mathbf{N}_0$ and $\mathbf{B}_0$.

We can now take the base curve Γ featured in the statement of Theorem 13 under some additional assumptions, including the existence of a weak curvature vector

$$D_1\dot{\gamma} = (D_1\dot{\gamma}_1, D_1\dot{\gamma}_2, \ldots, D_1\dot{\gamma}_n),$$

as the subject of geometric mollification. More precisely, we assume Γ is endowed with a differentiable adapted frame so that

$$\{\dot{\gamma}(s), \mathbf{M}(s), \mathbf{C}(s)\}$$

is an orthonormal collection in $T_{\gamma(s)}\mathbb{R}^n$, and there exist functions $\kappa, \tau \in L^1_{loc}(-L, M)$ satisfying the relations

$$\begin{cases} D_1\dot{\gamma} = k\mathbf{M} \\ \dot{\mathbf{M}} = -k\dot{\gamma} - \tau\mathbf{C} \\ \dot{\mathbf{C}} = \tau\mathbf{M} \\ \mathbf{C} = \dot{\gamma} \times \mathbf{M} \end{cases}$$

almost everywhere.

When $n = 3$, $\gamma \in C^2((-L, M) \to \mathbb{R}^3)$ and $k = |\ddot{\gamma}| > 0$ the vectors $\mathbf{M}$ and $\mathbf{C}$ are the uniquely determined principal normal and binormal respectively. In this case the functions k and τ are the uniquely determined curvature and torsion, and there is only one mollification parameter, namely $\sigma > 0$. In other cases, especially when $\gamma \in C^2((-L, M) \to \mathbb{R}^n)$ for some $n \geq 3$ with intervals on which $|\ddot{\gamma}| = 0$, the function τ will not be uniquely determined at all points. To the extent that different choices of τ are possible, formula (81) below gives distinct mollifications for which the function τ should be considered an additional mollification parameter. We do not explore the conceivable results of varying τ in such cases.

For $\sigma > 0$ we define the helical mollification $\nu \in C^\infty((-L + \sigma, M - \sigma) \to \mathbb{R}^n)$ by

$$\nu(t) = \int_{s \in (-L, M)} \mu_\sigma(t - s)\ \alpha(s, t - s) \tag{81}$$

where $\alpha = \alpha(s_0, \eta)$ is the adapted local helix given by Theorem 13 determined as stated in the theorem by taking the orthonormal frame with

$$\begin{aligned} \mathbf{N}_0 &= \mathbf{M}(s_0) \qquad \text{and} \\ \mathbf{B}_0 &= \mathbf{B}(s_0) \end{aligned}$$

and the constants $k = k(s_0)$ and $\tau = \tau(s_0)$. Naturally the latter relations are assumed to hold almost everywhere.

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