Mathematical Capillarity
variational formulation of stability;
application to floating bodies

with topics in differential geometry and partial differential equations

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Contents

Chapter 1

The First Problem(s)

1.1 Mise en Scène: A List

We wish to model (mathematically) a volume of liquid in equilibrium, and perhaps a place to start is by thinking about some everyday physical systems typical of those we wish to model. One of the simplest would be a glass of water sitting (vertically) on a table top. More generally, we could imagine the glass is replaced with a more exotic shaped vessel, perhaps a flower vase. The liquid could also be replaced with a different liquid, like mercury. Each of these changes, it might be expected, will make a difference in our modeling. In these cases we imagine the gravitational field of the earth should also play an important role. It is natural to ask how the gravitational force/field will be included in our mathematical model, and if we are being adventurous or imaginative we might ask what would happen without it.

We start below to formally compile a list of mathematical structures relevant to the modeling of these physical systems and many others. It may come as no surprise that the observed surface of a volume of liquid in equilibrium can be modeled, at least to some degree of accuracy, by a mathematical surface, and in some sense our primary interest will be in the shape of this surface which we will generally denote by $\mathcal{S}$. In technical language $\mathcal{S}$ is called the **free surface interface** or free surface for short. Observation of even the simple systems mentioned above suggests there is something nominally mysterious about how the surface of the liquid "curves" or bends at the edges. Eventually, we attempt to capture this mysterious behavior in terms of various "energies" in our model. See conditions/properties **6** and **7** of the list in section 1.3 below.

In the physical systems we have mentioned above the surface of the liquid volume actually has at least two pieces or two different kinds of pieces. To explain this more carefully, but still in rather general terms, let us imagine that a volume of liquid is modeled by an open subset $\mathcal{V}$ of three dimensional Euclidean space $\mathbb{R}^3$. The surface $\mathcal{S}$, or free surface interface, mentioned above is intended to model the surface separating the liquid of the volume from a complementary volume modeled somewhere in $\mathbb{R}^3\backslash\mathcal{V}$. We might expect, at least at first, this complementary volume to be empty.[1] Careful consideration of our environment here on the surface of planet Earth suggests that we rarely encounter truly empty space. What passes for everyday empty space on planet Earth is mostly some mixture of gasses we can conveniently, though perhaps rather inaccurately, refer to as "the atmosphere." The fact that this region of physical space is not empty, consists of some kind of "vapor," and supports a vapor pressure in particular is actually crucial physically to the appearance of the physical surface modeled by $\mathcal{S}$. In our model however, for most practical purposes, we can imagine this complementary set to be empty.

The other piece of the "surface" of the water in a glass is the surface separating the liquid of the volume from the glass itself. This surface is "under the surface" of the liquid so to speak and might be imagined to have a very different nature. Its shape, for example is not so mysterious: It is determined by the shape of the glass or more generally the vessel. It turns out that this wetted region on the vessel also plays an important role in the modeling and should be included. In our model we will have the boundary of $\mathcal{V}$ containing $\mathcal{S}$ and what we will refer to as the **wetted region** generally denoted by $\mathcal{W}$:

$$\partial\mathcal{V} = \mathcal{S} \cup \mathcal{W} \cup \Gamma.$$

The set Γ models the set where the free surface $\mathcal{S}$ meets the wetted region $\mathcal{W}$ and is called the **contact line**, though obviously Γ is usually not geometrically an actual (straight) line. Hopefully, at least in some instances, the region between the observed physical free surface and the wetted region can be modeled by a (smooth) curve.

It may occur to you (and it certainly occurs to me) that the modeling of a liquid volume in equilibrium might be substantially simpler if there were

[1] That is, we might expect some of the physical space outside the physical liquid volume to contain no physical matter. (I'm not talking about the mathematical empty set here.)

only free surface and liquid volume with

$$\partial\mathcal{V} = \mathcal{S}.$$

Mathematicaly, of course, one always wishes to consider simple (and especially the absolute simplest) special cases. With gravity, on the earth, one just doesn't run into this case of a free floating liquid volume very often. Free floating liquid volumes tend to drop down in the gravity field and hit something, for example, the earth itself. In outer space (I hear) one can see a pretty good approximation of a free floating drop in equilibrium. Eventually when we have developed our mathematical model for equilibrium capillary surfaces, we will be able to adapt our model pretty easily and directly to this case with zero gravity and prove a theorem, called Alexandrov's theorem, which asserts $\mathcal{S}$ should be a spherical surface in this case. This is in pretty good agreement with what people who might happen to live in an environment with little or no gravity might see.

With this general introduction, I will attempt two final comments about the list below before comencing with the details. First, the numbering is purely for reference and is in no way canonical. In fact, very few people approach the mathematical modeling of capillary surfaces even in the roughly axiomatic manner I've taken up below. Each condition or property should be considered individually and critically compared to observations with the expectation that more accurate conditions and mathematical constructions may be required for better modeling. In addition to that, I've chosen some conditions specifically because they simplify the exposition. I do intend that the result is a theory of mathematical capillarity which captures/models reasonably well some physical systems, but it should also be kept in mind that the list is not canonical in the sense that many equilibrium capillary systems are modeled using slight modifications of the list below. This point is also discussed in more detail below as the list takes shape.

First elements

In some generality one may model a physical liquid volume in equilibrium by an open connected set $\mathcal{V} \subset \mathbb{R}^3$ satisfying the following conditions:

1. (compactness) The closure $\overline{\mathcal{V}}$ is compact.

2. (nature) The boundary $\partial\mathcal{V}$ can be written as a disjoint union

$$\partial\mathcal{V} = \mathcal{S} \cup \Gamma \cup \mathcal{W}$$

with

a. $\mathcal{S}$ is a smooth surface embedded in $\mathbb{R}^3$ with smooth boundary Γ,

b. Γ is a smooth curve embedded in $\mathbb{R}^3$ with components

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_k,$$

c. $\mathcal{W}$ can be written as a disjoint union

$$\mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_\ell \cup \Gamma_{k+1} \cup \Gamma_{k+2} \cup \cdots \cup \Gamma_{k+m}$$

where each $\mathcal{W}_j$ is a smooth surface embedded in $\mathbb{R}^3$ with smooth boundary so that $\mathcal{W}$ is a piecewise smooth surface[2] with boundary Γ. Generally, the integers k, ℓ and m and the ordering of these various pieces have no particular physical meaning, but should be specified for each physical system being modeled. For flexibility, mathematical consistency, and convenience we generally allow some (or many) of these sets to be empty, but in a particular modeling application the nonempty representatives are obviously those of interest. The physical points to keep in mind, however, are that Γ, and hence $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$, models the contact line and $\Gamma_{k+1}, \Gamma_{k+2}, \ldots \Gamma_{k+m}$ are used to model internal wetted singular curves where smooth wetted regions meet.

3. (technical) The disjoint surfaces $\mathcal{S}$ and $\mathcal{W}$ each extend smoothly across Γ to surfaces $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{W}}$ intersecting transversally along

$$\tilde{\mathcal{S}} \cap \tilde{\mathcal{W}} = \Gamma.$$

That is, for each $P \in \Gamma = \partial\mathcal{S} = \partial\mathcal{W}$ there exists an open set $\mathcal{V}_P \subset \mathbb{R}^3$ and functions[3]

$$X, Y \in C^\infty(B_1(\mathbf{0}) \to \mathbb{R}^3), \tag{1.1}$$

with

[2]Some technicalities of the definition of piecewise smooth surface with boundary we have in mind here will be addressed below, and most importantly illustrative examples will be given in detail.

[3]See Appendix A for some notes on the notation used here.

a.

$$A = \frac{\partial X}{\partial x_1} \times \frac{\partial X}{\partial x_2} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.2}$$

b.

$$B = \frac{\partial Y}{\partial x_1} \times \frac{\partial Y}{\partial x_2} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.3}$$

c.

$$A \times B \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.4}$$

d. $X(\mathbf{0}) = Y(\mathbf{0}) = P$,

e. $X(B_1(\mathbf{0})) = \tilde{\mathcal{S}} \cap \mathcal{V}_P$, and

f. $Y(B_1(\mathbf{0})) = \tilde{\mathcal{W}} \cap \mathcal{V}_P$.

Note(s): The cross product appearing in (1.2), (1.3), and (1.4) is the usual one so that if $A = (a_1, a_2, a_3)^T$ and $B = (b_1, b_2, b_3)^T$ as in (1.4) then

$$A \times B = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{pmatrix}. \tag{1.5}$$

Quite often we will ignore transposes and consider a particular vector as a column vector or a row vector interchangeably without comment. For example, as a general rule $\mathbb{R}^2$ and $\mathbb{R}^3$ denote the real vector spaces of row vectors so that X and Y in (1.1) as well as the vectors $\partial X/\partial x_1$, $\partial X/\partial x_2$, $\partial Y/\partial x_1$, and $\partial Y/\partial x_2$ in (1.2) and (1.3) should nominally all be row vectors, but the equalities in (1.2), (1.3), and (1.5) suggest the consideration of column vectors. This may be viewed as a slight abuse of notation. Also when I write $A = (a_1, a_2, a_3)^T$ and $B = (b_1, b_2, b_3)^T$ as I have done above, I am simply emphasizing that A and B are considered as column vectors in the definition (1.5).

The technical conditions given in (1.2) and (1.3) are associated with the assumption that $\mathcal{S}$ and $\mathcal{W}$ are regular surfaces up to and including their surface boundaries. Notice that surface boundaries are, in principle, slightly different from topological boundaries like $\partial\mathcal{V}$. If, however, $\tilde{\mathcal{S}}$ is considered as a topological space, then $\partial\mathcal{S}$ is the topological boundary of $\mathcal{S}$ considered as a topological subspace of $\tilde{\mathcal{S}}$.

The technical condition (1.4) is associated with the assumption that $\mathcal{S}$ and $\mathcal{W}$ meet at a nonzero and non-π angle, very often unambiguously measured within the volume $\mathcal{V}$ modeling the liquid. See however Exercise 1.2 below.

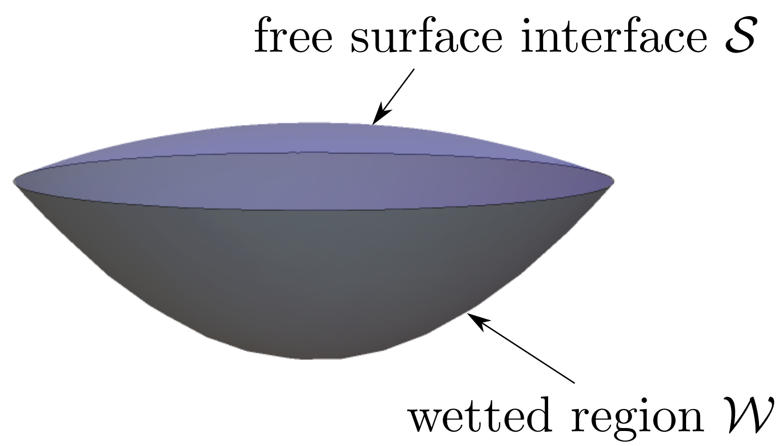

Figure 1.1: Modeling a volume of liquid (blue) in a bowl.

Configurations

The surface $\mathcal{W}$ in the discussion above models the region on a physical rigid object in contact with the liquid. We will model such objects, generally considered rigid and fixed and often called "support structures," in more detail below. The surface $\mathcal{S}$ models the surface of the liquid not in contact with any rigid support structure, and is variously known as the **free surface interface** or the (generalized) **meniscus**. Figure 1.1 illustrates the kind of situation we have in mind. Here we imagine a parabolic bowl with wetted surface satisfying $z = x^2 + y^2 < 1$ with the volume of liquid capped above by an axially symmetric meniscus $\mathcal{S}$. Assuming the presence of a downward gravity field, a primary objective in the study of mathematical capillarity is to determine the shape and other properties of the surface $\mathcal{S}$.

We refer to a volume $\mathcal{V}$ with designated boundary sets $\mathcal{S}$ and $\mathcal{W}$ satisfying the natural and technical conditions **1-3** above as a **configuration**. On the one hand, the basic conditions **1-3** allow some "exotic" configurations one might wish to exclude. On the other hand, it is also natural to consider certain more general configurations excluded by conditions **1-3**. One important direction of generalization is represented by the consideration of configurations with less regularity. In a certain sense the main text on the subject of mathematical capillarity [Fin86] showcases and is focused on situations where the contact line Γ is not a smooth curve. See comments at the end of the next section below.

Another important generalization involves attempts to model liquids, like helium at extremely low temperatures, which are inadequately modeled under the transversality condition (1.4). On the other hand, the restrictive conditions **1-3** provide a simple well-defined framework from which one can attempt these more complicated generalizations as well as a generalization which, in a certain sense, represents the focus of this text, namely attempting to model the configurations associated with the equilibrium of floating bodies in which the natural assumption that the wetted region is on a fixed rigid "support" structure must be relaxed or generalized.

1.2 Capillary Tube(s)

Conditions **1-3** do allow certain important configurations (or at least some versions of them) including that associated with the insertion of a circular tube into a bath of liquid under the influence of gravity. This may be called the **circular capillary tube problem**, and it is chosen as a kind of starting point (in some form) by Finn [Fin86]. Our version of the problem may be more properly referred to as the **circular capillary tubes problem** or just the capillary tubes problem if we happen to be restricting attention to a circular container/tube geometry as we now describe.

We begin here with the rigid support structure referred to informally in the previous section. Let a, R, d_0, and t_0 be positive real numbers with

$$0 < a < a + t_0 < R.$$

Let

$$C = \{(x, y, z) : x^2 + y^2 \geq R^2, z \geq 0\} \cup \{(x, y, z) : z \leq 0\}$$

and

$$T = \{(x, y, z) : a^2 \leq x^2 + y^2 \leq (a + t_0)^2,\ z \geq d_0\}.$$

Together $\mathcal{R} = C \cup T$ represents an excluded region modeling the **rigid support structure** consisting of a solid cylindrical tube (modeled by T) and a container (modeled by C). The complementary open region

$$E = \mathbb{R}^3 \backslash (C \cup T)$$

provides an environment for the liquid volume $\mathcal{V}$ we wish to consider in this elementary configuration. Figure 1.2 illustrates the rigid support structure described above.

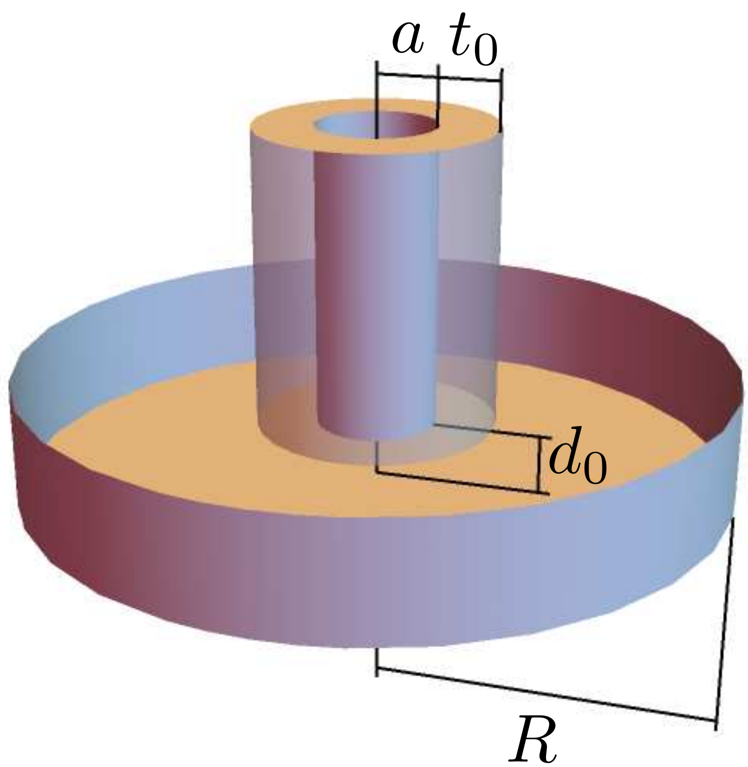

Figure 1.2: The environment for a capillary meniscus determined by a rigid structure. Here we have a circular cylindrical outer container C of radius R with closed bottom. Into the cavity (or bath region) determined by C descends a solid circular cylinder with inner radius a and thickness t_0. The tube descends to a depth d_0.

We consider then a model volume $\mathcal{V}$ satisfying conditions **1-3** above and

$$\overline{\mathcal{V}} \subset \overline{E}, \quad \mathcal{S} \subset E, \quad \mathcal{W} \subset \partial(C \cup T).$$

The technical condition **3** requires the free surface interface $\mathcal{S}$ and the wetted region $\mathcal{W}$ to have extensions along their surface boundaries. It is often the case that the rigid structure associated with the particular configuration under consideration suggests a kind of natural extension of the wetted region $\mathcal{W}$. For example, taking $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ where

$$\mathcal{S}_0 = \{(x, y, u(x, y)) : (x, y) \in B_a(\mathbf{0})\},$$
$$\mathcal{S}_1 = \left\{(x, y, u(x, y)) : (x, y) \in B_R(\mathbf{0})\backslash\overline{B_{a+t_0}(\mathbf{0})}\right\},$$

and

$$u \in C^\infty(\overline{U}), \qquad U = B_a(\mathbf{0}) \cup [B_R(\mathbf{0})\backslash\overline{B_{a+t_0}(\mathbf{0})}] \tag{1.6}$$

is a positive function with

$$u\Big|_{\partial B_a(\mathbf{0})} > d_0 \qquad \text{and} \qquad u\Big|_{\partial B_{a+t_0}(\mathbf{0})} > d_0,$$

we may consider a volume $\mathcal{V}$ given by

$$\mathcal{V} = \{(x, y, z) : (x, y) \in U,\ 0 < z < u(x, y)\}$$
$$\cup \{(x, y, z) : a^2 \le x^2 + y^2 \le (a + t_0)^2,\ 0 < z < d_0\}.$$

Notice that in this case we can write

$$\Gamma = \partial\mathcal{S} = \partial\mathcal{S}_0 \cup \partial\mathcal{S}_1 = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$$

with $\Gamma_0 = \partial\mathcal{S}_0 = \{(x, y, u(x, y)) : x^2 + y^2 = a^2\}$ and

$$\Gamma_1 = \{(x, y, u(x, y)) : x^2 + y^2 = (a + t_0)^2\} \text{ and} \tag{1.7}$$
$$\Gamma_2 = \{(x, y, u(x, y)) : x^2 + y^2 = R^2\}; \tag{1.8}$$

$$\mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3 \cup \mathcal{W}_4 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$$

with

$$\mathcal{W}_0 = \{(x, y, z) : x^2 + y^2 = a^2,\ d_0 < z < u(x, y)\}, \tag{1.9}$$
$$\mathcal{W}_1 = \{(x, y, z) : x^2 + y^2 = (a + t_0)^2,\ d_0 < z < u(x, y)\}, \tag{1.10}$$
$$\mathcal{W}_2 = \{(x, y, z) : x^2 + y^2 = R^2,\ 0 < z < u(x, y)\}, \tag{1.11}$$
$$\mathcal{W}_3 = \{(x, y, d_0) : a^2 < x^2 + y^2 < (a + t_0)^2\}, \tag{1.12}$$
$$\mathcal{W}_4 = \{(x, y, 0) : x^2 + y^2 < R^2\}, \tag{1.13}$$
$$\Gamma_3 = \{(x, y, d_0) : x^2 + y^2 = a^2\}, \tag{1.14}$$
$$\Gamma_4 = \{(x, y, d_0) : x^2 + y^2 = (a + t_0)^2\}, \text{ and} \tag{1.15}$$
$$\Gamma_5 = \{(x, y, 0) : x^2 + y^2 = R^2\}. \tag{1.16}$$

Notice conditions **2b** and **2c** in relation to these sets. Note that $k = 2$, $\ell = 4$, and $m = 3$. See also Exercises 1.3-1.5.

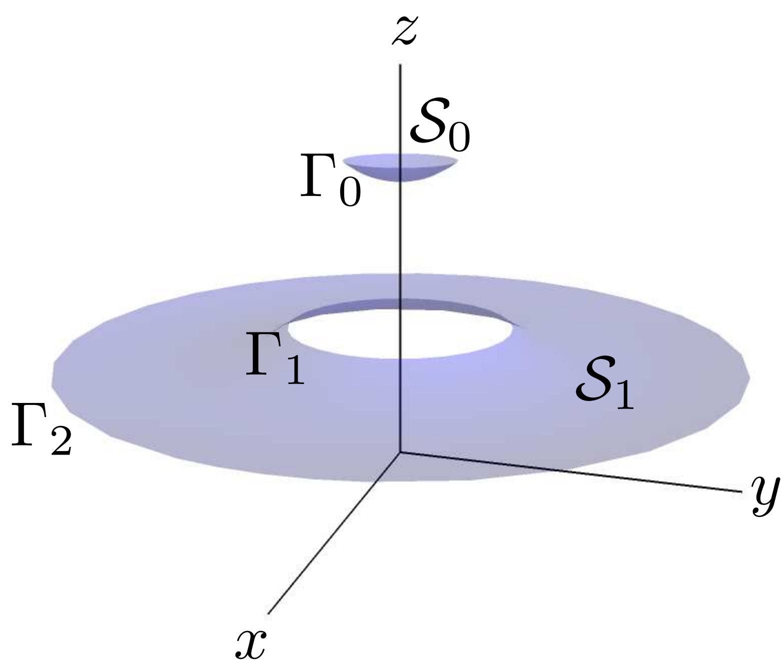

Figure 1.3: The free surface interfaces associated with the insertion of a capillary tube in a bath of liquid—the capillary tubes problem. In this case the inner tube interface $\mathcal{S}_0$ is bounded by a circle, and the outer (bath) interface $\mathcal{S}_1$ is bounded by two circles. A good question to ask at this point might be: Do we know these interfaces must be axially symmetric? We will encounter a theorem below giving the answer.

1.3 Energies

The previous section gives general geometric conditions defining what we have called a "capillary configuration," but naturally some additional criterion is needed to pick out from among all possible capillary configurations one corresponding to, or which can be reasonably considered to model, a physical capillary configuration that is actually observed. The basic additional criteria was suggested by Carl Gauss in 1830, though Thomas Young and Pierre Simon Laplace had obtained a primary consequence of the basic criterion without the benefit of having it to start what may be considered the beginning of the study of mathematical capillarity in 1805 and 1806. Gauss' idea was that the observed capillary configuration should be one minimizing a certain "potential" energy, perhaps subject to some natural additional constraints as described below. In terms of our general geometric conditions the energy associated to an elementary capillary configuration takes the form

$$\mathcal{E} = \int_{\mathcal{S}} \sigma - \int_{\mathcal{W}} \sigma\beta + \int_{\mathcal{V}} \Upsilon \tag{1.17}$$

where the last term is the conventional potential energy associated with the position in space of the volume of liquid $\mathcal{V}$ subject to whatever potential fields Υ, for example a potential field due to gravitational attractions perhaps, happen to be present, and the first two integral terms are distinctive to (mathematical) capillarity. The first is called the **free surface energy** and is associated with the formation of an interface separating the liquid from its vapor. Notice that if σ is a constant function on the surface, then the free surface energy becomes

$$\sigma \int_{\mathcal{S}} 1 = \sigma \ \text{area}(\mathcal{S})$$

which is proportional to the area of the free surface. That is, the amount of energy required to form a particular free surface interface is proportional to the area of that free surface. The constant of proportionality σ in this case is called **surface tension**, which of course has physical dimensions

$$\frac{\text{energy}}{\text{area}} = \frac{\text{force} \ \times \ \text{length}}{\text{area}} = \frac{\text{force}}{\text{length}}$$

or symbolically $(ML/T^2)/L = M/T^2$. Consequently, it is also natural to integrate σ along a curve Γ within the free surface $\mathcal{S}$ to obtain a force. If one imagines the free surface as a kind of "membrane" holding in the volume $\mathcal{V}$ and with a cut along Γ, then this calculated force may be imagined to be that required to hold the membrane together along the cut. In this way also we see that while the constant σ is called surface tension, it is more properly a tension per length or a kind of tension/force density.

More generally, one may encounter a **surface tension function** in more complicated (and potentially more accurate modeling). From the mathematical point of view, this function is most easily considered spatially dependent

surface tension	temperature
75.64	0
71.97	25
67.91	50
58.85	100

Table 1.1: The surface tension of water given in dynes per centimeter (or equivalently millinewtons per meter). Temperatures are given in degrees Celsius.

on the free surface so that we write $\sigma : \mathcal{S} \to (0, \infty)$ with values always assumed to be positive. When the physical mechanism of the modeling is contemplated, however, it is natural to imagine σ primarily dependent on other (varying) physical parameters on the surface, most notably (or commonly) temperature and/or some kind of density but very likely some other kind of varying molecular property like a chemical concentration for example. For the presentation here we will essentially always make the assumption that the surface tension σ is a constant function with physical origin, to the extent we contemplate it, determined simply by the liquid one happens to be modeling. Thus, water will have associated with it a different surface tension "constant" than mercury, though if you look up these numbers in a table, then at least a temperature dependence will usually be noted. For example, one can find the values given in Tables 1.1 and 1.2. The values in such tables should be taken as approximate, and in fact reported values of the surface tension for a given liquid may be found to vary substantially. One possibility is to view this diversity of opinion as resulting from the presence of various unavoidable contaminants with significant effect on the value. This is the usual explanation. It may be noted in addition, that the reported value is always obtained indirectly through some "method," and the reported value may depend on the method or how it is applied. We will discuss some of the relevant methods below. Finn certainly viewed the existence of a surface tension constant as phenomenological, that is representative of the possibility of producing reasonable quantitative and qualitative prediction—suggestive of actual physical existence but with the actual physical existence nevertheless unverified—and perhaps impossible to verify directly.

The second term in the capillary energy (1.17) is also important in deriv-

substance	surface tension	temperature
acetone	27.70	20
blood	55.89	22
ethyl alcohol	22.27	25
40% ethyl alcohol	29.63	25
11.1% ethyl alcohol	46.03	25
helium (superfluid)	0.37	-273
nitrogen (liquid)	8.85	-196
oxygen (liquid)	13.2	-182
mercury	487.00	15
55% sucrose	76.45	20

Table 1.2: The surface tension of various liquids given in dynes per centimeter (or equivalently millinewtons per meter). Temperatures are given in degrees Celsius. Percentages are relative to the indicated solute in water.

ing the main consequences related to the modeling of mathematical capillarity as we will see below. This term, given in the case where $\sigma\beta$ is a constant by

$$-\sigma\beta \, \text{area}(\mathcal{W}),$$

is called the **wetting energy**. When σ and β are both constant, β is called the **adhesion coefficient**. As we will see below, it is at least somewhat natural to assume $-1 < \beta < 1$ or at least $-1 \leq \beta \leq 1$ though certain underlying mechanisms, both physical and mathematical, associated with this assumption may be viewed as slightly obscure.

Even when the surface tension σ is considered constant, it is often necessary to consider $\beta : \mathcal{W} \to (-1, 1)$ to be some function with values varying spatially with respect to various spatial, physical, and chemical (and whateer other kind of) inhomogeneities. Roughly speaking, variations in the material of the container (or rigid structures determining the modeling of the geometric environment E considered above) or variations in the attractions between the molecules of the liquid and the molecules of the rigid structures may necessitate variations in the value of β. In practice, the adhesion coefficient is generally considered to be piecewise constant and constant with respect to a portion of the rigid structures of a given material in particular. For example in the capillary tubes problem with initial geometric structure given above,

substrate	contact angle
glass	0°-30°
quartz	25°-45°
aluminum	15°-25°
copper	80°-90°
teflon	100°-110°
silicon	20°-30°
nylon	50°-70°

Table 1.3: The contact angle/adhesion coefficient of water given in degrees. Note particularly that while all the data above is summarized from various published values and quartz should be a particular kind of glass, e.g., a typical labaroatory graduated cylinder is made of quartz, there is an obvious inconsistency in the first two ranges of values.

if the portion of the container holding the liquid bath into which the tube is dipped is plastic and the tube itself is glass, then the usual modeling assumption is that β takes one constant value β_1 on $\mathcal{W}_2$ and a different constant value β_0 on the surface $\mathcal{W}_0 \cup \mathcal{W}_1$ modeling the wetted surface on the glass tube.

Reported values of adhesion coefficients vary much more wildly than those of surface tension with inevitable (and so far unmeasurable) issues of "contamination" viewed as the main source of inaccessibility, though methods of measurement certainly also play some role. From the simplistic (phenomenological) point of view adopted here the surface tension "constant" β is dependent on the liquid involved and the substance of the solid support structures contacted by the liquid, usually referred to as the **substrate**. Reported values are usually given in terms of the **contact angle** $\gamma = \cos^{-1}\beta$ as indicated in Tables 1.3 and 1.4.

Before we attempt to derive some of the main consequences of energy minimizaiton, consider a common form of the potential energy term and some elementary observations about the capillary tubes problem. Let us model the gravitational force near the surface of the earth as constant, that is, a "point mass" of magnitude $m > 0$ experiences a vertical downward force mg where $g > 0$ is taken to be approximately 9.8 m/s^2. Under this assumption

liquid	substrate	contact angle
ethanol	copper	10°-25°
ethanol	quartz	0°-5°
glycerine	teflon	10°-25°
mercury	glass	120°-140°
mercury	quartz	130°-140°
acetone	quartz	2°-10°

Table 1.4: Published contact angle ranges for various liquid/substrate combinations.

we partition the volume $\mathcal{V}$ associated with a capillary configuration[4] into volumes $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_N$ with each $\mathcal{V}_j$ satisfying

$$\mathcal{V}_j \subset B_r(\mathbf{p}_j) = \{\mathbf{x} = (x_1, x_2, x_3) : |\mathbf{x} - \mathbf{p}_j| < r\}$$

where $r > 0$ is some (small) constant and $\mathbf{p}_j$ is a point in the capillary environment E. We assume further the liquid under consideration has constant density ρ. If coordinates are chosen so that $U_0 = \{\mathbf{x} = (x_1, x_2, x_3) : x_3 > 0\}$ models the region above the surface of the earth $\Sigma = \{\mathbf{x} = (x_1, x_2, x_3) : x_3 = 0\}$ with $E \subset U_0$ and Σ taken to be at zero gravitational potential, then we can associate to each volume $\mathcal{V}_j$ the approximate potential energy

$$\rho\ g\ \ \mathrm{vol}(\mathcal{V}_j)\ z_j = \int_0^{z_j} \left(\int_{\mathcal{V}_j} \rho \right) g\, dx_3$$

where z_j is the third component of the point $\mathbf{p}_j$ since

$$\int_{\mathcal{V}_j} \rho = \rho\ \ \mathrm{vol}(\mathcal{V}_j)$$

is the mass associated with the volume $\mathcal{V}_j$. An infinetesimally correct approximation of the gravitational energy in (1.17) is thus given by the sum

$$\sum_{j=1}^{N} \rho\ g\ \ \mathrm{vol}(\mathcal{V}_j)\ z_j = \rho\ g\ \sum_{j=1}^{N} z_j\ \ \mathrm{vol}(\mathcal{V}_j)$$

[4]Recall basic conditions **1** and **2**.

in the sense that

$$\int_{\mathcal{V}} \Upsilon = \rho\ g\ \lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{N} z_j\ \ \mathrm{vol}(\mathcal{V}_j)$$

where $\mathcal{P} = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_N\}$ denotes the partition of $\mathcal{V}$ and

$$\|\mathcal{P}\| = \max_{1\le j\le N} \mathrm{diam}(\mathcal{V}_j).$$

The limiting value, since the region $\mathcal{V}$ is an open set with well-behaved piecewise smooth boundary by assumption, is

$$\int_{\mathcal{V}} \Upsilon = \rho\ g\ \int_{\mathcal{V}} z \tag{1.18}$$

where z now denotes the third component of the spatial variable within $\mathcal{V}$. See Exercise 1.6.

Thus specializing to capillary configurations modeling physical systems in a vertical gravitational potential field of acceleration g and involving a single connected liquid volume $\mathcal{V}$ of density ρ, the simple energy (1.17) becomes

$$\mathcal{E} = \int_{\mathcal{S}} \sigma - \int_{\mathcal{W}} \sigma\beta + \rho\ g \int_{\mathcal{V}} z. \tag{1.19}$$

We assume also a constant surface tension σ so minimization of $\mathcal{E}$ is equivalent to minimization of $\mathcal{E}_\sigma = \mathcal{E}/\sigma$ or

$$\mathcal{E}_\sigma = \mathrm{area}(\mathcal{S}) - \int_{\mathcal{W}} \beta + \kappa \int_{\mathcal{V}} \tag{1.20}$$

where $\kappa = \rho g/\sigma$ is called the **capillary constant** and we are assuming β may be spatially dependent primarily for the purpose of allowing β to be piecewise constant corresponding to the various pieces of the wetted region $\mathcal{W}$ on materially different portions of the container surface.

At this point also let us consider the special case of the capillary tubes problem contemplated in section 1.2. It will be observed that essentially no no notational changes are needed to adapt the designations of $\mathcal{S}$, $\mathcal{W}$, and $\mathcal{V}$ in that case to a somewhat more general situation in which the inner vertical wall of the tube, the outer wall of the tube and the vertical wall of the container project onto the boundaries of three general simply connected nested domains

$$\Omega \subset\subset \Omega_1 \subset\subset \Omega_2.$$

In the case of a circular tube concentric with a circular container considered in section 1.2 one has $\Omega = B_a(\mathbf{0})$, $\Omega_1 = B_{a+t_0}(\mathbf{0})$, and $\Omega_2 = B_R(\mathbf{0})$. In general, the domains Ω and $\Omega_{\text{out}} = \Omega_2 \backslash \overline{\Omega_1}$ will be of particular interest, and we consider a pair of meniscus interfaces $\mathcal{S}_0$ and $\mathcal{S}_1$ determined as the graph of a function $u : U \to \mathbb{R}$ where $U = \Omega \cup \Omega_{\text{out}}$ and $u \in C^\infty(\overline{U})$.

Under this assumption the gravitational energy from (1.17) expressed in a particular case in (1.18) may be further expressed in terms of u as

$$\begin{aligned}\int_{\mathcal{V}} \Upsilon &= \rho\ g \int_{\Omega} \left(\int_0^u z\, dz \right) + \rho\ g \int_{\Omega_1 \backslash \Omega} \left(\int_0^{d_0} z\, dz \right) \\ &\quad + \rho\ g \int_{\Omega_{\text{out}}} \left(\int_0^u z\, dz \right) \\ &= \frac{\rho\ g}{2} \int_{\Omega} u^2 + \frac{\rho\ g}{2} \int_{\Omega_{\text{out}}} u^2 + \frac{\rho\ g}{2}\ \text{area}(\Omega_1 \backslash \Omega)\ d_0^2.\end{aligned}$$

Note that the last term

$$C_g = \frac{\rho\ g}{2}\ \text{area}(\Omega_1 \backslash \Omega)\ d_0^2$$

remains constant corresponding to a fixed volume of liquid in the container directly under the rigid structure of the tube for all of these configurations.

Under the same assumptions, the free surface energy becomes

$$\int_{\mathcal{S}} \sigma = \sigma \int_{\Omega} \sqrt{1 + |Du|^2} + \sigma \int_{\Omega_{\text{out}}} \sqrt{1 + |Du|^2}$$

where

$$Du = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$$

is the gradient of u and $\sqrt{1 + |Du|^2}$ is the corresponding scaling factor for area obtained, for example, as $|\ \det[(DX)^T DX]\ |$ from the parameterization $X(x, y) = (x, y, u(x, y))$. Note that we have used the regularity assumption (1.6) here or at least that $u \in C^1(\overline{U})$ where $U = \Omega \cup \Omega_{\text{out}}$.

Assuming the adhesion coefficient β is piecewise constant so that

$$\beta = \beta(\mathbf{x}) = \begin{cases} \beta_0, & \mathbf{x} \in \mathcal{W}_0 \\ \beta_1, & \mathbf{x} \in \mathcal{W}_1 \\ \beta_2, & \mathbf{x} \in \mathcal{W}_2, \\ \beta_3, & \mathbf{x} \in \mathcal{W}_3, \\ \beta_4, & \mathbf{x} \in \mathcal{W}_4, \end{cases} \tag{1.21}$$

he wetting energy takes the form

$$-\int_{\mathcal{W}} \sigma\beta = -\sigma\beta_0 \int_{\partial\Omega} (u - d_0) - \sigma\beta_3[\text{area}(\Omega_1) - \text{area}(\Omega)]$$
$$- \sigma\beta_1 \int_{\partial\Omega_1} (u - d_0) - \sigma\beta_4 \,\text{area}(\Omega_2)$$
$$- \sigma\beta_2 \int_{\partial\Omega_2} u.$$

For the identification/organization of $\mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_2 \cup \mathcal{W}_1 \cup \mathcal{W}_4 \cup \mathcal{W}_3$, labeling Figure 1.2 based on (1.9)-(1.13) may be helpful; see Exercise 1.3.

As long as we remain within this family of two-graphical-meniscus configurations, the regions on the bottom of the tube and the floor of the container remain entirely wetted for all configurations under consideration corresponding to the simple additive constants

$$C_h = -\sigma\beta_3[\text{area}(\Omega_1) - \text{area}(\Omega)] - \sigma\beta_4 \,\text{area}(\Omega_2).$$

Similarly, the integral terms

$$C_v = \sigma\beta_0 \int_{\partial\Omega} d_0 + \sigma\beta_1 \int_{\partial\Omega_1} d_0 = \sigma\beta_0 d_0 \,\text{area}(\Omega) + \sigma\beta_1 d_0 \,\text{area}(\Omega_1)$$

may be considered constant corresponding to always wetted "virtual vertical cylinders" between $z = 0$ and $z = d_0$ over $\partial\Omega$ and $\partial\Omega_1$. By these considerations, the remaining variable terms (and hence the important terms in the wetting energy) take the form

$$-\sigma\beta_0 \int_{\partial\Omega} u - \sigma\beta_1 \int_{\partial\Omega_1} u - \sigma\beta_2 \int_{\partial\Omega_2} u,$$

and only the piecewise constant values $\beta(\mathbf{x}) = \beta_j$ for $\mathbf{x} \in \mathcal{W}_j$, $j = 0, 1, 2$ in (1.21) play a role.

Collecting the various expressions for the energies in terms of the meniscus functions u the total capillary energy becomes

$$\begin{aligned}\mathcal{E} = \sigma \int_{\Omega} \sqrt{1 + |Du|^2} + \sigma \int_{\Omega_{\text{out}}} \sqrt{1 + |Du|^2} \\ - \sigma\beta_0 \int_{\partial\Omega} u - \sigma\beta_1 \int_{\partial\Omega_1} u - \sigma\beta_2 \int_{\partial\Omega_2} u \\ + \frac{\rho\, g}{2} \int_{\Omega} u^2 + \frac{\rho\, g}{2} \int_{\Omega_{\text{out}}} u^2 + C_g + C_h + C_v. \qquad (1.22)\end{aligned}$$

Dividing by σ and subtracting the constants we arrive at a simpler energy quantity

$$\begin{aligned}\mathcal{F} &= \mathcal{E}_\sigma - (C_g + C_h + C_v)/\sigma \\ &= \int_\Omega \sqrt{1+|Du|^2} + \int_{\Omega_{\text{out}}} \sqrt{1+|Du|^2} \\ &\qquad - \beta_0 \int_{\partial\Omega} u - \beta_1 \int_{\partial\Omega_1} u - \beta_2 \int_{\partial\Omega_2} u \\ &\qquad\qquad + \frac{\kappa}{2} \int_\Omega u^2 + \frac{\kappa}{2} \int_{\Omega_{\text{out}}} u^2. \end{aligned} \tag{1.23}$$

the minimization of which is equivalent to the minimization of $\mathcal{E}$.

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In most cases, the capillary energy is augmented with a secondary condition associated with the assumption that a given volume of liquid is under consideration. In the capillary tubes problem as we have formulated it, such a **volume constraint** condition can take the form

$$\int_{\mathcal{V}} 1 = \int_\Omega u + d_0[\text{area}(\Omega_1) - \text{area}(\Omega)] + \int_{\Omega_{\text{out}}} u = V \tag{1.24}$$

where V is a prescribed constant. As noted above the volume of liquid in the container directly below the tube is fixed for all configurations under consideration so that (1.24) may be expressed in the form

$$\int_\Omega u + \int_{\Omega_{\text{out}}} u = C_V \tag{1.25}$$

where

$$C_V = V - d_0[\text{area}(\Omega_1) - \text{area}(\Omega)]$$

is a given constant. It is convenient to denote the **admissible class** of all configurations $\mathcal{C}$ having the basic geometric properties described above and satisfying (1.25) by $\mathcal{A}$. It is also convenient to let

$$\widetilde{\text{vol}} : \mathcal{A} \to \mathbb{R} \qquad \text{by} \qquad \widetilde{\text{vol}}[\mathcal{C}] = \int_\Omega u + \int_{\Omega_{\text{out}}} u$$

give the quanity on the left in (1.25) for these particular configurations so that $\widetilde{\text{vol}}[\mathcal{C}] = C_V$ is equivalent to $\text{vol}[\mathcal{V}] = V$.

Thus, at the suggestion of Gauss, we seek to find a configuration $\mathcal{C}_0 \in \mathcal{A}$ determining and determined by a meniscus function

$$u : U = \Omega \cup \Omega_{\text{out}} \to (0, \infty)$$

for which

$$\mathcal{E}[\mathcal{C}_0] \le \mathcal{E}[\mathcal{C}] \tag{1.26}$$

for all other configurations $\mathcal{C} \in \mathcal{A}$, that is

$$\mathcal{E}[\mathcal{C}_0] = \min_{\mathcal{C} \in \mathcal{A}} \mathcal{E}[\mathcal{C}]. \tag{1.27}$$

The family of configurations with prescribed volume is, generally speaking, still very large, and in order to bring the techniques of calculus to bear on a minimization condition like (1.26) or (1.27) one may focus on a finite dimensional subfamily of admissible configurations.

As an illustration of a one parameter family, let us imagine a meniscus function $u_0 : U = \Omega \cup \Omega_{\text{out}} \to (0, \infty)$ is given and is assumed to correspond to (and determine) a minimizing configuration $\mathcal{C}_0 \in \mathcal{A}$. For h a small number satisfying an estimate $|h| < \epsilon$ consider the competing meniscus function $u : U = \Omega \cup \Omega_{\text{out}} \to (0, \infty)$ given by

$$u(x, y) = \begin{cases} u_0(x, y) + h, & (x, y) \in \Omega \\ u_0(x, y) + k, & (x, y) \in \Omega_{\text{out}} \end{cases} \tag{1.28}$$

where $k = k(h)$ is chosen so that (1.25) holds, that is

$$\int_\Omega (u_0 + h) + \int_{\Omega_{\text{out}}} (u_0 + k) = C_V$$

or

$$k = -\frac{\text{area}(\Omega)}{\text{area}(\Omega_{\text{out}})}\ h$$

since $u_0 \in \mathcal{A}$ and consequently

$$\int_\Omega u_0 + \int_{\Omega_{\text{out}}} u_0 = C_V.$$

It follows that for $\epsilon > 0$ small enough (see Exercise 1.7) one obtains for each h with $|h| < \epsilon$ an admissible $\mathcal{C}_h \in \mathcal{A}$ with

$$\mathcal{E}[\mathcal{C}_0] \leq f(h) = \mathcal{E}[\mathcal{C}_h] \qquad \text{for} \qquad -\epsilon < h < \epsilon.$$

If the function $f : (-\epsilon, \epsilon) \to \mathbb{R}$ is differentiable, then there must hold

$$f'(0) = \frac{d}{dh}\mathcal{E}[\mathcal{C}_h]\Big|_{h=0} = 0$$

as a necessary condition of minimization from calculus. In fact, f is differentiable, and

$$\begin{aligned} f'(0) = \sigma \Bigg[& -\beta_0 \,\text{length}(\partial\Omega) - \frac{dk}{dh}\left[\beta_1 \,\text{length}(\partial\Omega_1) + \beta_2 \,\text{length}(\partial\Omega_2)\right] \\ & +\kappa \left(\int_\Omega u_0 + \frac{dk}{dh} \int_{\Omega_{\text{out}}} u_0 \right) \Bigg] \\ = \sigma \Bigg[& -\beta_0 \,\text{length}(\partial\Omega) + \frac{\text{area}(\Omega)}{\text{area}(\Omega_{\text{out}})}\left[\beta_1 \,\text{length}(\partial\Omega_1) + \beta_2 \,\text{length}(\partial\Omega_2)\right] \\ & +\kappa \left(\int_\Omega u_0 - \frac{\text{area}(\Omega)}{\text{area}(\Omega_{\text{out}})} \int_{\Omega_{\text{out}}} u_0 \right) \Bigg]. \end{aligned} \tag{1.29}$$

See Exercise 1.8. Notice the derivatives of the free surface energy terms in this calculation are zero.

The condition $f'(0) = 0$ with $f'(0)$ calculated above yields an interesting formula for the quantity

$$\int_\Omega u_0$$

which is sometimes referred to as the **lifted volume**. Specifically, we find a necessary condition on the lifted volume is given by

$$\int_\Omega u_0 = \frac{\text{area}(\Omega)}{\text{area}(\Omega_{\text{out}})} \left(\int_{\Omega_{\text{out}}} u_0 - \frac{1}{\kappa}\left[\beta_1 \,\text{length}(\partial\Omega_1) + \beta_2 \,\text{length}(\partial\Omega_2)\right] \right).$$

$$+ \qquad \frac{\beta_0}{\kappa}\ \text{length}(\partial\Omega) \tag{1.30}$$

See Exercises 1.8, 1.9, and 1.10. Roughly speaking (1.30) determines the lifted volume inside the capillary tube as a function of the adhesion (coefficient) on the inner boundary of the tube relative to an "outer height" of the liquid u_0 in Ω_{out} outside the circular tube. We will make this explanation more precise later, but notice that under what might be called the *Archimedean bath hypothesis* that $\beta_1 = \beta_2 = 0$ and $u_0 \equiv u_{\text{out}}$ (a positive constant) on Ω_{out} the term

$$\frac{\text{area}(\Omega)}{\text{area}(\Omega_{\text{out}})}\left(\int_{\Omega_{\text{out}}} u_0 - \frac{1}{\kappa}\left[\beta_1 \,\text{length}(\partial\Omega_1) + \beta_2 \,\text{length}(\partial\Omega_2)\right]\right).$$

of (1.30) becomes simply $u_{\text{out}}\ \text{area}(\Omega)$ so that

$$\int_{\Omega} u_0 - u_{\text{out}}\ \text{area}(\Omega)$$

may be immediately interpreted as the volume inside the tube above the level u_{out}.

As noted above, the condition

$$\frac{d}{dh}\mathcal{E}[\mathcal{C}_h]\Big|_{h=0} = 0$$

is equivalent to

$$\frac{d}{dh}\mathcal{F}[\mathcal{C}_h]\Big|_{h=0} = 0,$$

and in fact it may be observed that

$$f'(0) = \frac{d}{dh}\mathcal{E}[\mathcal{C}_h]\Big|_{h=0} = \sigma\frac{d}{dh}\mathcal{F}[\mathcal{C}_h]\Big|_{h=0}.$$

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1.4 First Variations

We now turn to what may be considered the main consequences of Gauss' hypothesis concerning energy in our special case of the capillary tubes problem with a particular admissible configuration $\mathcal{C}_0 \in \mathcal{A}$ minimizing among configurations determined by an inner meniscus and an outer meniscus given by the graph of a function $u_0 : U = \Omega \cup \Omega_{\text{out}} \to (0, \infty)$. The initial method is again to seek a family of competing admissible configurations depending on some finite number of parameters. In this case, the situation will be nominally a little more complicated, though we will again use two real parameters h and k with k dependent on h so that effectively we have a one parameter dependence. We concentrate first on the inner meniscus.

Let $\phi, \psi \in C_c^\infty(\Omega)$ with ϕ otherwise arbitrary and ψ satisfying

$$\int_\Omega \psi \neq 0.$$

See Appendix A for the meaning and notation associated with $C_c^\infty(\Omega)$. For small real numbers h and k with h satisfying an explicit estimate $|h| < \epsilon$, consider a potentially competing meniscus function

$$u = u_0 + h\phi + k\psi.$$

The volume

$$\widetilde{\text{vol}}[\mathcal{C}] = \int_\Omega u + \int_{\Omega_{\text{out}}} u$$

relative to the constant $C_V = V - \pi\,[\text{area}(\Omega_1) - \text{area}(\Omega)]$ associated to u is

$$\widetilde{\text{vol}}[\mathcal{C}] = C_V + h \int_\Omega \phi + k \int_\Omega \psi.$$

Thus, taking

$$k = k(h) = -\frac{\int_\Omega \phi}{\int_\Omega \psi}\, h$$

we obtain for each fixed ϕ a one parameter family of admissible (volume preserving) mensicus functions $u_0 + h\phi + k(h)\psi$ corresponding to admissible configurations $\mathcal{C} \in \mathcal{A}$. In this context, the family of functions

$$\{v = u_0 + h\phi + k(h)\psi\}_{|h|<\epsilon}$$

is called a **variation** of the unknown minimizer u_0 of the functional $\mathcal{E}$ and the difference

$$v - u_0 = h\phi + k(h)\psi$$

is called for each h a **perturbation**. Again minimization of $\mathcal{E}$ is equivalent to minimization of the simpler functional $\mathcal{F}$, and the function $f : (-\epsilon, \epsilon) \to \mathbb{R}$ by

$$f(h) = \mathcal{F}[\mathcal{C}]$$

has a minimum value $\mathcal{F}[\mathcal{C}_0]$ at $h = 0$, and if f is differentiable the usual necessary condition from one-variable calculus tells us

$$f'(0) = \frac{d}{dh}\mathcal{F}[\mathcal{C}]\Big|_{h=0} = 0.$$

In order to calculate the derivative in question, we begin by writing the value of the associated energy $\mathcal{F}[\mathcal{C}]$ using the expression

$$\begin{aligned} \mathcal{F} = \int_\Omega \sqrt{1+|Du|^2} + \int_{\Omega_{\text{out}}} \sqrt{1+|Du|^2} \\ - \beta_0 \int_{\partial\Omega} u - \beta_1 \int_{\partial\Omega_1} u - \beta_2 \int_{\partial\Omega_2} u \\ + \frac{\kappa}{2} \int_\Omega u^2 + \frac{\kappa}{2} \int_{\Omega_{\text{out}}} u^2 \end{aligned} \tag{1.31}$$

in (1.23) where we recall $\kappa = \rho g/\sigma$ is the **capillary constant**. The particular variation $u_0 + h\phi + k\psi$ we have constructed leaves the terms associated with anything in the closed exterior $\mathbb{R}^2\backslash\Omega$ constant, so setting $f(h) = \mathcal{F}[\mathcal{C}]$ we find

$$f'(h) = \frac{d}{dh}\left(\int_\Omega \sqrt{1+|Du|^2} + \frac{\kappa}{2}\int_\Omega u^2\right).$$

We take each derivative in turn: Note that $Du = Du_0 + hD\phi + kD\psi$. Therefore

$$\begin{aligned} \frac{d}{dh}\int_\Omega \sqrt{1+|Du|^2} &= \frac{d}{dh}\int_\Omega \sqrt{1+|Du_0+hD\phi+kD\psi|^2} \\ &= \int_\Omega \frac{(Du_0+hD\phi+kD\psi)\cdot D\phi}{\sqrt{1+|Du_0+hD\phi+kD\psi|^2}} \\ &\qquad + \int_\Omega \frac{(Du_0+hD\phi+kD\psi)\cdot D\psi}{\sqrt{1+|Du_0+hD\phi+kD\psi|^2}}\,\frac{dk}{dh} \\ &= \int_\Omega \frac{(Du_0+hD\phi+kD\psi)\cdot D\phi}{\sqrt{1+|Du_0+hD\phi+kD\psi|^2}} \\ &\qquad - \frac{\int_\Omega \phi}{\int_\Omega \psi}\int_\Omega \frac{(Du_0+hD\phi+kD\psi)\cdot D\psi}{\sqrt{1+|Du_0+hD\phi+kD\psi|^2}}. \end{aligned}$$

Note the differentiation under the integral sign here is justified because the difference quotient/integrand

$$\frac{1}{v}\left(\sqrt{1+|Du_0+(h+v)D\phi+k(h+v)\ D\psi|^2}\right.$$
$$\left.-\sqrt{1+|Du_0+hD\phi+k(h)\ D\psi|^2}\right)$$

converges pointwise uniformly to the derivative

$$\frac{(Du_0+hD\phi+kD\psi)\cdot D\phi}{\sqrt{1+|Du_0+hD\phi+kD\psi|^2}}-\frac{\int_\Omega \phi}{\int_\Omega \psi}\frac{(Du_0+hD\phi+kD\psi)\cdot D\psi}{\sqrt{1+|Du_0+hD\phi+kD\psi|^2}}$$

as v tends to 0. This differentiation under the integral sign may also be justified using the dominated convergence theorem; see Exercise 1.11.

The other integral is differentiated similarly:

$$\begin{aligned}\frac{d}{dh}\int_\Omega u^2 &= \frac{d}{dh}\int_\Omega (u_0+h\phi+k\psi)^2\\ &= \int_\Omega 2(u_0+h\phi+k\psi)\ \frac{\partial}{\partial h}(h\phi+k\psi)\\ &= 2\int_\Omega (u_0+h\phi+k\psi)\ \phi - 2\frac{\int_\Omega \phi}{\int_\Omega \psi}\int_\Omega (u_0+h\phi+k\psi)\ \psi.\end{aligned}$$

Evaluating at $h=0$ (which gives also $k=0$) and combining these expressions we obtain the necessary condition

$$0=f'(0)=\int_\Omega \frac{Du_0}{\sqrt{1+|Du_0|^2}}\cdot D\phi-\int_\Omega \lambda\ \phi+\kappa\int_\Omega u_0\ \phi \tag{1.32}$$

where

$$\lambda=\frac{1}{\int_\Omega \psi}\left(\int_\Omega \frac{Du_0\cdot D\psi}{\sqrt{1+|Du_0|^2}}+\kappa\int_\Omega u_0\ \psi\right). \tag{1.33}$$

The condition (1.32) should hold for every $\phi\in C_c^\infty(B_a(\mathbf{0}))$. In order to obtain more precise information about the minimizer u_0, observe that in light of the identity

$$\operatorname{div}\left(\phi\ \frac{Du_0}{\sqrt{1+|Du_0|^2}}\right)=\operatorname{div}\left(\frac{Du_0}{\sqrt{1+|Du_0|^2}}\right)\ \phi+\frac{Du_0}{\sqrt{1+|Du_0|^2}}\cdot D\phi$$

and the divergence theorem according to which

$$\int_{\Omega} \operatorname{div}\left(\phi \, \frac{Du_0}{\sqrt{1+|Du_0|^2}}\right) = \int_{\partial\Omega} \phi \, \frac{Du_0}{\sqrt{1+|Du_0|^2}} \cdot \mathbf{n} = 0$$

where $\mathbf{n} = (x,y)/\sqrt{x^2+y^2}$ is the outward unit conormal to $\partial\Omega$, one may write the first term on the right of (1.32) as

$$\int_{\Omega} \frac{Du_0}{\sqrt{1+|Du_0|^2}} \cdot D\phi = -\int_{\Omega} \operatorname{div}\left(\frac{Du_0}{\sqrt{1+|Du_0|^2}}\right) \phi$$

and (1.32) becomes

$$\int_{\Omega} \left[\operatorname{div}\left(\frac{Du_0}{\sqrt{1+|Du_0|^2}}\right) - (\kappa u_0 - \lambda)\right] \phi = 0 \tag{1.34}$$

for every $\phi \in C_c^\infty(B_a(\mathbf{0}))$. By the fundamental lemma of vanishing integrals[5] we arrive at the first main consequence of Gauss' suggestion:

Theorem 1 If $u_0 \in C^\infty(\overline{U})$ where $U = \Omega \cup \Omega_{\text{out}}$ corresponds to an admissible minimizing configuration $\mathcal{C}_0 \in \mathcal{A}$ for the capillary energy

$$\begin{aligned} \mathcal{E}[\mathcal{C}] &= \sigma\left(\operatorname{area}(\mathcal{S}) - \int_{\mathcal{W}} \beta + \frac{\rho\, g}{\sigma} \int_{\mathcal{V}} x_3 \, dx_3\right) \\ &= \sigma\left(\int_U \sqrt{1+|Du|^2} - \int_{\partial U} \beta u + \frac{\kappa}{2} \int_U u^2\right) + C, \end{aligned}$$

for the capillary tubes problem (where $\kappa = \rho g/\sigma$ and C is a constant) then there is some constant $\lambda \in \mathbb{R}$ such that on the inner tube region Ω the function u_0 satisfies the partial differential equation

$$\operatorname{div}\left(\frac{Du_0}{\sqrt{1+|Du_0|^2}}\right) = \kappa u_0 - \lambda. \tag{1.35}$$

The equation (1.35) is often referred to as the **capillary equation**. The following may be evident at this point concerning solutions of (1.35):

[5] a.k.a. the fundamental lemma of the calculus of variations; see Exercise 1.12.

1. Solutions of (1.35) should be considered as **possible** minimizers of the capillary energy of Gauss and potential candidates for modeling the actual physical meniscus observed in a vertical capillary tube.

2. It is worth studying solutions $u \in C^2(U)$ of the equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \kappa u - \lambda \tag{1.36}$$

where U is an open connected subset of $\mathbb{R}^2$ (or perhaps even $U \subset \mathbb{R}^n$) in general and for all values of the constants κ and λ. As a corollary, it is worth understanding the meaning of the second order partial differential operator $M : C^2(U) \to C^0(U)$ given by

$$Mu = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right). \tag{1.37}$$

In short, (1.36) is an important PDE and the operator M given in (1.37) is an important partial differential operator.

At any rate we will now take these two assertions as given. The operator M is not linear like the (perhaps) familiar spatial Laplacian

$$\Delta u = \operatorname{div}(Du) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

It is not necessarily expected that the reader is familiar with the properties of the Laplace operator or PDEs like

$$\begin{array}{rcll} \Delta u & = & 0 & \quad\text{(Laplace's equation) or} \\ \Delta u & = & f & \quad\text{(Poisson's equation)} \end{array}$$

but certain aspects of the approach presented for consideration of the capillary equation below are naturally considered (and even motivated by) the extensive and in some ways simpler theory of these linear PDE. Accordingly, we will not hesitate to digress and discuss the properties of these equations in comparison with the main theory developed for (1.36). See Exercises 1.13 through 1.16 for some initial elementary comparisons. Note that in Poisson's equation f is a given function of the **spatial variables**, e.g., x and y in $\mathbb{R}^2$. The equation

$$\Delta u = \kappa u - \lambda \tag{1.38}$$

where κ and λ are constants may also be considered for comparison. In this case, the constants κ and λ should not be connected to the physical meanings (as the capillary constant and a Lagrange parameter for volume) but should simply be taken as given constants. In fact, in the special case $\lambda = 0$, the constant κ is often taken as an eigenvalue for the Laplace operator, and in this case is considered an additional unknown in the problem. In this context, the PDE $\Delta u = \kappa u$ is sometimes called Helmholtz' equation. Roughly speaking the eigenvalue problem is of interest for the Laplace operator because the operator is linear. Since the operator M is not linear the eigenvalue problem is not of particular interest, and in our comparison to the Laplace operator we will simply consider κ a given and known constant.

The nonlinear operator M also has a name. M is called the **mean curvature operator** and features in some other well-known PDE:

$$\begin{aligned} Mu &= 0 && \text{(minimal surface equation)} \\ Mu &= 2H \text{ (constant)} && \text{(constant mean curvature equation)} \\ Mu &= f && \text{(prescribed mean curvature equation)} \end{aligned}$$

The minimal surface equation is of course a special case of the equation of constant mean curvature, which in turn may be thought of as a kind of "zero gravity" special case of the capillary equation. The minimal surface equation is also associated with soap films, and the equation of constant mean curvature is also associated with soap bubbles, or soap films with a nonzero pressure difference across the film. In fact, curved liquid interfaces may be considered to model a differential in pressure across the interface, and the capillary equation may be interpreted to express that the pressure across the interface changes as an affine function of height.

Let us consider here for a moment the geometric meaning of the value

$$Mu = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right).$$

We have called this expression a curvature. Indeed, if $u(x, y) = u(x)$ is the graph of a cylindrical surface, then

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right). \tag{1.39}$$

On the one hand, this expression may be recognized as the familiar formula

$$\frac{u''}{(1+u'^2)^{3/2}}$$

for the curvature of the curve which is the graph of $u = u(x)$. See Exercise 1.17. On the other hand,

$$\psi = \sin^{-1}\left(\frac{u'}{\sqrt{1+u'^2}}\right) \in (-\pi/2, \pi/2)$$

is the inclination angle of the graph of u. See Exercise 1.18. One definition of the signed curvature of the graph of $u \in C^2(a,b)$ for $a, b \in \mathbb{R}$ with $a < b$ is

$$k = \frac{d\psi}{ds}$$

where s is an arclength parameter. In particular, up to an additive constant

$$s = \int_{x_0}^{x} \sqrt{1+u'(\xi)^2}\, d\xi$$

so

$$\begin{aligned} \frac{d\psi}{ds} &= \frac{d\psi}{dx}\,\frac{dx}{ds} \\ &= \frac{d\psi}{dx}\,\frac{1}{1+u'(x)^2} \\ &= \frac{d\psi}{dx}\ \cos\psi \\ &= \frac{d}{dx}\sin\psi \\ &= Mu \end{aligned}$$

for the cylindrical graph of $u = u(x)$ by (1.39). Cylindrical capillary surfaces of the form

$$\{(x, y, u(x)) : x \in (a,b),\ y \in \mathbb{R}\}$$

are considered in more detail in Section 1.8 below. More complicated surfaces can have more complicated curvature.

1.5 Digression on curvature of surfaces

Exercise 1.17 gives some basics about the curvature of a C^2 curve in a plane. Such a curve can be expressed locally near each point P_0 as a graph over the tangent line to the curve at P_0. For a C^2 curve in $\mathbb{R}^3$, the situation can be somewhat more complicated. In order to capture this potentially more complicated geometric situation, it is convenient to associate with each point on a C^2 curve not only a number that measures the curvature but also a vector. For the graph of a function $u : (a, b) \to \mathbb{R}$ given on an interval (a, b) with $a, b \in \mathbb{R}$ and $a < b$ and satisfying $u'(x_0) = 0$, this vector at the point $(x_0, u(x_0))$, the **curvature vector**, is taken to be $u''(x_0)(0, 1)$. In the case of the local parametric representation

$$\gamma(t) = (x_0, v(x_0)) + t\ \frac{(1, v'(x_0))}{\sqrt{1 + v'(x_0)^2}} + g(t)\ \frac{(-v'(x_0), 1)}{\sqrt{1 + v'(x_0)^2}}$$

given in part **(c)** of Exercise 1.17 the curvature vector at $\gamma(0) = (x_0, v(x_0))$ is taken to be

$$\gamma''(0) = g''(t)\ \frac{(-v'(x_0), 1)}{\sqrt{1 + v'(x_0)^2}}.$$

Note this vector is orthogonal to the unit tangent vector

$$\frac{(1, v'(x_0))}{\sqrt{1 + v'(x_0)^2}}$$

and has length the absolute value of the signed curvature. For a general parameterized curve it is convenient to require an additional condition in order to make sense of the curvature at a point. The reason for this additional condition is illustrated by the C^2 curve parameterized by $\alpha(t) = (t^3, t^2)$ when considered at the point $(0, 0)$; see Exercise 1.19. The condition is called "regular parameterization." Given $a, b \in \mathbb{R}$ with $a < b$ and $n \in \{1, 2, 3, \ldots\}$, a curve

$$\Gamma = \{\alpha(t) : t \in (a, b)\}$$

where $\alpha \in C^2((a, b) \to \mathbb{R}^n)$ is said to be **regularly parameterized** by α if $|\alpha'(t)| \neq 0$ for $t \in (a, b)$. A regularly paraemeterized C^2 curve may be reparameterized by arclength as follows: We set

$$s = \int_{x_0}^{t} |\alpha'(t)|\, dt \tag{1.40}$$

where x_0 is some point in (a, b). This relation defines s as a differentiable function of t with

$$\frac{ds}{dt} = |\alpha'(t)| > 0.$$

In particular, $s : (a,b) \to s(a,b) = (-\ell, m)$ is bijective onto some interval $(-\ell, m)$ with $-\ell < 0 < m$, differentiable and strictly increasing. Therefore s has an increasing differentiable inverse traditionally denoted by $t = s^{-1} : (-\ell, m) \to (a,b)$ with $t = t(s)$. In fact, both $s : (a,b) \to (-\ell, m)$ and $t : (-\ell, m) \to (a,b)$ are twice differentiable, and setting

$$\gamma(s) = \alpha \circ t(s)$$

we obtain a regular parameterization $\gamma \in C^2((-\ell, m) \to \mathbb{R}^n)$ of Γ. Differentiating the relation (1.40) again with s as the independent variable, one finds

$$\frac{dt}{ds} = \frac{1}{|\alpha'(t)|} = \frac{1}{|\alpha' \circ t(s)|}.$$

Consequently,

$$\frac{d\gamma}{ds} = \dot{\gamma} = \frac{\alpha'(t)}{|\alpha'(t)|} \qquad \text{with} \qquad \left|\frac{d\gamma}{ds}\right| \equiv 1,$$

and we define the **curvature vector** of Γ at $\alpha(t) = \gamma(s)$ to be the vector

$$\vec{k} = \ddot{\gamma}(s) = \frac{d^2\gamma}{ds^2}(s).$$

Thus, the curvature (vector) of a space curve is **the rate of change**

$$\frac{d}{ds}\,\dot{\gamma}$$

of the unit tangent $\dot{\gamma}$ with respect to arclength along the curve. In addition, the curvature vector is orthogonal to the unit tangent since $|\dot{\gamma}| \equiv 1$ and

$$0 = \frac{d}{ds}|\dot{\gamma}|^2 = \frac{d}{ds}(\dot{\gamma} \cdot \dot{\gamma}) = 2\ddot{\gamma} \cdot \dot{\gamma}.$$

Thus, the unit tangent vector at least potentially defines a normal vector to the curve. Under the current assumptions of a regular curve (or a curve admitting a regular parameterization) we leave open the possibility that that $\vec{k} = \ddot{\gamma} = \mathbf{0}$ is the zero vector. In this case, of course, $\vec{k}$ does not determine a specified normal vector to the curve. Otherwise, it is traditional to set

$$k = |\vec{k}| > 0$$

and define this non-negative number to be the (scalar or numerical) curvature of the curve, and define the **principal normal** at $\gamma(s)$ to be

$$\mathbf{n} = \frac{\vec{k}}{k}. \tag{1.41}$$

The number $k = |\vec{k}|$, however, is not always the same as the signed curvature k discussed above for planar curves.

If we do specialize once again to the special case of planar curves $\gamma = (\gamma_1, \gamma_2) \in C^2((-\ell, m) \to \mathbb{R}^2)$ and assume $\ddot{\gamma}(s_0) \neq 0$, then there is some $\epsilon > 0$ for which

$$\mathbf{n}(s) = \frac{\ddot{\gamma}(s)}{|\ddot{\gamma}(s)|}$$

is well-defined for $s_0 - \epsilon < s < s_0 + \epsilon$, and there must hold either

$$\mathbf{n}(s) = \dot{\gamma}^\perp = (-\dot{\gamma}_2, \dot{\gamma}_1) \qquad \text{for } s_0 - \epsilon < s < s_0 + \epsilon$$

or

$$\mathbf{n}(s) = -\dot{\gamma}^\perp = (\dot{\gamma}_2, -\dot{\gamma}_1) \qquad \text{for } s_0 - \epsilon < s < s_0 + \epsilon.$$

In either case, $\mathbf{n} \in C^1((s_0 - \epsilon, s_0 + \epsilon) \to \mathbb{R}^2)$, and we find the interesting relation

$$\dot{\mathbf{n}} = -k\,\dot{\gamma}. \tag{1.42}$$

See Exercise 1.20. Comparing the two relations (1.41) and (1.42), that is

$$\begin{aligned} \frac{d}{ds}\,\dot{\gamma} &= \quad k\,\mathbf{n}, \qquad \text{and} \\ \frac{d}{ds}\,\mathbf{n} &= -\,k\,\dot{\gamma}, \end{aligned} \tag{1.43}$$

we see (1.42) is a kind of parallel and alternative formulation of the fundamental curvature relation (1.41). In particular, curvature may also be expressed alternatively in terms of **the rate of change of the unit principal normal vector with respect to arclength along the curve**.

Note: The alternative formulation of curvature of planar curves encapsulated in (1.43) is especially important, or at least useful, to understand in relation to the formulation of curvature for a two-dimensional surface in $\mathbb{R}^3$. In that case instead of a regular parameterization $\alpha \in C^2((a, b) \to \mathbb{R}^2)$ one typically has a regular local parameterization $X \in C^2(U \to \mathbb{R}^3)$ of a surface $\mathcal{S}$ defined on some open set $U \subset \mathbb{R}^2$. In the latter case there is no specified tangent vector directly comparable to $\dot{\gamma}$ but rather one has a well-defined tangent plane at each point on the surface. Consequently one alternative for the formulation of curvature, at least at an intuitive or heuristic level is as a measure of the "rate" of change of the tangent plane with respect to arclength measured within the surface. On the other hand, for a surface $\mathcal{S}$ there is a direct analog N of a (specified) unit normal and/or the principal unit normal $\mathbf{n} \in C^1((s_0 - \epsilon, s_0 + \epsilon) \to \mathbb{R}^2)$ to a curve Γ, and from the quantitative or technical point of view formulating curvature as encapsulating the rate of change of a surface normal $N \in C^1(U \to \mathbb{R}^3)$ with respect to unit speed motion on the surface is usually the simpler and more desirable alternative.

Before I attempt to extend this discussion of the curvature of surfaces further and in more detail, let me attempt to wrap up my discussion of the curvature of space curves and leave it in a more complete state.

The situation with the principal normal $\mathbf{n}$ to a space curve can be somewhat more complicated than that considered above for a planar curve. Let us say a space curve Γ is regularly parameterized by arclength on an interval $(-\ell, m)$ as described above by $\gamma \in C^2((-\ell, m) \to \mathbb{R}^3)$. In most expositions two additional assumptions are made concerning the parameterization γ. The first is $\ddot{\gamma} \neq \mathbf{0} \in \mathbb{R}^3$ so the principal unit normal is well-defined. I made this assumption locally in our discussion of planar curves above in the form $\ddot{\gamma}(s_0) \neq \mathbf{0} \in \mathbb{R}^2$. One reason for attempting greater generality than usual is that ultimately I am primarily interested in curves on a surface $\mathcal{S}$, and in certain instances it is the case (and it is important to notice it is the case) that the curvature of the surface itself forces the condition $\ddot{\gamma} \neq \mathbf{0}$ at every point on every regularly parameterized curve on $\mathcal{S}$. Thus, for these curves the condition $\ddot{\gamma} \neq \mathbf{0}$ should not be an assumption but rather a necessary consequence of being on the surface $\mathcal{S}$.

The second "usual" assumption is $\gamma \in C^3((-\ell, m) \to \mathbb{R}^3)$. It is somewhat interesting that no additional regularity assumption was needed to obtain the fundamental relation $\dot{n} = -k\,\dot{\gamma}$ in the case of planar curves, and I would like to make some attempt to understand to what extent this usual assumption is required (or perhaps just convenient) in the case of space curves. All (three) of these conditions are referred to as "regularity" for a space curve. Since I have taken it upon myself to consider somewhat greater generality than usual, I will give these conditions separate names: Let $\Gamma \subset \mathbb{R}^3$ be parameterized by

$\alpha \in C^2((a,b) \to \mathbb{R}^3)$ where $a, b \in \mathbb{R}$ with $a < b$. The curve Γ is said to have

1. a **regular parameterization** locally at $\alpha(t_0)$ for some $t_0 \in (a,b)$ if for some $\epsilon > 0$ there holds $\alpha'(t) \neq 0$ for $t_0 - \epsilon < t < t_0 + \epsilon$.

2. nonvanishing curvature locally at $\alpha(t_0)$ if Γ is regularly parameterized by α on $(t_0 - \epsilon, t_0 + \epsilon) \subset\subset (a,b)$, regularly parameterized by $\gamma : (-\ell, m) \to \mathbb{R}^3$ with $-\ell < 0 < m$,

$$\gamma(-\ell) = \alpha(t_0 - \epsilon), \qquad \gamma(0) = \alpha(t_0), \qquad \gamma(m) = \alpha(t_0 + \epsilon),$$

and

$$\ddot{\gamma} = \frac{d^2\gamma}{ds^2} \neq \mathbf{0}.$$

3. a C^3 **parameterization** if $\alpha \in C^3((a,b) \to \mathbb{R}^3)$.

If we begin with all three of these assumptions, the principle normal $\mathbf{n} = \mathbf{n}(s)$ is well-defined for $-\ell < s < m$ and one may calculate the derivative

$$\frac{d}{ds}\mathbf{n}.$$

In this case, we obtain a relation

$$\frac{d}{ds}\mathbf{n} = -k\ \dot{\gamma} + \tau\ (\dot{\gamma} \times \mathbf{n}),$$

which can be immediately seen to be more complicated than (1.42). See Exercise 1.21. With this calculation in mind, we recall that for planar C^2 curves satisfying the first two conditions, the third condition is not necessary for the differentiation of the principal unit normal $\mathbf{n}$ with respect to arclength. Let me add this as a fourth condition:

4. A C^2 space curve Γ with local regular parameterization

$$\gamma \in C^2((-\ell, m) \to \mathbb{R}^3)$$

by arclength near $\gamma(0) = \alpha(t_0)$ and nonvanishing curvature and well-defined principal unit normal $\mathbf{n}(s)$ at $\gamma(s)$ satisfying

$$\mathbf{n} \in C^0(-\ell, m) \to \mathbb{R}^3)$$

is said to have a **differentiable principal normal** at $\gamma(0) = \alpha(t_0)$ if the derivative

$$\dot{\mathbf{n}}(0) = \frac{d\mathbf{n}}{ds}(0) = \lim_{s\to 0} \frac{\mathbf{n}(s) - \mathbf{n}(0)}{s}$$

exists.

I wish to construct an example for which the first two conditions (regular C^2 parameterization and nonvanishing curvature) hold but for which the well-defined principal unit normal is not differentiable. The example is given by $\alpha \in C^2(\mathbb{R} \to \mathbb{R}^3)$ with

$$\alpha(t) = \begin{cases} (t, t^2/2, -t^3/6), & t \le 0 \\ (t, t^2/2, t^3/6), & t \ge 0. \end{cases} \tag{1.44}$$

See Exercises 1.22 through 1.25. In spite of the example determined by (1.44), one is still inclined to believe some manifestation of the differentiability of the principal normal of plane curves persists for space curves of nonvanishing curvature so that the curvature function $k = |\ddot{\gamma}|$ can be recovered by differentiating some quantity obtained from the principle normal. Here is one possibility:

Conjecture 1 *Given a C^2 curve Γ with nonvanishing curvature and regular parameterization $\gamma \in C^2((-\ell, m) \to \mathbb{R}^3)$ by arclength, the unit vector field*

$$\nu(s) = \frac{(\mathbf{n}(s)\cdot\dot{\gamma}(0))\ \dot{\gamma}(0) + (\mathbf{n}(s)\cdot\mathbf{n}(0))\ \mathbf{n}(0)}{|(\mathbf{n}(s)\cdot\dot{\gamma}(0))\ \dot{\gamma}(0) + (\mathbf{n}(s)\cdot\mathbf{n}(0))\ \mathbf{n}(0)|}$$

obtained by projecting the principal normal onto the osculating plane spanned by $\dot{\gamma}(0)$ and $\mathbf{n}(0)$ satisfies

$$\dot{\nu}(0) = \frac{d\nu}{ds}(0) = \lim_{s\to 0} \frac{\nu(s) - \nu(0)}{s}$$

exists and $\dot{\nu}(0) = -k\ \dot{\gamma}(0)$.

At any rate we will henceforth primarily restrict attention to C^2 curves admitting a regular parameterization $\gamma \in C^2((-\ell, m) \to \mathbb{R}^n)$ for some $\ell, m > 0$ and some $n \in \{1, 2, 3\}$ satisfying $|\dot{\gamma}| \equiv 1$. Thus, in most cases, it will be natural to begin with such a parameterization though the notation may change.

closed curves

We have focused on a curve $\Gamma \subset \mathbb{R}^n$, mostly for $n = 2, 3$, given parametrically by a function $\alpha : (a, b) \to \mathbb{R}^n$ defined on an interval $(a, b) \subset \mathbb{R}$. Among the conditions satisfied by such a parameterized curve it is often required that $\alpha(t_1) \neq \alpha(t_2)$ for $a < t_1 < t_2 < b$. When this condition holds, the curve is said to be **embedded**. Roughly speaking a curve that may not satisfy the embeddedness condition is said to be immersed. It can be a little difficult to give a precise definition of an immersed curve as a subset of $\mathbb{R}^n$ as suggested in Exercise 1.26. A curve essentially defined by a specific single parameterization $\alpha : (a, b) \to \mathbb{R}^n$ is said to be **strictly immersed** if there exist $t_1 \neq t_2$ in (a, b) with $\alpha(t_1) = \alpha(t_2)$. In order to allow the possibility of a smooth **closed curve**, we can allow some condition like the following: There exists some $\epsilon > 0$ with $\epsilon < b - a$ such that

(i) $\alpha(t_1) \neq \alpha(t_2)$ for $a < t_1 < t_2 < b - \epsilon$, and

(ii) $\alpha(t) = \alpha(b - \epsilon + t - a) = \alpha(b - a - \epsilon + t)$ for $a < t < a + \epsilon$.

Technically, if $\Gamma = \{\alpha(t) : a < t < b\}$ *for some* $\alpha \in C^1((a, b) \to \mathbb{R}^n)$ satisfying **(i)** and **(ii)**, we say Γ is a **simple closed curve**. See Exercise 1.28.

An important fact about a (simple) closed curve Γ in $\mathbb{R}^2$ is contained in the famous Jordan curve theorem which asserts that $\mathbb{R}^2 \backslash \Gamma$ consists of precisely two connected components C_1 and C_2 exactly one of which, say C_1, is bounded in $\mathbb{R}^2$ in the sense that the entire component C_1 is a subset of a ball $B_R(\mathbf{0})$ for some $R > 0$ and exactly one of which, say C_2 is unbounded in the sense that C_2 is not a subset of any ball $B_M(\mathbf{0})$ for any $M > 0$, and both components are bounded by Γ in the set/topological sense that $\partial C_1 = \partial C_2 = \Gamma$. There is a nice generalization of the Jordan curve theorem to higher dimensions. This result is referred to as the Jordan-Brouwer separation theorem in the next section.

1.5.1 a formal definition of surface

Having gotten a taste of what can happen and how the curvature of curves works when regularity may be limited, we turn to our main topic which is understanding something about the curvature of a surface by considering certain C^∞ curves lying in or upon that surface. We begin with a definition:

Definition 1 (surface) A set $\mathcal{S} \subset \mathbb{R}^3$ is a (smooth compact embedded) surface (with boundary) if $\overline{\mathcal{S}}$ is compact and for each $p \in \overline{\mathcal{S}}$ there exists some open set $V \subset \mathbb{R}^3$ with $p \in V$, some open set $U \subset \mathbb{R}^2$ with $\mathbf{0} \in U$, and a function $X = (X_1, X_2, X_3) \in C^\infty(U \to \mathbb{R}^3)$ with $X(\mathbf{0}) = p$ such that

(i) $X : U \to X(U)$ is a homeomorphism,

(ii) $X_u \times X_v \neq 0$, where

$$X_u = \frac{\partial X}{\partial u} = \left(\frac{\partial X_1}{\partial u}, \frac{\partial X_2}{\partial u}, \frac{\partial X_3}{\partial u} \right), \qquad \text{and}$$
$$X_v = \frac{\partial X}{\partial v} = \left(\frac{\partial X_1}{\partial v}, \frac{\partial X_2}{\partial v}, \frac{\partial X_3}{\partial v} \right),$$

and the following conditions hold:

1. (interior points) If $p \in \mathcal{S}$

(iii) $\mathcal{S} \cap V = X(U) = \{X(u,v) : (u,v) \in U\}$.

2. (smooth boundary points) If $p \in \partial\mathcal{S} = \overline{\mathcal{S}} \backslash \mathcal{S}$ with the possible exception of a finite number of points $\{q_1, q_2, \ldots, q_k\} \subset \partial\mathcal{S}$,

(iv) $\mathcal{S} \cap V = X(U^+) = \{X(u,v) : (u,v) \in U,\ v > 0\}$,

(v) $X : U \cap \{(u,0) : u \in \mathbb{R}\} \to \partial\mathcal{S} \cap V$ determines a regular parameterization of a C^∞ curve in $\mathbb{R}^3$ by $\alpha(u) = X(u,0)$.

3. (corner points) If $p = q_j$, $j = 1, 2, \ldots, k$,

(vi) $\mathcal{S} \cap V = X(U^{++}) = \{X(u,v) : (u,v) \in U,\ u, v > 0\}$,

(vii) $X : U \cap \{(u,0) : u > 0\} \to \partial\mathcal{S} \cap V$ determines a regular parameterization of a C^∞ curve in $\mathbb{R}^3$ by $\alpha(u) = X(u,0)$, and

(viii) $X : U \cap \{(0,v) : v > 0\} \to \partial\mathcal{S} \cap V$ determines a regular parameterization of a C^∞ curve in $\mathbb{R}^3$ by $\beta(v) = X(0,v)$.

Notes: The topological context of the use of the symbol $\partial\mathcal{S}$ in the case of a surface and some of the accompanying terminology should be clarified. In Definition 1 the **boundary** $\partial\mathcal{S}$ is not the topological boundary of $\mathcal{S}$ with respect to $\mathbb{R}^3$ but rather the topological boundary of $\overline{\mathcal{S}}$ as a topological (metric) subspace of $\mathbb{R}^3$. The closure $\overline{\mathcal{S}}$, however, does denote the closure of

$\mathcal{S}$ as a subset of $\mathbb{R}^3$. If one refers to the **interior** of a surface $\mathcal{S}$, again, this does not refer to the interior of $\mathcal{S}$ as a subset of $\mathbb{R}^3$ but rather the interior of $\mathcal{S}$ as a topological subspace of $\mathbb{R}^3$ or simply as the set $\mathcal{S}$ itself in our definition. Consequently, when $\mathcal{S}$ is referred to as a **compact surface with boundary** it is not intended that $\mathcal{S}$ is necessarily compact nor that $\partial\mathcal{S} \subset \mathcal{S}$, but only that $\overline{\mathcal{S}}$ is compact. Of course in the case $\overline{\mathcal{S}} = \mathcal{S}$, i.e., when the surface boundary is empty, then such a surface $\mathcal{S}$ is required to be compact in our definition.[6,7]

Technically, it may be necessary to specify in Definition 1 that $\partial\mathcal{S} \subset \overline{\mathcal{S}}\backslash\mathcal{S}$. In short, $\mathcal{S}$ may be referred to as an **open surface** without boundary,[8] and the surface boundary of $\mathcal{S}$, the points of which satisfy conditions **2.** and **3.** of the definition, is given explicitly by $\partial\mathcal{S} = \overline{\mathcal{S}}\backslash\mathcal{S}$.

In spite of our somewhat lengthy discussion of curves a formal definition of an embedded regular curve as a subset of $\mathbb{R}^n$ analogous to the definition of a surface above was never given. See Exercise 1.26. It may be noted that the definition of Exercise 1.26 is used at least implicitly in conditions **2(v)**, **3(vii)** and **3(viii)**.

In the case where the surface boundary is empty the Jordan-Brouwer separation theorem holds, that is $\mathbb{R}^3\backslash\mathcal{S}$ consists of precisely two connected

[6] Unfortunately, one doesn't often have the luxury of avoiding surfaces with boundary in problems of mathematical capillarity; if a free surface interface without boundary is encountered, this essentially means one is dealing with a free floating drop(let) having no contact with any rigid support structures (tubes, walls, containers, etc.). In the case of zero gravity such a liquid drop in equilibrium is usually modeled by a rounds sphere. This is a famous result of A.D. Alexandrov from 1955. It is an interesting result, and we will discuss it when we discuss capillary surfaces in zero gravity, but from some point of view that is a rather special case.

[7] The condition that $\overline{\mathcal{S}}$ is compact may also be dropped from the definition altogether to obtain a more general definition of surface with boundary. For the physical modeling of liquid interfaces in mathematical capillarity, the restriction to compact surfaces strikes this author as appropriate. This is for the simple reason that I have never encountered a physical volume of liquid I perceive to be infinite in extent. There are other mathematical motivations for this restriction as well. Most authors have found it convenient to include mathematical surfaces of infinite extent, and to imagine such surfaces are appropriate to model capillary interfaces in some vague manner that for the most part has been able to avoid serious scrutiny or any form of actual quantitative comparison for that matter, though there are a couple notable exceptions.

[8] The formal definition of an open surface is obtained from the definition above by simply leaving out the discussion of $\overline{\mathcal{S}}\backslash\mathcal{S}$ and conditions **2.** and **3.** in particular. In the context of our usage, namely physical modeling of capillary surfaces, one may also wish to retain the condition that $\overline{\mathcal{S}}$ is a compact subset of $\mathbb{R}^3$.

components C_1 and C_2 exactly one of which, say C_1, is bounded in $\mathbb{R}^3$ in the sense that the entire component C_1 is a subset of a ball $B_R(\mathbf{0})$ for some $R > 0$ and exactly one of which, say C_2 is unbounded in the sense that C_2 is not a subset of any ball $B_M(\mathbf{0})$ for any $M > 0$, and both components are bounded by $\mathcal{S}$ in the set/topological sense that $\partial C_1 = \partial C_2 = \mathcal{S}$. This generalization holds for hypersurfaces ($n-1$ dimensional submanifolds) without boundary in $\mathbb{R}^n$ for $n = 4, 5, \ldots$, as well, but as mentioned briefly above, in most problems the free surface interface has nontrivial boundary and fits together with various wetted regions to enclose a set modeling a liquid volume. In short the situation can be complicated as suggested by Exercises 1.1 and 1.2.

1.5.2 smooth curves on a surface

At each interior point $p \in \mathcal{S}$, we can take a local parameterization $X : U \to \mathbb{R}^3$ of $\mathcal{S}$ with $X(0,0) = p$ as in Definition 1. Recall that there is an open set $V \subset \mathbb{R}^3$ in this case for which $X : U \to X(U) = \mathcal{S} \cap V$ is a homeomorphism. It follows that for all $r > 0$ small enough

$$X : X^{-1}(B_r(p) \cap \mathcal{S}) \to B_r(p) \cap \mathcal{S}$$

is a homeomorphism. Thus, one way to express the points $X \in B_r(p) \cap \mathcal{S}$ is as the image $X(u,v)$ for some points (u,v) in an open set U containing $\mathbf{0} = (0,0)$ in $\mathbb{R}^2$. In particular, the local parameterization X determines a correspondence between points near p in $\mathcal{S}$ and the points in an open subset of $\mathbb{R}^2$.

Also in this case there is a well-defined tangent plane associated with $p = X(0,0)$ given by

$$T_p\mathcal{S} = \text{span}\{X_u(0,0), X_v(0,0)\}$$

where we note the vectors $X_u(0,0)$ and $X_v(0,0)$ are linearly indepenent in $\mathbb{R}^3$ by virtue of condition **(ii)** in Definition 1. Thus, $T_p\mathcal{S}$ is a two dimensional (abstract) vector space and/or a two-dimensional vector subspace of $\mathbb{R}^3$ with basis $\{X_u(0,0), X_v(0,0)\}$. The affine plane (sometimes called an "affine subspace")

$$\mathcal{T}_p = \{p + aX_u(0,0) + bX_v(0,0) : a, b \in \mathbb{R}\}$$

in $\mathbb{R}^3$ is also a surface of interest in comparison with $\mathcal{S}$. The local parameterization X also gives a correspondence between vectors in $T_{\mathbf{0}}\mathbb{R}^2$ and vectors in $T_p\mathcal{S}$. Specifically, $dX_{\mathbf{0}}(a,b) = aX_u + bX_v$ where $X_u = X_u(0,0)$ and $X_v = X_v(0,0)$.

Most of the basic assertions concerning the local representation of the surface $\mathcal{S}$ and the associated tangent plane can be adapted to any point $X = X(u,v) \in \mathcal{S}$ given by any local parameterization. We will use and refer to this correspondence freely in that generality below, and distinction between X, X_u, and X_v evaluated at $\mathbf{0} \in \mathbb{R}^2$ or a general point $(u,v) \in U \subset \mathbb{R}^2$ may need to be determined by context. Generally, the use of p and $T_p\mathcal{S}$ indicate evaluation at $\mathbf{0}$, while X and $T_X\mathcal{S}$ suggest the more general correspondence.

There are many smooth (space) curves Γ passing through an interior point p on a surface. Locally, these curves are also in one-to-one correspondence with curves Γ_0 passing through $(0,0) \in \mathbb{R}^2$. For the moment, it will be convenient to assume such a curve $\Gamma_0 \subset \mathbb{R}^2$ is parameterized by arclength so that the correspondence of curves is given by $\alpha(t) = X \circ \gamma_0(t)$ where $\gamma_0 \in C^\infty((-\epsilon,\epsilon) \to \mathbb{R}^2)$ for some $\epsilon > 0$ is a parameterization of Γ_0 by arclength satisfying $\gamma_0(0) = (0,0)$ and the usual conditions for a regular plane curve.

Exercise 1.29 suggests there are many smooth curves, with many different curvature vectors, passing through $\mathbf{0} = (0,0) \in \mathbb{R}^2$. In order to understand something about the smooth curves passing through a point p on a surface $\mathcal{S}$, we might first wish to find a formula for the curvature vector $\ddot{\gamma}(0)$ of the curve determined by $\alpha(s) = X \circ \gamma_0(s)$ for $s \in (-\epsilon,\epsilon)$. Presumably this curvature value should be determined by the direction $\dot{\gamma}_0(0)$ and the curvature vector $\ddot{\gamma}_0(0)$ of Γ_0, but the dependence may be complicated. First of all we have $\alpha'(s) = DX\ \dot{\gamma}_0(s)$, and the arclength along $\Gamma \subset \mathcal{S}$ is given by

$$\sigma = \int_0^t |DX\ \dot{\gamma}_0(s)|\, ds \tag{1.45}$$

where $DX = DX \circ \gamma_0(s)$. From this we conclude

$$\frac{dt}{d\sigma} = \frac{1}{|DX\ \dot{\gamma}_0(t)|}$$

which is well-defined because $dX : T_{\mathbf{0}}\mathbb{R}^2 \to T_p\mathcal{S}$ is nonsingular, and as usual we have an arclength parameterization of Γ given by $\gamma(\sigma) = X \circ \gamma_0(t(\sigma))$ with

$$\frac{d\gamma}{d\sigma} = \dot{\gamma} = \frac{DX\ \dot{\gamma}_0}{|DX\ \dot{\gamma}_0|} = \frac{DX\ \dfrac{d\gamma_0}{ds}}{\left|DX\ \dfrac{d\gamma_0}{ds}\right|}.$$

If we wish to be more explicit we may write $\dot{\gamma}_0 = (\mu_0, \nu_0)$ so that

$$DX\ \dot{\gamma}_0 = \mu_0\ \frac{\partial X}{\partial u} \circ \gamma_0 + \nu_0\ \frac{\partial X}{\partial v} \circ \gamma_0. \tag{1.46}$$

See Exercise 1.30. The main calculation we need to make is

$$\begin{aligned}
\ddot{\gamma}(0) &= \frac{d}{d\sigma}\ \frac{DX\ \dot{\gamma}_0}{|DX\ \dot{\gamma}_0|} \\
&= \left(\frac{d}{ds}\ (DX\ \dot{\gamma}_0)\ \frac{1}{|DX\ \dot{\gamma}_0|} - DX\ \dot{\gamma}_0\ \frac{\frac{d}{ds}\ (DX\ \dot{\gamma}_0)\ \cdot\ DX\ \dot{\gamma}_0}{|DX\ \dot{\gamma}_0|^3} \right) \frac{dt}{d\sigma} \\
&= \frac{d}{ds}\ (DX\ \dot{\gamma}_0)\ \frac{1}{|DX\ \dot{\gamma}_0|^2} - DX\ \dot{\gamma}_0\ \frac{\frac{d}{ds}\ (DX\ \dot{\gamma}_0)\ \cdot\ DX\ \dot{\gamma}_0}{|DX\ \dot{\gamma}_0|^4}.
\end{aligned}$$

Thus, we see the calculation is essentially reduced to

$$\begin{aligned}
\frac{d}{ds} DX\ \dot{\gamma}_0 = \dot{\mu}_0 \frac{\partial X}{\partial u} + \dot{\nu} \frac{\partial X}{\partial v} \\
+ \mu_0 \left(\frac{\partial^2 X}{\partial u^2}\ \mu_0 + \frac{\partial^2 X}{\partial u \partial v}\ \nu_0 \right) \\
+ \nu_0 \left(\frac{\partial^2 X}{\partial u \partial v}\ \mu_0 + \frac{\partial^2 X}{\partial v^2}\ \nu_0 \right)
\end{aligned}$$

which we write as

$$\frac{d}{ds}(DX\ \dot{\gamma}_0) = DX\ \ddot{\gamma}_0 + \beta(D^2 X\ \dot{\gamma}_0, \dot{\gamma}_0)$$

where $\beta : (\mathbb{R}^3)^2 \times \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\beta((\mathbf{v}, \mathbf{w}), (\mu, \nu)) = \mu \mathbf{v} + \nu \mathbf{w}$$

and

$$D^2 X\ \dot{\gamma}_0 = \mu_0 \begin{pmatrix} X_{uu} \\ X_{uv} \end{pmatrix} + \nu_0 \begin{pmatrix} X_{uv} \\ X_{vv} \end{pmatrix}.$$

Clearly, β is bilinear, i.e., linear in each argument, as is

$$B : T_{\gamma_0}\mathbb{R}^2 \times T_{\gamma_0}\mathbb{R}^2 \to \mathbb{R}^3 \qquad \text{by} \qquad B(\mathbf{v}, \mathbf{w}) = \beta(D^2 X\ \mathbf{v}, \mathbf{w})$$

where $D^2X = D^2X \circ \gamma_0$. In particular, B is symmetric.

The quantity $|DX\ \dot\gamma_0|$ may also be expressed more explicitly in terms of $\dot\gamma_0 = (\mu_0, \nu_0)$ as

$$|DX\ \dot\gamma_0| = \sqrt{E\mu_0^2 + 2F\mu_0\nu_0 + G\nu_0^2} = \sqrt{A(\dot\gamma_0, \dot\gamma_0)}$$

where $E = X_u \cdot X_u$, $F = X_u \cdot X_v$, $G = X_v \cdot X_v$, and $A : T_{\gamma_0}\mathbb{R}^2 \times T_{\gamma_0}\mathbb{R}^2 \to \mathbb{R}$ by

$$A(\mathbf{v}, \mathbf{w}) = DX\ \mathbf{v} \cdot DX\ \mathbf{w}; \qquad DX = DX \circ \gamma_0.$$

From this expression, it is clear A is bilinear and symmetric.

In terms of A and B we can write

$$\frac{d}{ds}(DX\ \dot\gamma_0)\ \cdot\ DX\ \dot\gamma_0 = A(\ddot\gamma_0, \dot\gamma_0) + B(\dot\gamma_0, \dot\gamma_0) \cdot DX\ \dot\gamma_0.$$

The last dot product may also be written as $C(\dot\gamma_0, \dot\gamma_0, \dot\gamma_0)$

$$C : (T_{\gamma_0}\mathbb{R}^2)^3 \to \mathbb{R} \qquad \text{by} \qquad C(\mathbf{v}, \mathbf{w}, \mathbf{z}) = \beta(D^2X\ \mathbf{v}, \mathbf{w})\ \cdot\ DX\ \mathbf{z}$$

is trilinear. Evaluating at $\sigma = 0$ we find the curvature vector of Γ at $p \in \mathcal{S}$ is given by

$$\ddot\gamma = \frac{DX\ \ddot\gamma_0 + B(\dot\gamma_0, \dot\gamma_0)}{A(\dot\gamma_0, \dot\gamma_0)} - \frac{A(\ddot\gamma_0, \dot\gamma_0) + C(\dot\gamma_0, \dot\gamma_0, \dot\gamma_0)}{A(\dot\gamma_0, \dot\gamma_0)^2} DX\ \dot\gamma_0 \tag{1.47}$$

where A, B, and C are multilinear functions depending on $DX(\mathbf{0})$ and $D^2X(\mathbf{0})$, and in this case

$$\begin{aligned} \ddot\gamma &= \ddot\gamma(0), \\ \ddot\gamma_0 &= \ddot\gamma_0(0), \text{ and} \\ \dot\gamma_0 &= \dot\gamma_0(0). \end{aligned}$$

See Exercise 1.31.

While some interesting information can be derived from the formula (1.47) for the curvature of the space curve $\Gamma = \alpha(-\epsilon, \epsilon) = X \circ \gamma_0(-\epsilon, \epsilon)$, and this information indeed contains essentially all the information about the curvature of the surface $\mathcal{S}$ itself at the point p, that information is not packaged or isolated in such a convenient form. The problem with (1.47) is that a great deal of information about the parameterization X is included which is not directly related to the geometry of $\mathcal{S}$ but really has only to do with how $\mathcal{S}$ has been parameterized.

1.5.3 curvature(s) of a surface

In order to isolate the information of interest in (1.47) and put that information in a more convenient form turns out to be relatively easy, though the means of doing so are not necessarily obvious. The basic idea is due to Meusnier who suggested the following: Instead of considering $\ddot{\gamma} = \ddot{\gamma}(0)$ as a function of $\mathbf{u} = \dot{\gamma}_0(0) \in \mathbb{S}^1 \subset T_{\mathbf{0}}\mathbb{R}^2$ and the curvature vector $\ddot{\gamma}_0(0) \in T_{\mathbf{0}}\mathbb{R}^2$, consider instead the quantity

$$k_n = \ddot{\gamma}(0) \cdot N \tag{1.48}$$

as a function of $\mathbf{u} = \dot{\gamma}(0) \in \mathbb{S}^1_p \subset T_p\mathcal{S}$ where

$$\mathbb{S}^1_p = \{\mathbf{v} \in T_p\mathcal{S} : |\mathbf{v}| = 1\}$$

and $N = X_u \times X_v/|X_u \times X_v|$ is the unit normal to $\mathcal{S}$ at p. The real valued quantity k_n given in (1.48) is called the **normal curvature** of the surface $\mathcal{S}$ at p in the direction $\mathbf{u} = \dot{\gamma}(0)$. First of all, we should say that while the formula in (1.48) certainly gives a well-defined real number associated with every regular curve with parameterization $\gamma \in \mathfrak{AP}_p(\mathcal{S})$ where the notation for the class $\mathfrak{AP}_p\mathcal{S}$ of arcwise parameterizations is given in Exercise 1.31, it is not quite clear that $k_n : \mathbb{S}^1_p \to \mathbb{R}$ has a properly defined domain. In fact as it stands right now in the current formulation, this assertion is not quite true. With a relatively minor modification, however, this assertion will make sense, though it will take some additional work to see clearly what is going on.

On the other hand, the quantity suggested by Meusnier has some properties one can immediately recognize as, at the very least, convenient. First of all, turning to formula (1.47)

$$k_n = \frac{1}{A(\dot{\gamma}_0, \dot{\gamma}_0)}\ B(\dot{\gamma}_0, \dot{\gamma}_0) \cdot N. \tag{1.49}$$

This is the case because $DX\ \dot{\gamma}_0$ and $DX\ \ddot{\gamma}_0$ are both in $T_p\mathcal{S}$ to which N is orthogonal. In fact, every vector

$$DX(\mathbf{0})\ \mathbf{v} = DX(\mathbf{0}) \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \mu\frac{\partial X}{\partial u}(\mathbf{0}) + \nu\frac{\partial X}{\partial v}(\mathbf{0})$$

is a linear combination of the tangent vectors $X_u(\mathbf{0})$ and $X_v(\mathbf{0})$ in $T_p\mathcal{S}$ for every $\mathbf{v} = (\mu, \nu) \in \mathbb{R}^2$.

Considering the formula (1.49) along with the bijection $dX_{\mathbf{0}} : T_{\mathbf{0}}\mathbb{R}^2 \to T_p\mathcal{S}$ according to which

$$dX_{\mathbf{0}}(\dot{\gamma}_0) = \frac{d\alpha}{ds}(0)$$

where $\alpha(s) = X \circ \gamma_0$ strongly suggests k_n should induce some kind of function on $\mathbb{S}^1_p$ or more generally on $T_p\mathcal{S}$. In fact, setting $\mathbf{w} = dX_{\mathbf{0}}^{-1}(\dot{\gamma})$ and $\alpha_0(\sigma) = X^{-1} \circ \gamma(\sigma)$ we have

$$\frac{d\alpha_0}{d\sigma}(0) \in T_{\mathbf{0}}\mathbb{R}^2 \backslash \{\mathbf{0}\} \qquad \text{and} \qquad dX_{\mathbf{0}}\left(\frac{d\alpha_0}{d\sigma}(0)\right) = \dot{\gamma}.$$

Thus we obtain a curve $\Gamma_0 \subset \mathbb{R}^2$ with an arclength parameterization $\gamma_0 \in \mathfrak{AP}_{\mathbf{0}}\mathbb{R}^2$ such that

$$dX_{\mathbf{0}}(\dot{\gamma}_0(0)) = dX_{\mathbf{0}}\left(\frac{\frac{d\alpha_0}{d\sigma}(0)}{\left|\frac{d\alpha_0}{d\sigma}(0)\right|}\right) = \frac{\dot{\gamma}}{|dX_{\mathbf{0}}^{-1}(\dot{\gamma})|}$$

where $\dot{\gamma} = \dot{\gamma}(0)$. That is, we can replace $\dot{\gamma}_0(0)$ with

$$\dot{\gamma}_0(0) = \frac{dX_{\mathbf{0}}^{-1}(\dot{\gamma})}{|dX_{\mathbf{0}}^{-1}(\dot{\gamma})|}$$

in (1.49) and take account of the bilinearity of A and B to obtain

$$k_n = k_n(\dot{\gamma}) = \frac{1}{A(dX_{\mathbf{0}}^{-1}(\dot{\gamma}), dX_{\mathbf{0}}^{-1}(\dot{\gamma}))}\ B(dX_{\mathbf{0}}^{-1}(\dot{\gamma}), dX_{\mathbf{0}}^{-1}(\dot{\gamma})) \cdot N. \qquad (1.50)$$

For the sake of argument

Formula (1.50) shows at the very least the surprising fact that the normal curvature k_n is entirely independent of the curvature vector $\ddot{\gamma} = \ddot{\gamma}(0)$ and consequently of the associated curvature vector $\ddot{\gamma}_0 = \ddot{\gamma}_0(0)$ back in $T_{\mathbf{0}}\mathbb{R}^2$. Recall that there are many different curves with $\dot{\gamma}(0) = \mathbf{u} \in T_p\mathcal{S}$ having many different curvature vectors $\ddot{\gamma}(0)$, but once $\dot{\gamma}(0) = \mathbf{u} \in T_p\mathcal{S}$ is fixed, there is only one unique normal curvature…almost.

The problem is the normal N. If we take N given by $X_u \times X_v/|X_u \times X_v|$, which is what we have done, then we can switch the normal by changing the parameterization. See Exercise 1.32. This makes k_n dependent on the choice of normal and, as we have it phrased, dependent on the choice of local parameterization. More broadly, the formula (1.50) contains all kinds of additional formal dependence on the parameterization $X : U \to \mathcal{S}$ through the dependence in the functions A and B, not to mention dependence through the inverse differential $dX_{\mathbf{0}}^{-1} = (dX_{\mathbf{0}})^{-1}$.

I will deal with the apparent dependence on $X : U \to \mathcal{S}$ more comprehensively below, but for now let me focus on the choice of normal N. Informally, I now restrict attention to capillary configurations for which the model volume[9] (of liquid) $\mathcal{V} \subset \mathbb{R}^3$ satisfies

$$\partial\mathcal{V} = \mathcal{S} \cup \Gamma \cup \mathcal{W}$$

is a **piecewise** smooth embedded surface[10] satisfying the **conclusion** of the Jordan-Brouwer separation theorem in the sense that there exists an open spatially unbounded complement $C = \mathbb{R}^3 \backslash \overline{\mathcal{V}}$ with

$$\partial C = \mathcal{S} \cup \Gamma \cup \mathcal{W}.$$

For each point $p \in \partial\mathcal{V} = \partial C$ there is some $r > 0$ such that

(i) There is a homeomorphism $\psi : B_r(p) \to B_r(\mathbf{0})$ with

(ii)

$$\psi\big|_{\mathcal{V}\cap B_r(p)} : \mathcal{V} \cap B_r(p) \to B_r^-(\mathbf{0}) = \{\mathbf{x} = (x_1, x_2, x_3) : \mathbf{x} \in B_r(\mathbf{0}),\ x_3 < 0\}$$

is a homeomorphism,

(iii)

$$\psi\big|_{\partial\mathcal{V}\cap B_r(p)} : \partial\mathcal{V} \cap B_r(p) \to B_r^0(\mathbf{0}) = \{\mathbf{x} = (x_1, x_2, x_3) : \mathbf{x} \in B_r(\mathbf{0}),\ x_3 = 0\}$$

is a homeomorphism, and

(iv)

$$\psi\big|_{C\cap B_r(p)} : C \cap B_r(p) \to B_r^+(\mathbf{0}) = \{\mathbf{x} = (x_1, x_2, x_3) : \mathbf{x} \in B_r(\mathbf{0}),\ x_3 > 0\}$$

is a homeomorphism.

[9]See condition **2** on page 7.

[10]This is a concept we have not formally defined, and is a little complicated to define carefully.

As a consequence, the unit **outward** normal field N on $\mathcal{S}$ as well as the smooth portions of the wetted region $\mathcal{W}$ is always well-defined independent of any parameterization. Furthermore, we can always take a local parameterization $X : U \to \mathcal{S}$ at $p \in \mathcal{S}$ with $X(\mathbf{0}) = p$ and

$$N = \frac{X_u \times X_v}{|X_u \times X_v|}.$$

See Exercise 1.32. This takes care of the ambiguity concerning the sign of N in the definition/formula (1.50) for the normal curvature.[11] In reference to our application in mathematical capillarity, the convention introduced here should be emphasized: The normal N to a free surface $\mathcal{S}$ or to a wetted surface $\mathcal{W}$ is always taken to point out of the liquid volume.

With the outward unit normal field N assumed as above, it is natural to consider N as a function on the entire surface $\mathcal{S}$ with $N : \mathcal{S} \to \mathbb{R}^3$. This is in contrast to, though presumed nominally consistent with, our previous local definition of $N : U \to \mathbb{R}^3$ given by

$$N = N(u, v) = \frac{X_u \times X_v}{|X_u \times X_v|}. \tag{1.51}$$

Technically, we should choose one and define the other in terms of our choice. In particular, we can perhaps now agree to write

$$\frac{X_u \times X_v}{|X_u \times X_v|} = N \circ X,$$

though it is still natural to express this value simply as N with a suppression of the argument. Another similar practice is to use the symbol N to represent $N \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{R}^3$ when $\gamma \in \mathfrak{AP}_p\mathcal{S}$ or even $N \circ X \circ \gamma_0 : (-\epsilon, \epsilon) \to \mathbb{R}^3$ when $\gamma_0 \in \mathfrak{AP}_{\mathbf{0}}\mathbb{R}^2$. Generally, we rely on the context to make the usage clear, and we will try to include clarifying postscripts like "where $N = N(u, v)$", "where $N = N(0, 0)$", or "where $N = N \circ X^{-1}$."

Our specific interest here is to let N denote $N \circ \gamma$ so that the quantity

$$\frac{d}{d\sigma} N = \dot{N}$$

[11] More generally, one may restrict attention to **orientable** surfaces and make a choice of normal in order to define a notion of normal curvature with respect to a particular unit normal field, but the limited construction we are suggesting is natural and serves our purposes for modeling free surface interfaces and wetted regions at least in simple situations.

makes sense, where σ is an arclength parameter on a curve $\Gamma \subset \mathcal{S}$ and $\dot{\gamma} = \dot{\gamma}(0) = \mathbf{u} \in \mathbb{S}^1_p \subset T_p\mathcal{S}$. Notice then that as a function of σ, the quantity $\dot{\gamma} \cdot N \equiv 0$ and, consequently,

$$0 = \frac{d}{d\sigma}(\dot{\gamma} \cdot N) = \ddot{\gamma} \cdot N + \dot{\gamma} \cdot \dot{N}.$$

Evaluating at $\sigma = 0$ we conclude

$$k_n = -\mathbf{u} \cdot \dot{N} = -\dot{\gamma} \cdot \frac{d}{d\sigma} N. \tag{1.52}$$

This expression makes it clear at once that the normal curvature k_n at p in the direction $\mathbf{u}$ is independent of any local parameterization X of the surface $\mathcal{S}$. One sees furthermore a linear dependence in the first factor of k_n on $\mathbf{u} = \dot{\gamma} = \dot{\gamma}(0) \in T_p\mathcal{S}$, though the dependence of the second factor $\dot{N}$ is less clear. To make this dependence clear let us review the meaning of

$$\dot{N} = \frac{d}{d\sigma} N$$

and also consider the local structure of the surface $\mathcal{S}$ itself from a different point of view. The conclusion we wish to reach is the following:

Theorem 2 (bilinear form of normal curvature) There exists a real symmetric 3×3 matrix $\overline{M}$ and a unique corresponding bilinear quadratic form

$$II : T_p\mathcal{S} \times T_p\mathcal{S} \to \mathbb{R}$$

given by

$$II(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \overline{M}\mathbf{w} = \langle \mathbf{v}, \overline{M}\mathbf{w} \rangle_{\mathbb{R}^3}$$

so that

$$k_n = II(\dot{\gamma}, \dot{\gamma});$$

the matrix $\overline{M}$ itself is not uniquely determined, but the value $\overline{M}\mathbf{w}$ for $\mathbf{w} \in T_p\mathcal{S} \subset \mathbb{R}^3$ and the corresponding linear transformation $S : T_p\mathcal{S} \to T_p\mathcal{S}$ determined by

$$S\mathbf{w} = \overline{M}\mathbf{w}$$

are unique. In fact, for $\mathbf{w} = \dot{\gamma} = \dot{\gamma}(0) \in T_p\mathcal{S}$ with $|\dot{\gamma}| = 1$ there holds

$$S\dot{\gamma} = \overline{M}\dot{\gamma} = -\frac{d}{d\sigma}[N \circ \gamma(\sigma)]\Big|_{\sigma=0}$$

so that (1.52) may be written as

$$k_n = II(\dot{\gamma},\dot{\gamma}) = \langle \dot{\gamma}, S\dot{\gamma}\rangle_{\mathbb{R}^3}. \tag{1.53}$$

The function $S = S_p : T_p\mathcal{S} \to T_p\mathcal{S}$ is called the **shape operator** at $p \in \mathcal{S}$, and the function $II = II_p$ is called the **second fundamental form** of the surface.

Idea of the proof: The normal $N : \mathcal{S} \to \mathbb{R}^3$ has a smooth extension $\overline{N} : V \to \mathbb{R}^3$ locally near a point p to an open set $V \subset \mathbb{R}^3$ with $p \in V$. The extension mapping $\overline{N}$ has a full derivative $\overline{M} = D\overline{N}$ which is a real symmetric 3×3 matrix valued function on V. The matrix $\overline{M}$ is not uniquely determined because it depends on the extension $\overline{N}$, and the extension is not unique. See Exercises 1.33 and 1.34.

The directional derivative

$$D_{\dot{\gamma}}\overline{N}(p)$$

where $\mathbf{u} = \dot{\gamma} = \dot{\gamma}(0) \in T_p\mathcal{S}$ with $|\mathbf{u}| = 1$ or more generally $D_{\mathbf{u}}\overline{N}(p)$ for $\mathbf{u} \in \mathbb{S}^2 \subset T_p\mathbb{R}^3$ can be computed in various ways. One way is using the usual definition

$$D_{\mathbf{u}}\overline{N}(p) = \lim_{v\to 0} \frac{\overline{N}(p+v\mathbf{u}) - \overline{N}(p)}{v}.$$

By the multivariable mean value theorem/chain rule there is some v_* between 0 and v for which

$$\overline{N}(p+v\mathbf{u}) - \overline{N}(p) = D\overline{N}(p+v_*\mathbf{u})(v\mathbf{u}).$$

Thus,

$$D_{\mathbf{u}}\overline{N}(p) = \lim_{v\to 0} D\overline{N}(p+v_*\mathbf{u})\ \mathbf{u} = D\overline{N}(p)\ \mathbf{u}.$$

On the other hand, if $\gamma \in \mathfrak{AP}_p\mathbb{R}^3$ is an arclength parameterization of any space curve with $\dot{\gamma}(0) = \mathbf{u}$, then

$$\frac{d}{d\sigma}\overline{N}\circ\gamma(\sigma)\Big|_{\sigma=0} = D\overline{N}(p)\ \dot{\gamma} = D\overline{N}(p)\ \mathbf{u}$$

by the chain rule. In particular, if we take $\gamma \in \mathfrak{AP}_p\mathcal{S}$, then $\overline{N}\circ\gamma(\sigma) = N\circ\gamma(\sigma)$, and we see

$$\dot{N} = \frac{d}{d\sigma}[N\circ\gamma(\sigma)]\Big|_{\sigma=0} = D\overline{N}(p)\ \dot{\gamma}$$

as well. Most importantly, the value

$$\dot{N} = \frac{d}{d\sigma}[N \circ \gamma(\sigma)]\Big|_{\sigma=0}$$

is independent of the extension $\overline{N}$.

Finally then we can set $S(\mathbf{v}) = -D\overline{N}\mathbf{v}$ for $\mathbf{v} \in T_p\mathcal{S}$. This determines a linear function/operator $S : T_p\mathcal{S} \to T_p\mathcal{S}$ which we know is well-defined (independent of the extension $\overline{N}$) because

$$S(\mathbf{v}) = |\mathbf{v}| S\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = -|\mathbf{v}|\frac{d}{d\sigma}[N \circ \gamma(\sigma)]\Big|_{\sigma=0}$$

where $\gamma \in \mathfrak{AP}_p\mathcal{S}$ with $\dot{\gamma}(0) = \mathbf{v}/|\mathbf{v}|$. We define $II : T_p\mathcal{S} \times T_p\mathcal{S}$ by $II(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot S(\mathbf{w})$ so that (1.52) and (1.53) hold. These are essentially all the assertions of Theorem 2. $\Box$

The only real ingredient lacking in the discussion of the proof of Theorem 2 above is the construction of the smooth local extension $\overline{N} : V \to \mathbb{R}^3$ of the normal field N on a surface along the lines of the special cases considered in Exercises 1.33 and 1.34. Let me attempt here to tie up this loose end.

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I want to give an alternative way to express the surface $\mathcal{S}$ locally near a point p and in closer relation to the tangent plane to $T_p\mathcal{S}$. Intuitively it is clear that some subset $B_r(p) \cap \mathcal{S}$ for some $r > 0$ should be expressible as the graph of a function over some interior set in $\mathcal{T}_p$. More precisely, there should exist orthonormal vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in $T_p\mathcal{S}$ and a function $w \in C^\infty(W)$ for some open set $W \subset \mathbb{R}^2$ with $\mathbf{0} = (0, 0) \in W$ for which

$$\mathcal{S} \cap B_r(p) = \{p + x\ \mathbf{u}_1 + y\ \mathbf{u}_2 + w(x, y)\ N : (x, y) \in W\}$$

where $N = [X_u(0, 0) \times X_v(0, 0)]/|X_u(0, 0) \times X_v(0, 0)|$ is the unit normal $\mathcal{S}$ at p. Obtaining this local structure for $\mathcal{S}$ and illuminating the relation between the function w and the parameterization X in particular will require some careful attention.

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Taking $r > 0$ smaller if necessary, there is some $\delta > 0$ for which $X^{-1}(B_r(p) \cap \mathcal{S}) \subset B_\delta(\mathbf{0})$ where $B_\delta(\mathbf{0}) = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < \delta^2\}$ and $X : B_\delta(\mathbf{0}) \to X(B_\delta(\mathbf{0}))$ is a homeomorphism.

Associated with each $(u, v) \in U$ or alternatively for each $X(u, v) \in \mathcal{S} \cap V$, there is a smooth orthonormal frame $\{\mathbf{u}_1, \mathbf{u}_2, N\}$ for $\mathbb{R}^3$ determined by

$$\mathbf{u}_1 = \frac{X_u}{|X_u|},$$
$$\mathbf{u}_2 = \frac{X_v - (X_v \cdot \mathbf{u}_1)\mathbf{u}_1}{|X_v - (X_v \cdot \mathbf{u}_1)\mathbf{u}_1|}, \qquad \text{and}$$
$$N = \frac{X_u \times X_v}{|X_u \times X_v|}.$$

In particular,

$$|X_u \times X_v|^2 = |X_u|^2 |X_v|^2 - (X_u \times X_v)^2 > 0,$$

incorporating the orientation reversing[12] transformation $(u, v) \mapsto (-u, v)$ of the domain U if necessary we may assume $\{\mathbf{u}_1, \mathbf{u}_2, N\}$ is a right-handed orthonormal frame field, and there is a well-defined two-dimensional vector subspace of $\mathbb{R}^3$ given by

$$T_X\mathcal{S} = \{aX_u + bX_v : a, b \in \mathbb{R}\} = \{x\mathbf{u}_1 + y\mathbf{u}_2 : x, y \in \mathbb{R}\}.$$

In particular, the tangent space $T_p\mathcal{S}$ at $p \in \mathcal{S}$ has an orthonormal basis

$$\left\{ \frac{X_u}{|X_u|}, \frac{X_v - (X_u \cdot X_v)X_u/|X_u|^2}{|X_v - (X_u \cdot X_v)X_u/|X_u|^2} \right\}$$

where $X_u = X_u(0, 0)$ and $X_v = X_v(0, 0)$. The addition of

$$N = N(0, 0) = \frac{X_u \times X_v}{|X_u \times X_v|}$$

results in a right handed orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, N\}$ for $\mathbb{R}^3$. Fixing for the moment these values at $(u, v) = (0, 0)$ in the vector space basis $\{\mathbf{u}_1, \mathbf{u}_2, N\}$ at $p \in \mathcal{S}$, we consider two transformations of Euclidean space. The first is $\Xi : U \to \mathbb{R}^2$ given by

$$\Xi(u, v) = ([X(u, v) - p] \cdot \mathbf{u}_1, [X(u, v) - p] \cdot \mathbf{u}_2),$$

[12] See Exercise 1.32.

and the second is $\psi : U \times \mathbb{R} \to \mathbb{R}^3$ by

$$\psi(u, v, z) = X(u, v) + zN.$$

==========

Consider an open set $V = B_r(p)$ where $r > 0$ and the affine translate

$$\{p + x\mathbf{u}_1 + y\mathbf{u}_2 : x, y \in \mathbb{R}\}$$

of $T_p\mathcal{S}$. If $r > 0$ is small enough, then $\mathcal{S} \cap V$

I claim that for $r > 0$ small enough the equation/condition

$$x\mathbf{u}_1 + y\mathbf{u}_2 + zN = X(u, v)$$

Consider also the function $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\psi(x, y, z) = x\mathbf{u}_1 + y\mathbf{u}_2 + zN.$$

1.6 Digression on the calculus of variations

In section 1.4 we derived a partial differential equation

$$Mu = \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = \kappa u - \lambda$$

satisfied by any minimizer $u = u_0$ of the capillary energy suggested by Gauss and determining a mensicus surface

$$\mathcal{S} = \mathcal{S}_0 = \{(x, y, u(x, y)) : (x, y) \in \Omega\}$$

given as a graph over the domain Ω corresponding interior region of the horizontal cross-section of a vertical capillary tube. The same approach gives an equation

$$Mu = \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = \kappa u - \lambda_{\text{out}} \tag{1.54}$$

satisfied by the outer meniscus graph $\mathcal{S}_1 = \{(x, y, u(x, y)) : (x, y) \in \Omega_{\text{out}}\}$ over the outer (or remainder of the bath) domain $\Omega_{\text{out}} = \Omega_2 \backslash \overline{\Omega_1}$. In this section I am going to give a separate, and in several aspects somewhat different, derivation of (1.54). One of the objectives of this different derivation is to give a clear identification of the constants λ and λ_{out} as Lagrange parameters, or Lagrange multipliers, as they arise in the general theory of the calculus of variations. I will then explain how to combine elements of the discussion above to conclude that the two nominally different Lagrange parameters λ and λ_{out} must in fact be equal.

A careful statement of the principle of Lagrange multipliers as considered in the elementary calculus of functions of several variables is the following:

Theorem 3 (principle of Lagrange multipliers in calculus) Assume $f, g \in C^1(U)$ where U is an open subset of $\mathbb{R}^n$, and let

$$\mathcal{L} = \{\mathbf{x} \in U : g(\mathbf{x}) = c\}$$

for some constant $c \in \mathbb{R}$. If $\mathbf{x}_0 \in \mathcal{L}$ and there is some some $\delta > 0$ for which $\mathbf{x}_0$ satisfies

$$f(\mathbf{x}_0) \le f(\mathbf{x}) \qquad \text{for every } \mathbf{x} \in \mathcal{L} \cap B_\delta(\mathbf{x}_0)$$

then either $Dg(\mathbf{x}_0) = \mathbf{0} \in \mathbb{R}^n$ or there is some constant $\lambda \in \mathbb{R}$ such that

$$Df(\mathbf{x}_0) = \lambda Dg(\mathbf{x}_0). \tag{1.55}$$

The reader is encouraged to write down a proof of this assertion and especially to model a proof on that of the "infinite dimensional version" we give below. For the moment note two things: (1) The possibility that the gradient of the constrant function vanishes is not vacuous: The function $f(x, y) = x$ has a minimum subject to the constraint $g(x, y) = x^2 + y^2 = 0$ at the origin $\mathbf{0} = (0, 0) \in \mathbb{R}^2$, but the gradient of f is given by $Df = (1, 0)$ and never vanishes so that (1.55) cannot hold at the point $\mathbf{x}_0 = (0, 0)$ where the local minimum occurs. (2) An alternative framing for the second possitility (1.55) is that the modified function $f - \lambda g$ has an interior critical point at $\mathbf{x}_0$ which may be interpreted as a potential local minimum (without any constraint) for the function $f - \lambda g$. In practice, the modified function $f - \lambda g$ may not have an unconstrained local minimum at $\mathbf{x}_0$. See Exercise 1.35.

Let us begin consideration of the infinite dimensional minimization problem with a constraint by observing that in the case of capillary graphs, or standard meniscus minimizing shapes, in the simple capillary tubes problem the energy functional $\mathcal{E}$, and the simplified/modified energy functional $\mathcal{F}$, may be considered with domain a standard set of real valued functions. In more general situations, for example if one seeks a minimizing interface among parametric surfaces or especially if one considers situations in which rigid structures may be deformed (elastocapillarity) or have pieces moving relative to one another (as when one allows a floating object), then it may be more natural (and necessary) to consider $\mathcal{E} = \mathcal{E}[\mathcal{C}]$ as a function of an entire "configuration" on some kind of set modeling all the possible configurations, and a larger collection of configurations than those determined entirely by a free surface interface given as a graph in particular. So then we have considered $\mathcal{E} = \mathcal{E}[\mathcal{C}]$ with $\mathcal{C}$ a configuration above, but in the case of a vertical capillary tube under consideration it is natural to consider $\mathcal{E} = \mathcal{E}[u]$ and

$$\mathcal{E}, \mathcal{F} : \mathcal{A} = \left\{ u \in C^\infty(\overline{U}) : u > 0, \text{ and } u\Big|_{\partial\Omega\cup\partial\Omega_1} > d_0 \right\} \to \mathbb{R}$$

where as usual $\Omega \subset\subset \Omega_1 \subset\subset \Omega_2$ and $U = \Omega \cup \Omega_{\text{out}} = \Omega \cup (\Omega_2 \backslash \overline{\Omega_1})$.

Retaining the special case in which the admissible class $\mathcal{A}$ is assumed to satisfy $\mathcal{A} \subset C^2(\overline{U})$ for some bounded open set $U \subset \mathbb{R}^n$, we consider a general functional $\mathcal{F} : \mathcal{A} \to \mathbb{R}$. We assume in particular certain standard aspects of the calculus of variations. Specifically, we assume that for some $\epsilon > 0$ the perturbation $\phi \in C_c^\infty(U)$ with $\|\phi\|_{C^1} < \epsilon$ may be used to construct an admissible variation $u + \phi$. Thus, the standard perturbation space of functions $C_c^\infty(U)$ comes into play. We assume there is a second functional $\mathcal{G} : \mathcal{A} \to \mathbb{R}$, and to simplify the discussion we assume both functionals $\mathcal{F}$ and $\mathcal{G}$ are **integral functionals** of the form(s)

$$\mathcal{F}[u] = \int_{\mathbf{x}\in U} F(\mathbf{x}, u, Du) \qquad \text{and} \qquad \mathcal{G}[u] = \int_{\mathbf{x}\in U} G(\mathbf{x}, u, Du) \tag{1.56}$$

where $F, G \in C^\infty(U \times \mathbb{R} \times \mathbb{R}^n)$. The function F appearing in the integrand of $\mathcal{F}$ is called the **Lagrangian** of the functional $\mathcal{F}$ and will be considered as a function of the $2n + 1$ variables $\mathbf{x}$, z, and $\mathbf{p}$ so that $F = F(\mathbf{x}, z, \mathbf{p})$. A similar description applies to the functional $\mathcal{G}$. It will be noted that $\mathcal{F}$ does not contain boundary integral terms corresponding to the wetting energies

$$-\sigma \int_{\partial U} \beta\ u \qquad \text{and} \qquad -\int_{\partial U} \beta\ u$$

in the actual capillary functionals $\mathcal{E}$ and $\mathcal{F}$ respectively. Again, this is only to simplify the discussion. The argument in the proof uses only "interior variations" so that additional integral terms like these wetting energies would essentially constitute only additive constants. Specifically, there is no problem allowing the more general form

$$\mathcal{F}[u] = \int_{\mathbf{x}\in U} F(\mathbf{x}, u, Du) + \int_{\mathbf{x}\in\partial U} F_b(\mathbf{x}, u, Du) \tag{1.57}$$

as long as $\Gamma = \partial U$ and $F_b : \partial U \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are suitable for integration. If for example Γ is a union of disjoint C^1 curves of finite length and $F_b \in C^0(\partial U \times \mathbb{R} \times \mathbb{R}^n)$, then there is no problem. Perhaps the most general assumption would be that $\Gamma = \partial U$ is a rectifiable curve. We will consider "boundary variations" later. At that time we will focus on the wetting energies and assume considerably more regularity than required for this discussion.

The functional $\mathcal{G}$ may be compared to the volume functional

$$\mathcal{G}[u] = \int_U u$$

which of course has no boundary integral terms. For the reasons discussed above, there is no problem considering the more general form

$$\mathcal{G}[u] = \int_{\mathbf{x}\in U} G(\mathbf{x}, u, Du) + \int_{\mathbf{x}\in\partial U} G_b(\mathbf{x}, u, Du) \tag{1.58}$$

as long as ∂U and G_b are suitably regular.

The perturbation space $C_c^\infty(U)$ is an infinite dimensional linear space, and an integral functional like $\mathcal{F}$ does not, of course, have a standard "gradient" like a real valued function of several variables in calculus. Such a functional does admit, however, an analogue of the differential map $df_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{R}^n \to T_{\mathbf{x}}\mathbb{R}$ for a real valued function $f \in C^1(U)$. This "infinite dimensional" differential is called the **first variation** of $\mathcal{F}$ and is itself a functional defined on the infinite dimensional linear space of perturbations:

$$\delta\mathcal{F}_u : C_c^\infty(U) \to \mathbb{R}$$

by

$$\delta\mathcal{F}_u[\phi] = \int_{\mathbf{x}\in U} \frac{\partial F}{\partial z}(\mathbf{x}, u, Du)\ \phi + \int_{\mathbf{x}\in U} \sum_{j=1}^n \frac{\partial F}{\partial p_j}(\mathbf{x}, u, Du)\ \frac{\partial \phi}{\partial x_j}.$$

This value is also given by

$$\delta\mathcal{F}_u[\phi] = \frac{d}{dh}\mathcal{F}[u + h\phi]\Big|_{h=0}.$$

The functional $\delta\mathcal{F}_u$ is clearly linear on $C_c^\infty(U)$ so that $\delta\mathcal{F}_u$ is in the **algebraic dual space** associated to $C_c^\infty(U)$. It is somewhat more complicated to say $\delta\mathcal{F}_u : C_c^\infty(U) \to \mathbb{R}$ is continuous because this requires a topology on $C_c^\infty(U)$, and there is no natural norm or simple distance on the space. We will not discuss the topology on $C_c^\infty(U)$ in detail nor justify the assertion that $\delta\mathcal{F}_u$ is in the so called *continuous dual space* $(C_c^\infty(U))^*$ of continous linear functionals on $C_c^\infty(U)$, but one can roughly see this is true in the sense that if $\{\phi_j\}_{j=1}^\infty$ is a sequence of functions in $C_c^\infty(U)$ converging to the zero function in $C^k(\overline{U})$ for some $k \geq 1$, for example this is true for every k if $\phi_j = (1/j)\phi$ for some fixed $\phi \in C_c^\infty(U)$, then it is pretty clear that

$$\lim_{j\nearrow\infty} \delta\mathcal{F}_u[\phi_j] = 0.$$

In any case, we will go ahead and write $\delta\mathcal{F}_u \in (C_c^\infty(U))^*$. No implication of this assertion will be used in any explicit way anyway, but the exploration of $C_c^\infty(U)$ as a topological linear space is something that can be taken up as an interesting aside in an effort to strengthen the interpretation of the calculus of variations in terms of the familiar concepts from elementary calculus.

Since the perturbation space $C_c^\infty(U)$ is an infinite dimensional linear space, it is natural to consider the admissible class $\mathcal{A}$ as "infinite dimensional." In particular, we wish to prove the following "infinite dimensional" version of the principle of Lagrange multipliers in the calculus of variations:

Theorem 4 (principle of Lagrange multipliers in the calculus of variations) Assume $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathbb{R}$ are integral functionals of the form (1.57) and (1.58) where U is a bounded open subset of $\mathbb{R}^n$ and $\mathcal{A}$ has the property that given $u \in \mathcal{A}$ there is some $\epsilon > 0$ for which $u + \phi \in \mathcal{A}$ whenever $\phi \in C_c^\infty(U)$ with $\|\phi\|_{C^1} < \epsilon$. Let

$$\mathcal{L} = \{u \in \mathcal{A} : \mathcal{G}[u] = c\}$$

for some constant $c \in \mathbb{R}$. If $u_0 \in \mathcal{L}$ and there is some some $\delta > 0$ for which u_0 satisfies

$$\mathcal{F}[u_0] \leq \mathcal{F}[u] \qquad \text{for every } u \in \mathcal{L} \cap B_\delta(u_0)$$

where $B_\delta(u_0) = \{u \in \mathcal{A} : \|u - u_0\|_{C^1} < \delta\}$, then either $\delta\mathcal{G}_{u_0} = \mathbf{0} \in (C_c^\infty(U))^*$ or there is some constant $\lambda \in \mathbb{R}$ such that

$$\delta\mathcal{F}_{u_0} = \lambda\delta\mathcal{G}_{u_0}. \tag{1.59}$$

Proof: For any $\phi, \psi \in C_c^\infty(U)$ consider $u + h\phi + k\psi$ and the function $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(h,k) = \mathcal{G}[u + h\phi + k\psi]$$

in particular. Notice the function $u + h\phi + k\psi$ will be admissible, i.e., in $\mathcal{A}$, for h and k small enough, say on $B_\epsilon(0,0)$ for some $\epsilon > 0$. We wish to choose $k = k(h)$ so that in fact on some interval $-\epsilon < h < \epsilon$ the constraint condition

$$\mathcal{G}[u + h\phi + k\psi] = c$$

holds. This assertion was extremely easy in our discussion above when $\mathcal{G}$ happened to be the volume functional for a capillary graph. A generalization allowing more general constraint functionals $\mathcal{G}$ is one of the main points of this digression.

The function $g : B_\epsilon(\mathbf{0}) \to \mathbb{R}$ satisfies $g \in C^1(B_\epsilon(\mathbf{0}))$ and

$$Dg(\mathbf{0}) = \left(\frac{\partial g}{\partial h}(\mathbf{0}), \frac{\partial g}{\partial k}(\mathbf{0}) \right) = (\delta\mathcal{G}_u[\phi], \delta\mathcal{G}_u[\psi]) \,.$$

See Exercise 1.36. The dichotomy of the theorem arises here: Either $\delta\mathcal{G}_u$ is the zero functional, or we can choose $\psi \in C_c^\infty(U)$ so that $\delta\mathcal{G}_u[\psi] \neq 0$ and consiquently

$$Dg(0,0) \neq (0,0) \qquad \text{with} \qquad \frac{\partial g}{\partial k}(0,0) = \delta\mathcal{G}_u[\psi] \neq 0$$

in particular. In the latter case we conclude from the implicit function theorem (at least informally) that the equation $g(h,k) = c$ can be solved uniquely on some interval $-\epsilon < h < \epsilon$ for k as a function of h with $k = k(h)$ having regularity determined by and matching the regularity of g. Thus, if $g \in C^k(\mathbb{R}^2)$ for some $k \in \{1, 2, 3, \ldots, \}$, then $k \in C^k(-\epsilon, \epsilon)$ as well.

Let us attempt to give this assertion in a little more precise form using the inverse function theorem. For this let us define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi(h,k) = (h, g(h,k))$. Notice that $\Phi(0,0) = (0,c)$. Also,

$$D\Phi = \begin{pmatrix} 1 & 0 \\ g_h & g_k \end{pmatrix} \qquad \text{and} \qquad \det D\Phi(0,0) = \frac{\partial g}{\partial k}(0,0) = \delta\mathcal{G}_u[\psi] \neq 0.$$

Consequently, the inverse function theorem says that for some $\epsilon > 0$, the function $\Phi : B_\epsilon(0,0) \to \Phi(B_\epsilon(0,0))$ is one-to-one, onto, and has an inverse

$\Phi^{-1} \in C^\infty(\Phi(B_\epsilon(0,0)) \to B_\epsilon(0,0))$. Let us denote the coordinate functions of Φ^{-1} by id and $\beta = \beta(h,\eta)$. Given that

$$\Phi \circ \Phi^{-1}(h,\eta) = (\mathrm{id}(h,\eta), g(\mathrm{id}(h,\eta), \beta(h,\eta)) = (h,\eta),$$

it is clear that $\mathrm{id}(h,\eta) \equiv h$. Finally, we take $k : (-\epsilon,\epsilon) \to \mathbb{R}$ by $k(h) = \beta(h,c)$. This, it will be noted, is the value corresponding to a unique point $(h,k(h)) = (h,\beta(h,c)) \in B_\epsilon(0,0)$ for which

$$\Phi(h,\beta(h,c)) = (h,c).$$

In particular $g(h,k(h)) = c$. Thus, $k(h)$ is indeed a solution of the "equation" $g(h,k) = c$ for k as a function of h, and if $(h,\tilde{k})$ were any other element of $B_\epsilon(\mathbf{0})$ for which $g(h,\tilde{k}) = c$, then we would have

$$\Phi(h,\tilde{k}) = (h, g(h,\tilde{k})) = (h,c)$$

so that applying Φ^{-1} to both sides we have $(h,\tilde{k}) = (h,\beta(h,c)) = (h,k(h))$. Thus, $\tilde{h} = k(h) = \beta(h,c)$ is uniquely determined. This is the content of the assertion of the implicit function theorem. See Exercise 1.37.

The discussion above gives us a one parameter family of admissible functions $u + h\phi + k(h)\psi$ with

$$\mathcal{G}[u + h\phi + k(h)\psi] = c.$$

Furthermore, differentiating the defining relation $g(h,k(h)) = c$, we find

$$\frac{\partial g}{\partial h}(0,0) + \frac{\partial g}{\partial k}(0,0)\ k'(0) = 0,$$

so

$$k'(0) = -\frac{\delta\mathcal{G}_u[\phi]}{\delta\mathcal{G}_u[\psi]}.$$

Assuming u is a/the minimizer (called u_0 in the statement of the theorem) we know

$$\frac{d}{dh}\mathcal{F}[u + h\phi + k(h)\psi]\Big|_{h=0} = 0.$$

That is,

$$\delta\mathcal{F}_u[\phi] + k'(0)\ \delta\mathcal{F}_u[\psi] = \delta\mathcal{F}_u[\phi] - \frac{\delta\mathcal{F}_u[\psi]}{\delta\mathcal{G}_u[\psi]}\ \delta\mathcal{G}_u[\phi] = 0.$$

Setting

$$\lambda = \frac{\delta \mathcal{F}_u[\psi]}{\delta \mathcal{G}_u[\psi]},$$

we obtain the assertion of the theorem. $\square$

Applying Theorem 4 to the minimization of the capillary energy $\mathcal{E}$ (or $\mathcal{F}$) subject to the constraint of volume we can consider

$$\mathcal{F} = \int_U \sqrt{1+|Du|^2} = \int_{\partial U} \beta\ u + \frac{\kappa}{2}\int_U u^2$$

given in (1.23) subject to (1.25)

$$\int_U u = C_V$$

where C_V is a constant related to the total volume $\mathcal{V}$ and with $U = \Omega \cup \Omega_{\text{out}}$. As mentioned above, the boundary integral terms in $\mathcal{F}$ play no role in the techniques of interior variation used in the proof of Theorem 4. The result is that there is a constant $\lambda \in \mathbb{R}$ for which

$$\delta(\mathcal{F} - \lambda\mathcal{G})_u[\phi] = \int_U \frac{Du}{\sqrt{1+|Du|^2}} \cdot D\phi + \int_U (\kappa u - \lambda)\phi = 0$$

for all $\phi \in C_c^\infty(U)$ where $U = \Omega \cup \Omega_{\text{out}}$.

The usual integration by parts and application of the fundamental lemma of vanishing integrals, assuming $u \in C^2(U)$ now gives the capillary equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \kappa u - \lambda \tag{1.60}$$

on all of $U = \Omega \cup \Omega_{\text{out}}$ notably with a single constant λ. Reflection on the argument of section 1.4 leads to two observations: The two test functions ψ and ϕ in the argument were entirely independent of one another and there is/was no particular reason to restrict the test function ϕ (nor the test funtion ψ for that matter) to the class $C_c^\infty(\Omega)$. In fact, allowing $\phi \in C_c^\infty(U)$ where $U = \Omega \cup \Omega_{\text{out}}$ gives the same conclusion of (1.60).

It is in fact true that a minimizing function $u \in C^2(U)$ satisfies

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \kappa u - \lambda \qquad \text{on} \qquad \Omega$$

and

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \kappa u - \lambda_{\text{out}} \qquad \text{on} \qquad \Omega_{\text{out}}$$

for some constants λ and λ_{out}. These constants given according to (1.33) by

$$\lambda = \frac{1}{\int_\Omega \psi}\left(\int_\Omega \frac{Du \cdot D\psi}{\sqrt{1+|Du|^2}} + \kappa \int_\Omega u\ \psi\right)$$

and similarly by

$$\lambda_{\text{out}} = \frac{1}{\int_{\Omega_{\text{out}}} \psi}\left(\int_{\Omega_{\text{out}}} \frac{Du \cdot D\psi}{\sqrt{1+|Du|^2}} + \kappa \int_{\Omega_{\text{out}}} u\ \psi\right)$$

may be different if the volumes above the regions Ω and Ω_{out} are sepearately and independently kept fixed. If these separate volumes are allowed to vary independently subject to the constraint that they sum to a single constant, corresponding physically to the condition that the liquid under the two components of the meniscus are allowed to communicate with one another, then these two constants must be the same. This is the case no matter which function ψ is chosen, and in particular, the more general value

$$\lambda = \frac{1}{\int_U \psi}\left(\int_U \frac{Du \cdot D\psi}{\sqrt{1+|Du|^2}} + \kappa \int_U u\ \psi\right)$$

may be allowed corresponding to the more general choice of a fixed function $\psi \in C_c^\infty(U)$. See Exercise 1.38.

The boundary condition

We have now established that a minimizer $u \in C^2(\overline{U})$ must satisfy a partial differential equation of prescribed mean curvature

$$\operatorname{div} Tu = \kappa u - \lambda$$

where $T : C^1(U) \to C^0(U \to \mathbb{R}^n)$ by

$$Tu = \frac{Du}{\sqrt{1+|Du|^2}}.$$

This conclusion was given in Theorem 1 for the inner tube region Ω, and we have seen in various ways that the same conclusion holds on all of $U = \Omega \cup \Omega_{\text{out}}$ for some Lagrange parameter λ. We now augment this geometric partial differential equation with an appropriate boundary condition. To this end we assume ∂U is a set with adequate regularity to allow integration and certain continuity properties. One possibility is that $\partial\Omega$ consists of three C^2 simple closed curves $\partial\Omega$, $\partial\Omega_1$ and $\partial\Omega_2$. Upon these curves we recognize the unit normal ν pointing out of U, and consider a particular test function $\phi \in C_c^\infty(\partial U)$ by which we mean the restriction to ∂U of a function $\phi \in C_c^\infty(\mathbb{R}^2)$. We will give more detail on this point when the time comes. For the moment we observe that given the volume constraint condition on the perturbation, namely,

$$\int_U (u+\phi) = \int_U u \qquad \text{or simply} \qquad \int_U \phi = 0$$

we should have

$$\frac{d}{dh}\mathcal{F}[u+h\phi]\Big|_{h=0} = 0. \tag{1.61}$$

Accordingly, we compute

$$\begin{aligned}\frac{d}{dh}\mathcal{F}[u+h\phi] &= \frac{d}{dh}\left(\int_U \sqrt{1+|Du+hD\phi|^2}\right.\\ &\qquad\qquad \left. - \int_{\partial U} \beta(u+h\phi) + \frac{\kappa}{2}\int_U (u+h\phi)^2\right)\\ &= \int_U \frac{(Du+hD\phi)\cdot Dh}{\sqrt{1+|Du+hD\phi|^2}} - \int_{\partial U} \beta\ \phi + \kappa\int_U (u+h\phi)\phi.\end{aligned}$$

Setting $h = 0$, we obtain a first necessary condition for a minimizer associated with boundary variations:

$$\int_U Tu\cdot D\phi - \int_{\partial U} \beta\ \phi + \kappa\int_U u\ \phi \tag{1.62}$$

for all $\phi \in C_c^\infty(\mathbb{R}^2)$ for which

$$\int_U \phi = 0. \tag{1.63}$$

Integrating by parts, or more properly applying the divergence theorem in conjunction with the product rule

$$\operatorname{div}(\phi\ Tu) = Tu \cdot D\phi + \phi \operatorname{div} Tu,$$

we can write (1.62) in the form

$$\int_{\partial U} (Tu \cdot \nu)\phi - \int_U (\operatorname{div} Tu - \kappa u)\phi - \int_{\partial U} \beta\ \phi = 0 \tag{1.64}$$

where ν is the unit normal field on ∂U pointing out of U and again the condition holds for all $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying the volume constraint condition (1.63). We have shown above that the expression

$$\operatorname{div} Tu - \kappa u$$

takes a constant value $-\lambda$, so the area integral term vanishes:

$$-\int_U (\operatorname{div} Tu - \kappa u)\phi = \lambda \int_U \phi = 0.$$

More generally, if we wish to preserve the naive view that $\operatorname{div} Tu - \kappa u$ might be piecewise constant taking values $-\lambda$ on Ω and λ_{out} on Ω_{out}, a view that arose by considering volume preserving perturbations with support restricted to the domains Ω and Ω_{out} respectively, then we can impose the additional restriction(s)

$$\phi \in C_c^\infty(\mathbb{R}^2 \backslash \overline{\Omega_{\text{out}}}) \qquad \text{and alternatively} \qquad \phi \in C_c^\infty(\mathbb{R}^2 \backslash \overline{\Omega})$$

along with (1.63) which becomes in these cases

$$\int_\Omega \phi = 0, \qquad \text{alternatively} \qquad \int_{\Omega_{\text{out}}} \phi = 0.$$

In any case, we may conclude

$$\int_{\partial\Omega} (Tu \cdot \nu - \beta_0)\phi = 0 \tag{1.65}$$

if $\phi \in C_c^\infty(\mathbb{R}^2 \backslash \overline{\Omega_{\text{out}}})$ and

$$\int_\Omega \phi = 0,$$

while

$$\int_{\partial\Omega_{\text{out}}} (Tu\cdot\nu-\beta)\phi = \int_{\partial\Omega_1} (Tu\cdot\nu-\beta_1)\phi + \int_{\partial\Omega_2} (Tu\cdot\nu-\beta_2)\phi = 0 \quad (1.66)$$

if $\phi \in C_c^\infty(\mathbb{R}^2\backslash\overline{\Omega})$ and

$$\int_{\Omega_{\text{out}}} \phi = 0.$$

As usual the question is: What conclusions can be drawn concerning the minimizing solution $u : U \to \mathbb{R}$ from these integral conditions?

Theorem 5 (boundary condition(s)) If $u \in C^2(\overline{U})$ minimizes the capillary energy $\mathcal{E}$ determined by a capillary tubes domain $U = \Omega \cup \Omega_{\text{out}}$ as described above with ∂U consisting of C^2 embedded and nested simple closed curves $\partial\Omega$, $\partial\Omega_1$ and $\partial\Omega_2$ over which the piecewise constant values β_0, β_1, and β_2 as adhesion coefficients are assumed respectively, then

$$Tu\cdot\nu = \begin{cases} \beta_0, & \mathbf{x} \in \partial\Omega \\ \beta_1, & \mathbf{x} \in \partial\Omega_1 \\ \beta_2, & \mathbf{x} \in \partial\Omega_2. \end{cases}$$

Proof: Assume by way of contradiction that for some point $\mathbf{p} \in \partial\Omega$ there holds

$$Tu\cdot\nu \neq \beta_0$$

in the integral condition (1.65). Let us in fact assume as a first case that

$$Tu(\mathbf{p})\cdot\nu(\mathbf{p}) - \beta_0 = \alpha > 0. \quad (1.67)$$

Let $\mu \in C_c^\infty(\mathbb{R}^2)$ be a specific test function given as follows:

$$\mu(x,y) = \begin{cases} \frac{1}{c_0} e^{\frac{1}{x^2+y^2-1}}, & x^2+y^2<1 \\ 0, & x^2+y^2 \geq 1 \end{cases}$$

where

$$c_0 = \int_{B_1(0,0)} e^{\frac{1}{x^2+y^2-1}},$$

so that $\text{supp}(\mu) = \overline{B_1(0,0)}$, $\mu(x,y) > 0$ for $x^2 + y^2 < 1$, and

$$\int_{\mathbb{R}^2} \mu = 1.$$

The construction now becomes a little delicate. For each $h > 0$ consider $\phi \in C_c^\infty(\mathbb{R}^n)$ given by

$$\phi(\mathbf{x}) = \frac{1}{h^2}\mu\left(\frac{\mathbf{x}-\mathbf{p}}{h}\right) - \frac{c}{h^2}\mu\left(\frac{\mathbf{x}-(\mathbf{p}-2h\nu)}{h}\right)$$

where

$$c = \frac{1}{h^2}\int_{\mathbf{x}\in\Omega} \mu\left(\frac{\mathbf{x}-\mathbf{p}}{h}\right).$$

For h small $B_h(\mathbf{p}-2h\nu) \subset\subset \Omega$ and consequently

$$\int_{\mathbf{x}\in\Omega} \frac{c}{h^2}\mu\left(\frac{\mathbf{x}-(\mathbf{p}-2h\nu)}{h}\right) = \int_{\mathbf{x}\in B_h(\mathbf{p}-2h\nu)} \frac{c}{h^2}\mu\left(\frac{\mathbf{x}-(\mathbf{p}-2h\nu)}{h}\right) = c$$

and

$$\int_\Omega \phi = 0.$$

That is, ϕ is a volume preserving perturbation. Also for h small $\phi \in C_c^\infty(\mathbb{R}^2\backslash\overline{\Omega_{\text{out}}})$, so (1.65) holds. In view of (1.67) we may assume by continuity that

$$Tu(\mathbf{x})\cdot\nu(\mathbf{x}) - \beta_0 \geq \frac{\alpha}{2} > 0 \qquad \text{for} \qquad \mathbf{x} \in \partial\Omega \cap B_h(\mathbf{p}).$$

Thus we see

$$0 = \int_{\partial\Omega} (Tu\cdot\nu - \beta_0)\phi \geq \frac{\alpha}{2}\int_{\partial\Omega} \phi.$$

Here of course ϕ denotes the restriction

$$\phi = \phi\big|_{\partial\Omega}.$$

We obtain a contradiction as long as

$$\int_{\partial\Omega} \phi\big|_{\partial\Omega} > 0.$$

For this we only need a simple continuity/monotonicity property of integration on $\partial\Omega$: Note that the restriction satisfies

(i) $\phi \in C^0(\partial\Omega)$,

(ii) $\phi \geq 0$, and

(iii) $\phi(\mathbf{p}) > 0$.

Thus, the condition

$$\int_{\partial\Omega} \phi = \int_{B_h(\mathbf{p})\cap\partial\Omega} \phi > 0$$

is adequate, and this certainly holds if $\partial\Omega$ is a C^2 curve. See Exercise 1.39.

We reach a similar contradiction if $Tu(\mathbf{p}) \cdot \nu(\mathbf{p}) - \beta_0 \leq -\alpha/2$ for some $\alpha > 0$, namely for $h > 0$ small

$$\int_{\partial\Omega} (Tu \cdot \nu)\phi = \int_{\partial\Omega\cap B_h(\mathbf{p})} (Tu \cdot \nu)\phi \leq -\frac{\alpha}{2} \int_{B_h(\mathbf{p})\cap\partial\Omega} \phi < 0.$$

We leave it to the reader (Exercise 1.40) to formulate a more elegant proof giving the assertion $Tu \cdot \nu = \beta_j$ on $\partial\Omega_j$, when $j = 1, 2,$. □

A different proof showing $\lambda_{\text{out}} = \lambda$

We now offer another proof that $\lambda = \lambda_{\text{out}}$ using the simple variation (1.28) of section 1.3 and the formula (1.30) for the lifted volume in particular. We will also use the assertion/boundary condition of Theorem 5 which we note did not depend on the equality $\lambda_{\text{out}} = \lambda$. This proof illustrates more clearly the modeling of the communication of liquid between the physical liquid above the inner tube and the outer bath. We note also that the variation (1.28) gives an example of a variation that moves the boundary values.

Note that the formula for the lifted volume (1.30) may take the form

$$\kappa \, \frac{1}{\text{area}(\Omega)} \int_{\Omega} u \; - \; \beta_0 \, \frac{\text{length}(\partial\Omega)}{\text{area}(\Omega)} =$$
$$\kappa \, \frac{1}{\text{area}(\Omega_{\text{out}})} \int_{\Omega_{\text{out}}} u \; - \; \beta_1 \, \frac{\text{length}(\partial\Omega_1)}{\text{area}(\Omega_{\text{out}})} \; - \; \beta_2 \, \frac{\text{length}(\partial\Omega_2)}{\text{area}(\Omega_{\text{out}})}.$$

We proceed to obtain alternative expressions for λ and λ_{out} via integration. Integrating the equation $\operatorname{div} Tu = \kappa u - \lambda$ over Ω we find

$$\begin{aligned}
\lambda &= \frac{1}{\text{area}(\Omega)} \left(\kappa \int_\Omega u - \int_\Omega \operatorname{div} Tu \right) \\
&= \frac{1}{\text{area}(\Omega)} \left(\kappa \int_\Omega u - \int_{\partial\Omega} Tu \cdot \nu \right) \\
&= \frac{1}{\text{area}(\Omega)} \left(\kappa \int_\Omega u - \int_{\partial\Omega} \beta_0 \right) \\
&= \frac{1}{\text{area}(\Omega)} \left(\kappa \int_\Omega u - \beta_0 \ \text{length}(\partial\Omega) \right).
\end{aligned}$$

Similarly, integrating $\operatorname{div} Tu = \kappa u - \lambda_{\text{out}}$ over Ω_{out} yields

$$\begin{aligned}
\lambda_{\text{out}} &= \frac{1}{\text{area}(\Omega)} \left(\kappa \int_{\Omega_{\text{out}}} u - \int_{\Omega_{\text{out}}} \operatorname{div} Tu \right) \\
&= \frac{1}{\text{area}(\Omega_{\text{out}})} \left(\kappa \int_{\Omega_{\text{out}}} u - \int_{\partial\Omega_{\text{out}}} Tu \cdot \nu \right) \\
&= \frac{1}{\text{area}(\Omega_{\text{out}})} \left(\kappa \int_{\Omega_{\text{out}}} u - \int_{\partial\Omega_1} \beta_1 - \int_{\partial\Omega_2} \beta_2 \right) \\
&= \frac{1}{\text{area}(\Omega_{\text{out}})} \left(\kappa \int_{\Omega_{\text{out}}} u - \beta_1 \ \text{length}(\partial\Omega_2 - \beta_2 \ \text{length}(\partial\Omega_2) \right).
\end{aligned}$$

Thus, we see the lifted volume formula of section 1.3, when propertly interpreted, says precisely that $\lambda_{\text{out}} = \lambda$. $\square$

1.7 Simple observations about the capillary equation

By considering a vertical translation $u = u_0 + c$ we obtain a one-to-one correspondence between solutions of the PDE (1.35) and the equation $M[u - c] = \kappa(u - c) + \lambda$ for u. Since the mean curvature operator (like the Laplacian) is invariant under vertical translation, we have $M[u - c] = Mu$, and taking $c = \lambda/\kappa$ we can see every sulution of (1.35) is a vertical translate of a solution of

$$Mu = \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = \kappa u.$$

1.8 Axioms for 2-D mathematical capillarity

1.9 Exercises

Exercise 1.1 Let $\mathcal{D}$ denote the toroidal region

$$\mathcal{D} = \left\{ (x,y,z) : \left| (x,y,z) - \frac{2}{\sqrt{x^2+y^2}}(x,y,0) \right| < 1,\ 1 \le x^2+y^2 < 3 \right\},$$

and let $\mathcal{R} = \mathbb{R}^3 \backslash \overline{\mathcal{D}}$ model a rigid support structure with boundary containing the wetted region $\mathcal{W}$ determined by the interface

$$\mathcal{S} = \{2(\cos t, \sin t, 0) + r(\cos(t/2)\cos t, \cos(t/2)\sin t, \sin(t/2)) : \\ t \in \mathbb{R},\ -1 < r < 1\}. \quad (1.68)$$

Determine the following

(a) The wetted region $\mathcal{W}$ in ∂R. Find chart functions $Y : U \to \mathcal{W}$ giving regular local parameterizations of $\mathcal{W}$ on some open sets $U \subset \mathbb{R}^2$. By **regular** here, we mean $Y : U \to Y(U)$ is a homeomorphism and

$$\frac{\partial Y}{\partial x_1} \times \frac{\partial Y}{\partial x_2} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For example the restriction

$$X\Big|_{U_j} : U_j \to \mathcal{S}$$

where

$$X(t,r) = 2(\cos t, \sin t, 0) + r(\cos(t/2)\cos t, \cos(t/2)\sin t, \sin(t/2))$$

is given by the formula in (1.68) gives a regular parameterization of an open half of the Möbius strip $\mathcal{S}$ when

$$U_1 = (-\pi/2, \pi/2) \times (-1,1),$$
$$U_2 = (0, \pi) \times (-1,1),$$
$$U_3 = (\pi/2, 3\pi/2) \times (-1,1), \text{ or}$$
$$U_4 = (\pi, 2\pi) \times (-1,1)$$

with

$$\mathcal{S} = \bigcup_{j=1}^{4} X(U_j).$$

(b) The contact line $\Gamma = \overline{\mathcal{S}} \cap \overline{\mathcal{W}}$. Find a regular parameterization $\gamma : \mathbb{R} \to \Gamma$ of the contact line.

(c) The volume $\mathcal{V}$ bounded by by $\mathcal{S}$, Γ and $\mathcal{W}$.

(d) The angle at which $\mathcal{S}$ meets $\mathcal{W}$ along Γ.

Exercise 1.2 Let $\mathcal{V}$, $\mathcal{S}$, Γ, and $\mathcal{W}$ be as described in Exercise 1.1. Consider/answer the following questions:

(a) What is the angle at which $\mathcal{S}$ meets $\mathcal{W}$ along Γ measured **within the volume** $\mathcal{V}$?

(b) Do the wetted regions $\mathcal{S}$ and $\mathcal{W}$ constitute a configuration satisfying configuration properties **1-3** above?

(c) Can you modify this configuration to obtain a configuration in which the contact angle γ measured within the volume $\mathcal{V}$ is not well-defined?

(d) Can you think of some reasons to rule out configurations of this kind? Hint: What is peculiar about a/the physical capillary system modeled by this configuration?

(e) What is the mean curvature of $\mathcal{S}$?

Exercise 1.3 Make a composite drawing of the capillary tubes configuration containing the elements in both Figure 1.2 and Figure 1.3. Label all wetted regions $\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_4$ and associated boundaries (including contact lines) $\Gamma_0, \Gamma_1, \ldots, \Gamma_5$ as identified/discussed above, and identify the following:

(a) $\partial \mathcal{S}_1 = \Gamma_i \cup \Gamma_j$ where Γ_i and Γ_j are circles from among (1.7), (1.8), (1.14), (1.15), (1.16).

(b) $\partial \mathcal{W}_1 = \Gamma_i \cup \Gamma_j$ where Γ_i and Γ_j are circles from among (1.7), (1.8), (1.14), (1.15), (1.16).

(c) $\partial \mathcal{W}_2 = \Gamma_i \cup \Gamma_j$ where Γ_i and Γ_j are circles from among (1.7), (1.8), (1.14), (1.15), (1.16).

(d) $\partial\mathcal{W}_3 = \Gamma_i \cup \Gamma_j$ where Γ_i and Γ_j are circles from among (1.7), (1.8), (1.14), (1.15), (1.16).

(e) $\partial\mathcal{W}_4 = \Gamma_i \cup \Gamma_j$ where Γ_i and Γ_j are circles from among (1.7), (1.8), (1.14), (1.15), (1.16).

Exercise 1.4 Find smooth extensions of each of the wetted surfaces $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, and $\mathcal{W}_4$ from Exericse 1.3 above.

Exercise 1.5 What is the radius of each circle Γ_0, Γ_1, Γ_2, Γ_3, Γ_4 and Γ_5 from Exericse 1.3 above?

Exercise 1.6 How does the expression

$$\rho g \int_{\mathcal{V}} z$$

given in (1.18) change if instead of a a constant, the density of the liquid is assumed to be given by a spatially dependent function $\rho : \mathcal{V} \to (0, \infty)$?

Exercise 1.7 Let $u_0 \in C^2(\overline{U})$ be an admissible meniscus function for the capillary tubes problem in the sense that $U = \Omega \cup \Omega_{\text{out}}$ with $\Omega \subset\subset \Omega_1 \subset\subset \Omega_2$ nested domains,

$$\Omega_{\text{out}} = \Omega_2 \backslash \overline{\Omega_1},$$

$u_0 > 0$ and

$$u_0\Big|_{\partial(\Omega_1\backslash\Omega)} > d_0$$

where d_0 is a positive constant. Find the maximum value of $\epsilon > 0$ (and show there is such a value) for which the function $u \in C^2(\overline{U})$ given in (1.28) by

$$u(x,y) = \begin{cases} u_0(x,y) + h, & (x,y) \in \Omega \\ u_0(x,y) - h \operatorname{area}(\Omega)/\operatorname{area}(\Omega_{\text{out}}), & (x,y) \in \Omega_{\text{out}} \end{cases}$$

is admissible in the sense that u satisfies $u > 0$,

$$u\Big|_{\Omega_1\backslash\Omega} > d_0,$$

and

$$\int_\Omega u + \int_{\Omega_{\text{out}}} u = \int_\Omega u_0 + \int_{\Omega_{\text{out}}} u_0$$

for $h < \epsilon$. Hint: You should be able to express your answer in terms of the four positive numbers

$$\min\{u(x,y) - d_0 : (x,y) \in \partial\Omega\} = \min_{\partial\Omega} u - d_0,$$

$$\min\{u(x,y) - d_0 : (x,y) \in \partial\Omega_1\} = \min_{\partial\Omega_1} u - d_0,$$

$$\min_{\overline{\Omega}} u, \qquad \text{and} \qquad \min_{\overline{\Omega_{\text{out}}}} u.$$

Exercise 1.8 Substitute the expression for the competitor mensicus function u given in (1.28) into (1.22) and calculate

$$\frac{d}{dh}\mathcal{E}[\mathcal{C}_h] \qquad \text{and} \qquad \frac{d}{dh}\mathcal{E}[\mathcal{C}_h]\Big|_{h=0}$$

to verify (1.29) and hence the formula (1.30) for the lifted volume.

In certain instances, it is natural to designate a different quantity as the **lifted volume**. The next two exercises describe one such case.

Exercise 1.9 Assume the surface of the container and the outer surface of the capillary tube in the capillary tubes problem are coated with a substance so that the wetted regions $\mathcal{W}_1$ and $\mathcal{W}_2$ share a common liquid-surface adhesion coefficient $\beta_1 = \beta_2 = 0$ but the inner surface of the capillary tube maintains adhesion coefficient β_0. For notation, see Exercise 1.3 above.

(a) Derive a formula analogous to (1.30) for the quantity

$$\int_\Omega u_0$$

giving the integral of a minimizing inner meniscus function

$$u_0 = u_0\Big|_\Omega > 0$$

in terms of the integral of the outer minimizing mensiscus function

$$u_{\text{out}} = u_0\Big|_{\Omega_{\text{out}}} > 0.$$

(b) Assume the outer minimizing meniscus function u_{out} is a positive constant,[13] and use your formula from part **(a)** above to express the quantity

$$\int_\Omega u_0 - \int_\Omega u_{\text{out}} = \int_\Omega u_0 - u_{\text{out}} \ \text{area}(\Omega) \tag{1.69}$$

in terms of $\kappa = \rho\, g/\sigma$ and length$(\partial\Omega)$.

(c) Explain why the quantity in (1.69) is naturally designated the **lifted volume** in this case.

(d) Draw illustrations to accompany your explanation in part **(c)** above with at least one for the case $\beta_0 > 0$ and one for the case $\beta_0 < 0$. What might be a better name for the quantity in (1.69) in the case $\beta_0 < 0$?

Exercise 1.10 Assume the Archimedean bath hypothesis $\beta_1 = \beta_2 = 0$ and u_{out} is a positive constant of Exercise 1.9, but assume the inner adhesion coefficient β_0 may be spatially dependent with $\beta_0 \in C^0(\partial\Omega_0)$.

(a) Show the lifted volume does not depend on the outer tube domain Ω_1 nor the container domain Ω_2 in any way.

(b) How is the assertion of part **(a)** simply translated/applied to the situation of the concentric circular cylindrical tube and container? Hint: The raised volume depends only on. . . .

Exercise 1.11 Let u_0, ϕ, and ψ be fixed in $C^\infty(\overline{U})$ where U is a bounded open subset of $\mathbb{R}^n$ and let h be a fixed real number. Show an L^1 estimate

$$\left\| \frac{1}{v} \left(\sqrt{1 + |Du_0 + (h+v)D\phi + k(h+v)\ D\psi|^2} - \sqrt{1 + |Du_0 + hD\phi + k(h)\ D\psi|^2} \right) \right\| < C$$

that holds uniformly for $v \in \mathbb{R}$. Hint: The constant C may depend on. . .

Exercise 1.12 (fundamental lemma of vanishing integrals) Show that if $f \in C^0(U)$ where U is an open subset of $\mathbb{R}^n$ and

$$\int_U f\ \phi = 0 \qquad \text{for every } \phi \in C_c^\infty(U),$$

[13] This may be called the *Archimedean bath hypothesis*.

then $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in U$. Hint(s): Assume by way of contradiction that $f(\mathbf{x}_0) > 0$ for some $\mathbf{x}_0 \in U$. Choose a particular $\phi \in C_c^\infty(U)$ to obtain a contradiction.

Exercise 1.13 Let U be a fixed bounded, open, and connected subset of $\mathbb{R}^2$; let $M : C^2(U) \to C^0(U)$ denote the associated capillary operator and let $\Delta : C^2(U) \to C^0(U)$ denote the Laplace operator.

(a) Show minimizers of the **Dirichlet energy** $\mathcal{D} : C^2(\overline{U}) \to C^0(\overline{U})$ by

$$\mathcal{D}[u] = \int_U |Du|^2 \tag{1.70}$$

are solutions of Laplace's equation.

(b) Show minimizers of the **Dirichlet energy** (1.70) subject to the constraint

$$\int_U u = c$$

where c is some constant are solutions of some version of Poisson's equation.

(c) The assertions of parts **(a)** and **(b)** are said to give **variational formulations** for Laplace's equation and (a special case of) Poisson's equation. Can you find a variational formula for Poisson's equation in general where $f \in C^0(\overline{U})$ is a given function?

Exercise 1.14 Let M and Δ be as in Exercise 1.13.

(a) Show $\Delta[au + bv] = a\Delta u + b\Delta v$ for $a, b \in \mathbb{R}$ and $u, v \in C^2(U)$ so that Δ is linear.

(b) Show by example that $M[au]$ is not always aMu for $a \in \mathbb{R}$ and $u \in C^2(U)$ so that M is nonlinear.

(c) Show by example that $M[u+v]$ is not always $Mu+Mv$ for $u, v \in C^2(U)$ so that (again) M is nonlinear.

Exercise 1.15 Let M and Δ be as in Exercise 1.13 and let $\alpha > 0$ be a positive constant. Consider the scaling transform(ation) $A : C^2(U) \to C^2(\alpha U)$ given by $Au = v$ where

$$v(\mathbf{x}) = \alpha\ u\left(\frac{\mathbf{x}}{\alpha}\right)$$

and $\alpha U = \{\alpha\mathbf{x} : \mathbf{x} \in U\}$.

(a) Calculate $Mv = MAu$.

(b) Calculate $\Delta v = \Delta Au$.

Technically the operators M and Δ featured in parts **(a)** and **(b)** are nominally different operators because they have different domains with respect to the set αU, but of course the structural formulas for these nominally different operators are exactly the same as the corresponding operator of the same name from Exercise 1.13.

Exercise 1.16 ($\Delta u = \kappa u - \lambda$) Let κ and λ be fixed real constants.

(a) Show that every solution of $\Delta u = \kappa u - \lambda$ has the form

$$u = u_h + u_p$$

where u_p is a solution of the Poisson equation $\Delta u_p = -\lambda$ and u_h is a solution of the homogeneous PDE $\Delta u_h = \kappa u_h$.

(b) In one space dimension the PDE $\Delta u = \kappa u - \lambda$ becomes the ODE $u'' = \kappa u - \lambda$. Solve the Poisson equation $u_p'' = -\lambda$ in this case.

(c) Find the general solution of $u_h'' = \kappa u_h$

(i) when $\kappa < 0$,

(ii) when $\kappa = 0$, and

(iii) when $\kappa > 0$.

Exercise 1.17 (curvature) Consider the circle

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - r)^2 = r^2\}.$$

(a) Express $\Gamma \cap B_{r\sqrt{2}}(\mathbf{0})$ as the graph of a function $u : C^\omega(-r, r)$.

(b) Calculate $u''(0)$ for the function you found in part **(a)**.

(c) Let $\epsilon_0 > 0$ and $x_0 \in (-r, r)$. Consider a function $v \in C^2(-\epsilon_0, \epsilon_0)$ satisfying the following:

(i) $v(x_0) = u(x_0)$,

(ii) $v'(x_0) = u'(x_0)$, and

(iii) $v''(x_0) = u''(x_0)$.

It is reasonable to define the curvature of the graph

$$G = \{(x, v(x)) : -\epsilon_0 < x - x_0 < \epsilon_0\}$$

of v at $P_0 = (x_0, v(x_0))$ to be $k = 1/r$. Show that for some $\epsilon > 0$, some $a, b \in \mathbb{R}$ with $a < 0 < b$, and some $g \in C^2(a, b)$ the curve $G \cap B_\epsilon(P_0)$ can be expressed as

$$\left\{ (x_0, v(x_0)) + t\, \frac{(1, v'(x_0))}{\sqrt{1 + v'(x_0)^2}} + g(t)\, \frac{(-v'(x_0), 1)}{\sqrt{1 + v'(x_0)^2}} : t \in (a, b) \right\}.$$

(d) Calculate a formula for $g''(0)$ in terms of $v'(x_0)$ and $v''(x_0)$ where g is the function you found in part **(c)**; apply this calculation to the special case $v \equiv u$.

Exercise 1.18 Let $u \in C^2(a, b)$ where $a, b \in \mathbb{R}$ with $a < b$. Draw a picture illustrating the inclination angle ψ of the graph of u given by

$$\psi = \sin^{-1}\left(\frac{u'}{\sqrt{1 + u'^2}}\right).$$

Exercise 1.19 Consider $\alpha \in C^2(\mathbb{R} \to \mathbb{R}^2)$ by $\alpha(t) = (t^3, t^2)$.

(a) Give an accurate sketch of the image curve

$$\Gamma = \{\alpha(t) : t \in \mathbb{R}\}$$

determined by α.

(b) What happens if you attempt to reparameterize Γ by arclength?

(c) Show there exists no regular paraemeterization of Γ.

Exercise 1.20 (a Frenet's equation for a planar curve) Consider $\gamma \in C^2((a, b) \to \mathbb{R}^3)$ where $a, b \in \mathbb{R}$ with $a < b$ and $|\dot\gamma(s)| \equiv 1$. Assume

$$\vec{k} = \ddot\gamma = |\ddot\gamma|\ \mathbf{n} \neq \mathbf{0}.$$

Show $\dot{\mathbf{n}} = -k\ \dot\gamma = -|\ddot\gamma|\ \dot\gamma$.

Exercise 1.21 (a Frenet's equation) Consider $\gamma \in C^3((a,b) \to \mathbb{R}^3)$ where $a, b \in \mathbb{R}$ with $a < b$ and $|\dot{\gamma}(s)| \equiv 1$. Assume

$$\vec{k} = \ddot{\gamma}(s_0) \neq \mathbf{0}.$$

(a) Show $\{\dot{\gamma}, \mathbf{n}, \dot{\gamma} \times \mathbf{n}\}$ is an orthonormal basis for $\mathbb{R}^3$ at $\gamma(s_0)$.

(b) Show

$$\dot{\mathbf{n}} = -k\ \dot{\gamma} + \tau\ \dot{\gamma} \times \mathbf{n}$$

where $\tau = \tau(s)$ is some real valued function defined locally near $s_0 \in (a,b)$. Hint: $\dot{\mathbf{n}} = (\dot{\mathbf{n}} \cdot \dot{\gamma})\ \dot{\gamma} + (\dot{\mathbf{n}} \cdot \mathbf{n})\ \mathbf{n} + (\dot{\mathbf{n}} \cdot \dot{\gamma} \times \mathbf{n})\ \dot{\gamma} \times \mathbf{n}$.

Exercise 1.22 (C^2 curves) Consider the (space) curve Γ with parameterization $\alpha \in C^2(\mathbb{R} \to \mathbb{R}^3)$ given in (1.44).

(a) Calculate α' and verify in particular that $\alpha'(0)$ is well-defined.

(b) Reparameterize this curve by arclength and calculate the curvature vector $\mathbf{n} = \mathbf{n}(s)$ determined by the arclength parameterization $\gamma \in C^2(\mathbb{R} \to \mathbb{R}^3)$ with $\gamma(0) = \alpha(0)$. In particular, show

$$\ddot{\gamma} \in C^0(\mathbb{R} \to \mathbb{R}^3).$$

(c) Show Γ has nonvanishing curvature so that the principal unit normal $\mathbf{n} \in C^0(\mathbb{R} \to \mathbb{R}^3)$ given by

$$\mathbf{n}(s) = \frac{\ddot{\gamma}(s)}{|\ddot{\gamma}(s)|}$$

is well-defined with image in $\mathbb{S}^2 = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.

(d) Show the principal unit normal $\mathbf{n}$ from part **(c)** above satisfies

$$\lim_{s \to 0} \frac{\mathbf{n}(s) - \mathbf{n}(0)}{s} \qquad \text{does not exist.}$$

Exercise 1.23 (C^2 curves) Construct examples of planar curves Γ with C^2 regular parameterizations $\alpha \in C^2((a,b) \to \mathbb{R}^2)$ for some $a, b \in \mathbb{R}$ with $a < b$ such that $\alpha(t_0) \in \Gamma$ is an **isolated point of vanishing curvature**, that is a local parameterization $\gamma \in C^2((-\ell, m) \to \mathbb{R}^2)$ of Γ by arclength with $\gamma(0) = \alpha(t_0)$ satisfies $\ddot{\gamma}(0) = \mathbf{0}$ but $\ddot{\gamma}(s) \neq 0$ for $s \neq 0$, and having the following properties:

(a)

$$\lim_{s\to 0} \mathbf{n}(s)$$

does not exist.

(b)

$$\lim_{s\to 0} \mathbf{n}(s)$$

exists, and the function

$$\nu(s) = \begin{cases} \mathbf{n}(s), & s \neq 0 \\ \lim_{s\to 0} \mathbf{n}(s), & s = 0 \end{cases}$$

satisfies $\nu \in C^1((-\ell, m) \to \mathbb{R}^2)$.

(c)

$$\lim_{s\to 0} \mathbf{n}(s)$$

exists, and the function

$$\nu(s) = \begin{cases} \mathbf{n}(s), & s \neq 0 \\ \lim_{s\to 0} \mathbf{n}(s), & s = 0 \end{cases}$$

satisfies $\nu \in C^0((-\ell, m) \to \mathbb{R}^2)\backslash C^1((-\ell, m) \to \mathbb{R}^2)$.

(d)

$$\lim_{s\to 0} \mathbf{n}(s)$$

exists, and the function

$$\nu(s) = \begin{cases} \mathbf{n}(s), & s \neq 0 \\ \lim_{s\to 0} \mathbf{n}(s), & s = 0 \end{cases}$$

satisfies $\nu \notin C^0((-\ell, m) \to \mathbb{R}^2)$.

Exercise 1.24 (C^2 curves) Show that if Γ is a curve with C^2 regular parameterization $\alpha \in C^2((a, b) \to \mathbb{R}^3)$ for some $a, b \in \mathbb{R}$ with $a < b$ satisfying

(i) Γ has nonvanishing curvature, and

(ii) Γ is planar, that is,

$$\alpha(t) \in \Pi = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : (\mathbf{x} - \mathbf{p}) \cdot N = 0\}$$

for some $\mathbf{p} = (p_1, p_2, p_3)$ and $N = (N_1, N_2, N_3)$ in $\mathbb{R}^3$, then Γ has a well-defined principal normal $\mathbf{n} \in C^1((a,b) \to \mathbb{S}^2)$.

Exercise 1.25 (C^2 curves) Construct examples of curves Γ with C^2 regular parameterizations $\alpha \in C^2((a,b) \to \mathbb{R}^3)$ for some $a, b \in \mathbb{R}$ with $a < b$ such that $\alpha(t_0) \in \Gamma$ is an **isolated point of vanishing curvature**, that is a local parameterization $\gamma \in C^2((-\ell, m) \to \mathbb{R}^3)$ of Γ by arclength with $\gamma(0) = \alpha(t_0)$ satisfies $\ddot{\gamma}(0) = \mathbf{0}$ but $\ddot{\gamma}(s) \neq 0$ for $s \neq 0$, and having the following properties:

(a)

$$\lim_{s \to 0} \mathbf{n}(s)$$

does not exist.

(b)

$$\lim_{s \to 0} \mathbf{n}(s)$$

exists, and the function

$$\nu(s) = \begin{cases} \mathbf{n}(s), & s \neq 0 \\ \lim_{s \to 0} \mathbf{n}(s), & s = 0 \end{cases}$$

satisfies $\nu \in C^1((-\ell, m) \to \mathbb{R}^3)$.

(c)

$$\lim_{s \to 0} \mathbf{n}(s)$$

exists, and the function

$$\nu(s) = \begin{cases} \mathbf{n}(s), & s \neq 0 \\ \lim_{s \to 0} \mathbf{n}(s), & s = 0 \end{cases}$$

satisfies $\nu \in C^0((-\ell, m) \to \mathbb{R}^3) \backslash C^1((-\ell, m) \to \mathbb{R}^3)$.

(d)

$$\lim_{s \to 0} \mathbf{n}(s)$$

exists, and the function

$$\nu(s) = \begin{cases} \mathbf{n}(s), & s \neq 0 \\ \lim_{s \to 0} \mathbf{n}(s), & s = 0 \end{cases}$$

satisfies $\nu \notin C^0((-\ell, m) \to \mathbb{R}^3)$.

Exercise 1.26 (definition of a curve) Adapt the discussion of $\mathcal{S}$ in Definition 1 to formulate a formal definition of a curve Γ as a subset of $\mathbb{R}^n$ for $n = 2$ or $n = 3$. Note(s): You may wish to obtain also a definition of a curve with boundary (endpoints). You may also wish to distinguish the compactness properties of the curves you define.

Exercise 1.27 (extension of a surface) Given a regular surface $\mathcal{S}$ with boundary as described in Definition 1, show there exists an open surface $\mathcal{U} \subset \mathbb{R}^3$ with $\overline{\mathcal{S}} \subset \mathcal{U}$.

Exercise 1.28 (simple closed curve) Let $a, b, n, \epsilon \in \mathbb{R}$ with $0 < \epsilon < b - a$ and $n \in \mathbb{N}$. Assume $\Gamma = \{\alpha(t) : a < t < b\}$ with $\alpha \in C^1((a,b) \to \mathbb{R}^n)$ satisfying

(i) $\alpha(t_1) \neq \alpha(t_2)$ for $a < t_1 < t_2 < b - \epsilon$, and

(ii) $\alpha(t) = \alpha(b - a - \epsilon + t)$ for $a < t < a + \epsilon$.

Show

(a) $\alpha(t_1) \neq \alpha(t_2)$ for $a < t_1 < t_2 < b$ if and only if $t_2 = b - a - \epsilon + t_1$.

(b) $a < a - b + t + \epsilon < a + \epsilon$ and $\alpha(t) = \alpha(a - b + t + \epsilon)$ for $b - \epsilon < t < b$.

(c) $\alpha \in C^1([a,b] \to \mathbb{R}^n)$, and calculate $\alpha'(a)$ and $\alpha'(b)$.

(d) Is it necessarily the case that $\alpha'(a) = \alpha'(b)$?

Exercise 1.29 (arclength parameterization in the plane; curvature vectors) Let $\mathfrak{AP}_\mathbf{0}\mathbb{R}^2$ denote the collection of all parameterizations

$$\gamma_0 \in \bigcup_{\epsilon > 0} C^\infty((-\epsilon, \epsilon) \to \mathbb{R}^2)$$

satisfying

(i) $\gamma_0(0) = \mathbf{0} = (0,0)$.

(ii) $|\dot{\gamma}_0| \equiv 1$.

Let

$$\mathbb{S}^1 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$$

denote the unit circle in $\mathbb{R}^2$.

(a) Given $\mathbf{u} \in \mathbb{S}^1$ construct an arclength parameterization $\gamma_0 \in \mathfrak{AP}_{\mathbf{0}}\mathbb{R}^2$ for which

$$\Gamma_0 = \{\gamma_0(s) : s \in (-\epsilon, \epsilon)\}$$

is a regular embedded curve with $|\dot{\gamma}_0| \equiv 1$ and $\dot{\gamma}_0(0) = \mathbf{u}$.

(b) Determine

$$K = \{\ddot{\gamma}_0(0) \in T_{\mathbf{0}}(\mathbb{R}^2) : \gamma_0 \in \mathfrak{AP}_{\mathbf{0}}\mathbb{R}^2\}.$$

(c) Let

$$K_{\mathbf{u}} = \{\ddot{\gamma}_0(0) \in K : \gamma_0 \in \mathfrak{AP}_{\mathbf{0}}\mathbb{R}^2,\ \dot{\gamma}_0(0) = \mathbf{u}\}$$

Given any $\mathbf{u} \in \mathbb{S}^1$ and any curvature vector $\vec{k} \in K_{\mathbf{u}}$, construct an arclength parameterization $\gamma_0 \in \mathfrak{AP}_{\mathbf{0}}\mathbb{R}^2$ satisfying the conditions of part **(a)** and for which $\ddot{\gamma}_0(0) = \vec{k}$.

(d) Describe

$$\{(\dot{\gamma}_0(0), \ddot{\gamma}_0(0)) \in \mathbb{R}^4 : \gamma_0 \in \mathfrak{AP}_{\mathbf{0}}\mathbb{R}^2\}$$

as a submanifold of $\mathbb{R}^4$.

Exercise 1.30 Use (1.46) to describe

$$\{\alpha'(0) = DX\ \dot{\gamma}_0(0) \in T_p\mathcal{S} : \dot{\gamma}_0(0) \in \mathbb{S}^1\}$$

in $T_p\mathcal{S}$ where $X : U \to \mathbb{R}^3$ is a local parameterization of a surface $\mathcal{S}$ with $X(0,0) = p \in \mathcal{S} \subset \mathbb{R}^3$ and $\alpha(s) = X \circ \gamma_0(s)$.

Exercise 1.31 (arclength parameterization on a surface; curvature vectors) Let $\mathcal{S}$ be a surface in $\mathbb{R}^3$ with $p \in \mathcal{S}$. Let $\mathfrak{AP}_p\mathcal{S}$ denote the collection of all parameterizations

$$\gamma \in \bigcup_{\epsilon>0} C^\infty((-\epsilon, \epsilon) \to \mathcal{S})$$

satisfying

(i) $\gamma(0) = p$.

(ii) $|\dot{\gamma}| \equiv 1$.

Let $X : U \to \mathbb{R}^3$ be a local parameterization of $\mathcal{S}$ with $X(0,0) = p$. Use (1.47) to complete the following parts:

(a) Given a regular embedded curve $\Gamma_0 \subset U$ parameterized by $\gamma_0 \in \mathfrak{AP}_{\mathbf{0}}\mathbb{R}^2$, is the curvature vector $\ddot{\gamma}(0)$ where $\gamma \in \mathfrak{AP}_p\mathcal{S}$,

$$\Gamma = \{\gamma(\sigma) : \sigma \in -\sigma_0 < \sigma < \sigma_0\} = \{X \circ \gamma_0(s) : -\epsilon < s < \epsilon\},$$

and

$$\sigma_0 = \int_0^\epsilon |Dx \circ \gamma_0(s)\ \dot{\gamma}_0(s)|\, ds$$

a linear function of the curvature vector $\ddot{\gamma}_0(0) \in T_{\mathbf{0}}\mathbb{R}^2$ for $\dot{\gamma}_0(0) = \mathbf{u}$ fixed in $\mathbb{S}^1$? See Exercise 1.29.

(b) Describe

$$\{\ddot{\gamma}(0) : \gamma \in \mathfrak{AP}_p\mathcal{S}\} \subset T_p\mathbb{R}^3.$$

Hint: Consider various cases determiined by $DX(\mathbf{0})$ and $D^2X(\mathbf{0})$.

Exercise 1.32 (normals on a surface) Let $\mathcal{S}$ be a surface given locally by a regular parameterization $X : U \to \mathbb{R}^3$ near a point $p \in \mathcal{S}$ with U an open subset of $\mathbb{R}^2$. Consider $Y : \{(-u, v) : (u, v) \in U\} \to \mathbb{R}^3$ by $Y(u, v) = X(-u, v)$.

(a) Show $\{(-u, v) : (u, v) \in U\}$ is an open subset of $\mathbb{R}^2$ containing $(0, 0)$ and $Y(0, 0) = p$.

(b) Show Y is a regular local parameterization of $\mathcal{S}$ at $p \in \mathcal{S}$.

(c) Show the parameterization Y reverses the local unit normal field N in the sense that

$$\frac{Y_u \times Y_v}{|Y_y \times Y_v|} = -N = -\frac{X_u \times X_v}{|X_y \times X_v|}. \tag{1.71}$$

Provide for the relation (1.71) the appropriate compositions/arguments in order for it to make precise formal sense, i.e., so that the normal and the reverse normal are clearly located at the same point on $\mathcal{S}$.

Exercise 1.33 (planar surface) The plane

$$\mathcal{S} = \{(x, y, 0) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2\}$$

is a regular surface in $\mathbb{R}^3$.

(a) Find a regular parameterization $X : U \to \mathcal{S} \subset \mathbb{R}^3$ of the plane with $X(0,0) = (0,0,0)$ using some open set U in $\mathbb{R}^2$, and compute

$$X_u = \frac{\partial X}{\partial u}, \qquad X_v = \frac{\partial X}{\partial v}, \qquad \text{and} \qquad N = \frac{X_u \times X_v}{|X_u \times X_v|}.$$

(b) Find a smooth extension $\overline{N} \in C^\infty(V \to \mathbb{R}^3)$ of $N : \mathcal{S} \to \mathbb{R}^3$ with V an open set in $\mathbb{R}^3$ containing $(0,0,0)$. Calculate

$$D\overline{N} = \begin{pmatrix} \dfrac{\partial \overline{N}_1}{\partial x_1} & \dfrac{\partial \overline{N}_1}{\partial x_2} & \dfrac{\partial \overline{N}_1}{\partial x_3} \\[2ex] \dfrac{\partial \overline{N}_2}{\partial x_1} & \dfrac{\partial \overline{N}_2}{\partial x_2} & \dfrac{\partial \overline{N}_2}{\partial x_3} \\[2ex] \dfrac{\partial \overline{N}_3}{\partial x_1} & \dfrac{\partial \overline{N}_3}{\partial x_2} & \dfrac{\partial \overline{N}_3}{\partial x_3} \end{pmatrix} \tag{1.72}$$

where $\overline{N} = (\overline{N}_1, \overline{N}_2, \overline{N}_3)$.

(c) Which entries in the matrix $\overline{M} = D\overline{N}$ are determined uniquely by N and which are arbitrary? Find an extension $\overline{N}$ including arbitrary dependence in those entries which are arbitrary.

Exercise 1.34 (graph) Given $f \in C^\infty(U)$ where U is an open subset of $\mathbb{R}^2$ with $(0,0) \in U$ and $f(0,0) = 0$, consider the surface

$$\mathcal{S} = \{(x, y, f(x,y)) \in \mathbb{R}^3 : (x,y) \in U\}.$$

(a) Find a regular parameterization $X : U \to \mathcal{S} \subset \mathbb{R}^3$ of $\mathcal{S}$ with $X(0,0) = (0,0,0)$, and compute

$$X_u = \frac{\partial X}{\partial u}, \qquad X_v = \frac{\partial X}{\partial v}, \qquad \text{and} \qquad N = \frac{X_u \times X_v}{|X_u \times X_v|}.$$

(b) Find a smooth extension $\overline{N} \in C^\infty(V \to \mathbb{R}^3)$ of $N : \mathcal{S} \to \mathbb{R}^3$ with V an open set in $\mathbb{R}^3$ containing $(0,0,0)$. Calculate $D\overline{N}$ as in (1.72).

(c) Which entries in the matrix $\overline{M} = D\overline{N}$ are determined uniquely by N and which are arbitrary? How about if you assume

$$Df(0,0) = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right) = (0,0)$$

and specialize to $D\overline{N}(0,0)$? Find an extension $\overline{N}$ including arbitrary dependence in those entries which are arbitrary.

Exercise 1.35 (Lagrange multipliers) Prove Theorem 3 and give an example in which the function f has a minimum at a point $\mathbf{x}_0$ with respect to the constraint $g(\mathbf{x}) = 0$ and $Dg(\mathbf{x}_0) \neq \mathbf{0}$, but the function $f - \lambda g$ where λ is the Lagrange parameter for which $D(f - \lambda g)(\mathbf{x}_0) = \mathbf{0}$ does not have a local minimum (i.e., an unconstrained local minimum) at $\mathbf{x}_0$.

Exercise 1.36 (Lagrange multipliers) Let $\mathcal{G} : \mathcal{A} \to \mathbb{R}$ be an integral functional with Lagrangian G so that

$$\mathcal{G}[u] = \int_{\mathbf{x}\in U} G(\mathbf{x}, u, Du)$$

for u in some set $\mathcal{A} \subset C^1(\overline{U})$. Show

$$Dg(\mathbf{0}) = \left(\frac{\partial g}{\partial h}(\mathbf{0}), \frac{\partial g}{\partial k}(\mathbf{0})\right) = (\delta\mathcal{G}_u[\phi], \delta\mathcal{G}_v[\psi])\,.$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(h,k) = \mathcal{F}[u + h\phi + k\psi].$$

Exercise 1.37 (inverse and implicit function theorems) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$g(h,k) = h + k.$$

(a) Draw the image of $\Phi(h,k) = (h, g(h,k))$ on $B_\epsilon(0,0)$.

(b) Draw the graph of the function $k = k(h)$ determined by $k(h) = \beta(h,0)$ where $\Phi^{-1} = (\mathrm{id}, \beta)$ is the inverse of

$$\Phi\Big|_{B_\epsilon(\mathbf{0})}.$$

Exercise 1.38 The claim was made in connection with the equation (1.60) that for a minimizing (solution) of the capillary tubes problem given by $u \in C^2(U)$ where $U = \Omega \cup \Omega_{\text{out}}$ the constant

$$\lambda = \frac{1}{\int_U \psi} \left(\int_U \frac{Du \cdot D\psi}{\sqrt{1+|Du|^2}} + \kappa \int_U u\ \psi \right)$$

is independent of the choice of test function $\psi \in C_c^\infty(U)$. Prove this.

Exercise 1.39 Let $\Gamma \subset \mathbb{R}^2$ be a regular C^1 simple closed curve in the plane. Recall that for each $\mathbf{p} \in \Gamma$, there exists some $\epsilon > 0$ and a local arclength parameterization $\gamma : (-\epsilon, \epsilon) \to \Gamma$ such that

(0) $\gamma(0) = \mathbf{p}$,

(i) $\gamma \in C^1((-\epsilon, \epsilon) \to \mathbb{R}^2)$,

(ii) $|\dot{\gamma}| \equiv 1$, and

(iii) $\gamma(s_2) \neq \gamma(s_2)$ for $s_1 < s_2$.

Show that if $f \in C^0(\Gamma)$ is a nonnegative function with $f(\mathbf{p}) > 0$, then for any $h > 0$

$$\int_\Gamma f \geq \int_{\Gamma \cap B_h(\mathbf{p})} f > 0.$$

Hint: Note that

$$\int_{\Gamma \cap B_h(\mathbf{p})} f \geq \int_{\gamma(-\epsilon,\epsilon) \cap B_h(\mathbf{p})} f$$

and change variables and use the continuity of γ to integrate an appropriate function on a subinterval of the interval $(-\epsilon, \epsilon)$.

Exercise 1.40 Complete the proof of Theorem 5 showing

$$\frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu = \beta_j \qquad \text{on} \qquad \partial\Omega_j,\ j = 1, 2$$

where $\nu = \nu(\mathbf{x})$ is the unit normal to $\partial\Omega_j$ pointing out of Ω_{out}.

Chapter 2

Circular capillary tubes

We have obtained two geometric boundary value problems associated with the capillary tubes problem. Specifically, when and if a meniscus shape modeled by the graph of a function $u : \Omega \to \mathbb{R}$ minimizes the capillary energy, then provided the function u possesses adequate regularity, say $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then u should satisfy

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \kappa u - \lambda, & \text{on } \Omega \\ \\ \cos\gamma = \beta_0, & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where

$$\cos\gamma = \frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu$$

and ν is the unit normal to $\partial\Omega$ pointing out of Ω. The quantity $Tu \cdot \nu$ where

$$Tu = \frac{Du}{\sqrt{1+|Du|^2}}$$

may also be recognized as the dot product of the downward pointing unit normal

$$\frac{(u_x, u_y, -1)}{\sqrt{1+|Du|^2}}$$

with the extension $(\nu_1, \nu_2, 0)$ of ν to $\mathbb{R}^3$. Thus, the quantity $\cos\gamma$ may be identified as the angle at which the free surface interface

$$\mathcal{S} = \{(x, y, u(x, y)) : (x, y) \in \Omega\}$$

meets the vertical surface modeling the inner wall of the capillary tube.

Similarly, we can imagine modeling the outer meniscus with a free interface

$$\mathcal{S}_{\text{out}} = \{(x, y, u(x, y)) : (x, y) \in \Omega_{\text{out}}\}$$

with u satisfying

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \kappa u - \lambda, & \text{on } \Omega_{\text{out}} = \Omega_2 \backslash \overline{\Omega_1} \\ \cos\gamma = \beta_j, & \text{on } \partial\Omega_j,\ j = 1, 2 \end{cases} \tag{2.2}$$

We have also remarked, or it should be obvious, that these problems may be considered independently as models for a liquid capillary interface in the corresponding cylinder with a closed bottom. For all of these problems it is convenient to choose coordinates so the level $z = 0$ corresponds to the level of the projection of the tube domain Ω (or alternatively Ω_{out}) onto the plane modeling the bottom/floor of the container or environment. Naturally, this value $z = 0$ is somewhat arbitrary, and it is interesting that there is a canonical horizontal level built into the partial differential equation. Specifically, if we take a solution u of one of the problems (2.1) or (2.2) and set $w = u - \lambda/\kappa$ then

$$\operatorname{div} Tw = \operatorname{div} Tu = \kappa u - \lambda = \kappa w.$$

Thus we see w satisfies the **first vertically normalized capillary tube system**

$$\begin{cases} \operatorname{div}\left(\dfrac{Dw}{\sqrt{1+|Dw|^2}}\right) = \kappa w, & \text{on } U \\ \cos\gamma = \beta, & \text{on } \partial U. \end{cases} \tag{2.3}$$

Here we can take $U = \Omega \cup \Omega_{\text{out}}$ as we have done above with β now considered as a function taking potentially piecewise constant, or potentially more complicated, values on ∂U. In order to generalize the considerations we may also let U denote one of the domains Ω or Ω_{out} separately. In all cases, there is a solution corresponding to $\beta = 0$ given by $w \equiv 0$ or equivalently

$$u \equiv h_b = \frac{\lambda}{\kappa}.$$

Let us call the number $h_b = \lambda/\kappa$ the **reference bath height**. These solutions are unique:

Theorem 6 If U is an open bounded subset of $\mathbb{R}^2$ with C^1 boundary and $w \in C^2(U) \cap C^1(\overline{U})$ satisfies (2.3) with $\beta \equiv 0$ where

$$\cos\gamma = \nu \cdot \frac{Dw}{\sqrt{1+|Dw|^2}}$$

and ν is the unit normal to ∂U pointing out of U, then $w \equiv 0$.

Proof: As usual we set

$$Tw = \frac{Dw}{\sqrt{1+|Dw|^2}}.$$

Then $Tw - \kappa w \equiv 0$ on U, so

$$\begin{aligned} 0 &= \int_u w(Tw - \kappa w) \\ &= \int_U \operatorname{div}(wTw) - \int_U Dw \cdot Tw - \kappa \int_U w^2 \\ &= \int_{\partial U} wTw \cdot \nu - \int_U \frac{|Dw|^2}{\sqrt{1+|Dw|^2}} - \kappa \int_U w^2 \\ &= -\int_U \frac{|Dw|^2}{\sqrt{1+|Dw|^2}} - \kappa \int_U w^2. \end{aligned}$$

We conclude

$$\int_U \frac{|Dw|^2}{\sqrt{1+|Dw|^2}} = \kappa \int_U w^2 = 0.$$

In particular, $w \equiv 0$. $\square$

This argument can be generalized in remarkable ways. We will consider certain simple generalizations below, but ultimately one should see Theorem 5.1 in [Fin86].

2.1 Inner meniscus

Here we consider the axially symmetric meniscus over the inner disk $B_a(\mathbf{0}) \subset \mathbb{R}^2$ and the meridian equation

$$\frac{u''}{(1+u'^2)^{3/2}} + \frac{1}{r}\,\frac{u'}{\sqrt{1+u'^2}} = u \tag{2.4}$$

in particular. The singular initial value problem

$$\begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} + \dfrac{1}{r}\,\dfrac{u'}{\sqrt{1+u'^2}} = u, \quad 0 < r < a \\ \\ u(0) = u_0 \\ u'(0) = 0 \end{cases} \tag{2.5}$$

admits a unique solution $u \in C^2(0,a) \cap C^1[0,a)$ at least for $a > 0$ small enough. This solution, furthermore, extends to an even real analytic function $u \in C^\omega(-a,a)$, again if a is small enough depending on u_0. There are various ways to justify the existence, uniqueness and regularity assertions just given, but we will postpone these justifications until later and proceed on the assumption that they are correct. One solution $u \equiv 0$ when $u_0 = 0$ is known, but no other solution is known explicitly.

Technically, for the purposes of the calculus of variations and specifically to ensure the free surface energy associated with the inner meniscus is well-defined, we need also $u \in C^1[0,a]$. We should look for solutions with this additional regularity. Recall that there are various equivalent notions of continuous differentiability of a real valued function on a closed interval. In this case, one may consider the left and right limits

$$u'(0^+) = \lim_{t\searrow 0} \frac{u(0+t)-u(0)}{t} \quad \text{and} \quad u'(0^-) = \lim_{t\nearrow 0} \frac{u(a+t)-u(a)}{t}. \tag{2.6}$$

We say $u \in C^1[0,a]$ if these limits exist as real numbers and the derivative

$$u'(r) = \lim_{t\to 0} \frac{u(r+t)-u(r)}{t}$$

for $0 < r < a$ extends to a continuous real valued function on $[0,a]$ with the left and right endpoint values given in (2.6). Alternatively, we can assume $u \in C^1(0,a)$ and there exists some $\epsilon > 0$ and a function $\overline{u} \in C^1(-\epsilon, a+\epsilon)$ for which

$$\overline{u}\big|_{(0,a)} = u.$$

See Exercise 2.1. Notice that for a function $u \in C^1[0,a)$, the condition $u'(0) = 0$ in (2.5) is more properly given by $u'(0^+) = 0$ though it is also true that $\overline{u}'(0) = 0$ for any C^1 extension of u.

In this special case of a function $u \in C^1[0, a)$ satisfying the initial condition $u'(0) = 0$ in (2.5) there is one particular C^1 extension of u that is natural to consider, namely the even extension. This particular extension will be seen to have additional interest when we consider justification of the existence and uniqueness assumptions. For now, however, it is natural whenever we are considering any solution u of (2.5) to denote (also) by u the even extension $u : (-a, a) \to \mathbb{R}$ satisfying $u(x) = u(-x)$ for $x < 0$. We will then have $u \in C^1(-a, a)$.

Notice that if instead of $u \in C^2(0, a)$ we only assume u is twice differentiable, then the continuity of u'' on $(0, a)$ follows immediately from the ordinary differential equation (ODE) in (2.5). The even extension is then seen to satisfy $u \in C^2((-a, 0) \cup (0, a))$, but the additional regularity at $x = 0$ requires some attention. For the moment we make the sweeping assumption that not only does the even extension u satisfy $u \in C^2(-a, a)$ but that for any $c \in [0, a)$ there is some $\epsilon > 0$ such that the series

$$\sum_{n=0}^{\infty} \frac{u^n(c)}{n!} (x - c)^n \qquad \text{converges to } u(x) \text{ for } |x - c| < \epsilon.$$

That is to say, we assume u is **real analytic** and we write $u \in C^\omega(-a, a)$. In particular, for the even extension and taking $c = 0$, we assume u has a power series expansion

$$u(x) = \sum_{k=0}^{\infty} \frac{u^{(2k)}(0)}{(2k)!} x^{2k}.$$

If we wish to consider all solutions of (2.5) and the interesting solutions with $u_0 \neq 0$ in particular then it is enough to consider only solutions with $u_0 > 0$. If we have a solution with $u_0 < 0$, then $-u$ is a solution of the same equation and is a solution of the associated initial value problem obtained from (2.5) by replacing u_0 with $-u_0 > 0$. Accordingly we introduce the condition $u_0 > 0$ as a standing assumption.

2.2 Initial comparison of solutions

In order to make the existence and uniqueness assumptions mentioned above more definite and explicit, let us say that for every real number $u_0 > 0$, there exists some $a_1 > 0$ and a function $u_1 \in C^2(0, a_1) \cap C^1[0, a_1)$ satisfying

$$\begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} + \dfrac{1}{r}\,\dfrac{u'}{\sqrt{1+u'^2}} = u, \quad 0 < r < a_1 \\ \\ u(0) = u_0 \\ u'(0) = 0. \end{cases} \tag{2.7}$$

In accord with the discussion above, we denote the even extension to $(-a_1, a_1)$ also by u_1 and assume $u_1 \in C^\omega(-a_1, a_1)$. As for uniqueness, if there is any other function $u \in C^2(0, a) \cap C^1[0, a)$ satisfying the singular initial value problem (2.5), then we must have

$$u(r) \equiv u_1(r) \qquad \text{for} \qquad 0 < r < \min\{a_1, a\}.$$

It follows of course that $u(0) = u_1(0)$ and $u(x) = u_1(x)$ for $x < 0$ as well. In particular, the even extension of u must be real analytic on $(-a, a)$ as well. In what follows, we will nominally consider a general solution $u \in C^2(0, a) \cap C^1[0, a)$ of (2.5). In view of the foregoing discussion/assumptions if $a \le a_1$, then we are only considering some restriction of u_1. The more interesting possibility is that of considering such a solution with $a > a_1$. In this case, the following existence, uniqueness, and regularity result for nonsingular ODEs has an interesting consequence:

Theorem 7 If $I = (\alpha, \beta)$ is an interval in the real line with $\alpha < \beta$ and $f \in C^1(I \times \mathbb{R}^2)$, then for any $x_0 \in I$, any $y_0 \in \mathbb{R}$ and any $p_0 \in \mathbb{R}$, there exists some $\epsilon > 0$ such that the initial value problem

$$\begin{cases} y'' = f(x, y, y'), \quad x_0 - \epsilon < x < x_0 + \epsilon \\ y(x_0) = y_0, \\ y'(x_0) = p_0, \end{cases} \tag{2.8}$$

has a unique solution $y \in C^2(x_0 - \epsilon, x_0 + \epsilon)$.

Given the solution y, if the structure function f enjoys extra regularity, then the solution will have extra regularity as well:

(k) If $f \in C^k(I \times \mathbb{R}^2)$ for some $k \ge 1$, then $y \in C^{k+2}(I)$.

(ω) If $f \in C^\omega(I \times \mathbb{R}^2)$, then $y \in C^\omega(I)$.

Note: The regularity conditions here are the usual ones for a function of several variables. If $f \in C^k(I \times \mathbb{R}^2)$, then all partial derivatives of order less than or equal to k exist and are continuous; if $f \in C^\omega(I \times \mathbb{R}^2)$, then for each $(x_0, z_0, p_0) \in I \times \mathbb{R}^2$, there exists a convergent multivariable power series in x, z, and p centered at (x_0, z_0, p_0) and convergent in some open ball with center (x_0, z_0, p_0) to the function f.

If $a > a_1$, then $r = a_1$ becomes an interior point in the domain of the solution u. By the uniqueness of u_1 one has that $u_1'(a_1^-) = u'(a_1)$ exists, the function u is real analytic in the interval $(0, a)$, and consequently the function u satisfies all the conditions assumed about the initial solution u_1. See Exercise 2.4. In this case, there seems little reason not to replace u_1 with the alternative function u having a larger domain. The question then arises: Is it possible to find some $a_1 > 0$, and a solution $u_1 \in C^2(0, a_1) \cap C^1[0, a_1)$ of (2.7) for which the problem (2.5) has no solutions when $a > a_1$? The answer turns out to be affirmative as we shall see shortly.

Presently, we consider various properties of solutions $u \in C^2(0, a) \cap C^1[0, a)$ of (2.5) which apply to the special case of u_1, and we do not assume $a \leq a_1$. Perhaps we can take as a first and simplest objective showing $u(r) \geq u_0$ for $0 \leq r < a$ with equality only for $r = 0$. This follows in a sense because $u''(r) > 0$ for $0 \leq r < a$ and consequently $u'(r) \geq 0$ for $0 \leq r < a$ with equality only for $r = 0$. Each of these assertions will need to be justified however. Figure 2.1 shows a numerical plot of a solution u of the singular inititial value problem (2.5) with $u_0 = 1$ and $a = 1$ and gives a visual indication of some of the properties one might wish to establish for the capillary meridian.

A first observation is that

$$\frac{u''}{(1 + u'^2)^{3/2}} + \frac{1}{r} \, \frac{u'}{\sqrt{1 + u'^2}} = k_m + k_\ell = \frac{1}{r} \, \frac{d}{dr} \left(\frac{ru'}{\sqrt{1 + u'^2}} \right)$$

where

$$\frac{d}{dr} \left(\frac{u'}{\sqrt{1 + u'^2}} \right) = k_m$$

is the curvature of the generating curve of the interface, or **meridian curvature**,

$$\frac{1}{r} \, \frac{u'}{\sqrt{1 + u'^2}} = k_\ell$$

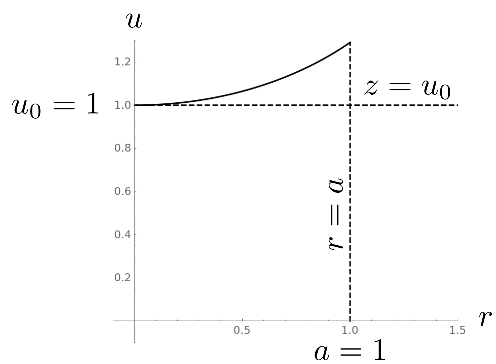

Figure 2.1: A capillary meridian or meniscus profile. This profile extends to be even and real analytic. We are assuming these properties at the moment. $u''(x) > 0$ for $-a < x < a$ so that $u'(r) > 0$ and $u(r) > u_0$ for $0 \leq r < a$. These properties are established below.

is the **latitudinal** (normal) **curvature**, and

$$\frac{u'}{\sqrt{1+u'^2}} = \sin\psi$$

is the sine of the inclination angle along the meridian. Under the assumption that u admits an even power series expansion

$$u(x) = \sum_{k=0}^{\infty} \frac{u^{(2k)}(0)}{(2k)!}\ x^{2k} = u_0 + \frac{u''(0)}{2}\ x^2 + \cdots,$$

we must have

$$u''(0) = \lim_{r\searrow 0}\left[u(r)\ (1+u'(r)^2) - \frac{1}{r}\ u'(r)\ (1+u'(r)^2)\right], \tag{2.9}$$

and each of the normal curvatures

$$k_m(0) = \lim_{r\searrow 0} \frac{u''}{(1+u'^2)^{3/2}} \qquad \text{and} \qquad k_\ell(0) = \lim_{r\searrow 0} \frac{1}{r}\ \frac{u'(r)}{\sqrt{1+u'(r)^2}}$$

must have a well-defined real limit in particular. Notice that by L'Hopital's rule

$$\lim_{r\searrow 0} \frac{u'(r)}{r} = u''(0),$$

and it follows from (2.9) that

$$u''(0) = u_0 - u''(0) \qquad \text{or} \qquad u''(0) = \frac{u_0}{2}.$$

We conclude further that

$$k_m(0) = k_\ell(0) = \frac{u_0}{2}.$$

One consequence of the fact that $u''(0) = u_0/2 > 0$ is that there exists some $\alpha > 0$ for which

$$u''(r) > 0 \qquad \text{for} \qquad 0 < r < \alpha. \tag{2.10}$$

The number $R > 0$ given by

$$\begin{aligned} R &= \sup\{\alpha \in \mathbb{R} : u''(r) > 0 \text{ for } 0 < r < \alpha\} & (2.11)\\ &= \sup\{\alpha \in \mathbb{R} : u'(r) > 0 \text{ and } u''(r) > 0 \text{ for } 0 < r < \alpha\} & (2.12)\\ &= \sup\{\alpha \in \mathbb{R} : u(r) > u_0,\ u'(r) > 0 \text{ and } u''(r) > 0 \text{ for } 0 < r < \alpha\} & (2.13)\end{aligned}$$

has special significance. In addition to the inequalities (2.11), (2.12), and (2.13) defining the radius R, a fourth condition is also implicit, namely that a twice differentiable solution u of the initial value problem (2.5) **exists** for $0 < r < \alpha$. Note this point carefully: In the definition of the set

$$E = \{\alpha \in \mathbb{R} : u''(r) > 0 \text{ for } 0 < r < \alpha\}$$

we are implicitly considering a family of solutions u each defined and twice differentiable on an interval $[0, \alpha)$. In view of the existence and uniqueness of solutions discussed above each such solution must also satisfy $u(x) = u_1(x)$ for $0 < x < \min\{a_1, \alpha\}$, and more generally any two such solutions must agree on their common interval of definition. In this way, we obtain a unique solution of the singular initial value problem

$$\begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} + \dfrac{1}{r}\,\dfrac{u'}{\sqrt{1+u'^2}} = u, \quad 0 < r < R \\[2ex] u(0) = u_0 \\ u'(0) = 0, \end{cases} \tag{2.14}$$

and by the previous observation, we might as well assume $a_1 \geq R$ (excepting the possibility that $R = +\infty$ which—it turns out—is not the case). That is, $u_1(x) = u(x)$ for $0 \leq x < R$. At the very least we can conclude at this point the following result:

Theorem 8 (convexity interval theorem) There is some unique extended real number $R > 0$, the one defined in (2.11), for which

(i) the singular initial value problem (2.14) admits a solution $u \in C^\omega(-R, R)$,

(ii) $u''(r) > 0$ for $0 < r < R$,

(iii) $u'(r) > 0$ for $0 < r < R$, and

(iv) $u(r) > u_0$ for $0 < r < R$.

Let us assume for a moment that $R < \infty$. The question to ask might be: What happens at $r = R$ which prevents any further extension of u with the same properties, or why, if the domain of u_1 extends beyond R with $a_1 > R$ which properties fail? Notice there are basically two possibilities: According to the definition (2.11), either

(i) The solution u_1 fails to exist, that is $a_1 = R$ or

(ii) $a_1 > R$, but the condition $u_1''(r) > 0$ fails for some $r \geq R$ (arbitrarily close to $r = R$).

In the latter case one must have $u''(R) = u_1''(R) = 0$. Since $u''(R) > 0$ for $0 < r < R$, one must also have in this case

$$u'''(R) = u_1'''(R) \leq 0. \tag{2.15}$$

Differentiating the ODE (2.4) we find

$$u''' = u' \ (1 + u'^2)^{3/2} + 3uu'u'' \ \sqrt{1 + u'^2} + \frac{1}{r^2} \ u'(1 + u'^2) - \frac{1}{r} \ u'' - \frac{3}{r} \ u'^2 u''.$$

Thus, in particular

$$\begin{aligned} u_1'''(R) &= u_1'(R) \ (1 + u_1'(R)^2)^{3/2} + \frac{1}{R^2} \ u_1'(R)(1 + u_1'(R)^2) \\ &= u_1'(R)(1 + u_1'(R)^2) \left(\frac{1}{R^2} + \sqrt{1 + u_1'(R)^2} \right). \end{aligned} \tag{2.16}$$

But

$$u_1'(R) = \lim_{r \nearrow R} u'(r).$$

Given the monotonicity $u''(r) > 0$ for $0 < r < R$, either the limit

$$\lim_{r \nearrow R} u'(r). \tag{2.17}$$

is a finite positive number or $+\infty$. Consider the situation in which

$$\lim_{r \nearrow R} u'(r) = u'(R^-) = u_1'(R)$$

is a positive real number. In this case we see from (2.16) that

$$u_1'''(R) \ge u_1'(R)\left(1 + \frac{1}{R^2}\right) > 0$$

contradicting (2.15). This means

$$\lim_{r \nearrow R} u'(r) = +\infty$$

and significantly

$$\lim_{r \nearrow R} u'(r) = \lim_{r \nearrow R} u_1'(r) = +\infty.$$

This contradicts the assumption of case **(ii)** altogether and suggests a more direct approach is to consider the limit in (2.17) from the beginning; see Exercise 2.7.

One consequence of the discussion above and/or Exercise 2.7 is that if $R < \infty$, then we know the singular initial value problem (2.5) has no solution on an interval $[0, a)$ with $a > R$. In particular, $a_1 \le R$ and the situation described in **(ii)** above never happens. This means the only possibility is **(i)** in which (we might as well take) $a_1 = R$ and the solution u_1 (always) satisfies

$$u_1''(r) > 0, \; u_1'(r) > 0, \text{ and } u_1(r) > u_0 \qquad \text{for} \qquad 0 < r < a_1.$$

In this case the unique global solution $u_1 \in C^\omega(-a_1, a_1)$ (somehow) fails to exist at or beyond $r = a_1$. In fact, we know something fairly precise about how that failure of existence occurs. We know

$$\lim_{r \nearrow R} u_1'(r) = +\infty.$$

At this point, all of these insightful assertions still leave open the possibility that $R = +\infty$, so let us now show that cannot happen. If $R = +\infty$, we have a (unique) solution of the singular initial value problem $u \in C^2(0, \infty)$ with

$$u''(r) > 0, \; u'(r) > 0, \text{ and } u(r) > u_0 \qquad \text{for} \qquad r > 0; \tag{2.18}$$

see Exercise 2.9. As a consequence of the last inequality in (2.18) we know also

$$\frac{d}{dr}\left(r \, \frac{u'}{\sqrt{1+u'^2}}\right) = ru > ru_0 \qquad \text{for} \qquad r > 0.$$

Let us make a comparison to a certain circular arc given by the graph of a function $c \in C^2[0, r_0) \cap C^0[0, r_0]$ with

$$r_0 = \frac{2}{u_0}$$

and

$$c(r) = u_0 + r_0 - \sqrt{r_0^2 - r^2}.$$

Notice that

$$c' = \frac{r}{\sqrt{r_0^2 - r^2}} \qquad \text{and} \qquad 1 + c'^2 = \frac{r_0^2}{r_0^2 - r^2}$$

so

$$\frac{c'}{\sqrt{1+c'^2}} = \frac{r}{r_0} = \frac{1}{2}\, ru_0$$

and

$$\frac{d}{dr}\left(r \, \frac{c'}{\sqrt{1+c'^2}}\right) = ru_0.$$

Evidently then, our comparison can take the form

$$\frac{d}{dr}\left(r \, \frac{c'}{\sqrt{1+c'^2}}\right) < \frac{d}{dr}\left(r \, \frac{u'}{\sqrt{1+u'^2}}\right).$$

This strict inequality holds for $0 < r < \min\{r_0, R\} = r_0$, and in fact there is no singularity in these expressions at $r = 0$ where they are both equal and, of course, take the value 0. Thus, we can integrate to find

$$\frac{c'}{\sqrt{1+c'^2}} < \frac{u'}{\sqrt{1+u'^2}} < 1 \qquad \text{for} \qquad 0 < r < \frac{2}{u_0}. \tag{2.19}$$

The inequality in (2.19) is interesting in several respects. First of all, notice it certainly justifies our assertion that the extended real number

$$R = \sup\{\alpha \in \mathbb{R} : u''(r) > 0 \text{ for } 0 < r < \alpha\}$$

is actually a finite real number since

$$\lim_{r \nearrow r_0} \frac{c'}{\sqrt{1 + c'^2}} = \frac{u_0}{2} \lim_{r \nearrow r_0} r = 1$$

so that

$$\lim_{r \nearrow r_0} \frac{u'}{\sqrt{1 + u'^2}} = 1$$

as well, meaning

$$\lim_{r \nearrow r_0} u'(r) = +\infty$$

contradicting our assumption $R = +\infty$ and specifically the consequence $u \in C^2(0, \infty)$. So that question is settled, and Theorem 8 can be strengthened substantially:

Theorem 9 (improved convexity interval theorem) There is a unique real number $R > 0$ satisfying

$$0 < R \leq \frac{2}{u_0} \tag{2.20}$$

for which the singular initial value problem (2.14) with $u_0 > 0$ has a unique solution $u \in C^2(0, R) \cap C^1[0, R)$. Furthermore, the solution u satisfies

(i) $u \in C^\omega[0, R)$ and u has an even real analytic extension $u \in C^\omega(-R, R)$,

(ii) There holds

$$u''(0) = \frac{u_0}{2}, \qquad \text{and} \qquad u''(r) > 0 \qquad \text{for} \qquad 0 \leq r < R$$

so that

$$u''(x) > 0, \qquad -R < x < R,$$

(iii)

$$u(x) \geq u_0, \qquad -R < x < R$$

with equality only for $x = 0$, and

(iv)

$$u'(r) \geq 0, \qquad 0 < r < R$$

with equality only for $r = 0$ and

$$\lim_{r \nearrow R} u'(r) = +\infty.$$

Notice also the geometric meaning of the quantities

$$\sin \psi = \frac{u'}{\sqrt{1 + u'^2}}$$

and

$$\sin \psi_c = \frac{c'}{\sqrt{1 + c'^2}}$$

appearing in (2.19). Here we recall that ψ is the **inclination angle** of the graph of u nominally measured with respect to the positive r direction and determined up to an additive multiple of 2π. In practice we can take $0 \leq \psi = \psi(r) \leq \pi/2$ for $0 \leq r < a_1 = R$. We have introduced also the inclination angle ψ_c associated with the circular arc which is the graph of c as indicated in Figure 2.2 where we have drawn the circular arc of radius $r_0 = 2$ associated with $u_0 = 1$ in relation to the solution plotted in Figure 2.1.

With all the virtues of Theorem 9, there are still many questions left unanswered. Two of the most obvious are the following:

(a) Can we assert that

$$R < r_0 = \frac{2}{u_0}$$

or is it the case that $R = r_0$?

(b) What about

$$\lim_{r \nearrow R} u(r)?$$

Is this value finite like

$$\lim_{r \nearrow r_0} c(r) = u_0 + \frac{2}{u_0},$$

or do we have

$$\lim_{r \nearrow R} u(r) = +\infty?$$

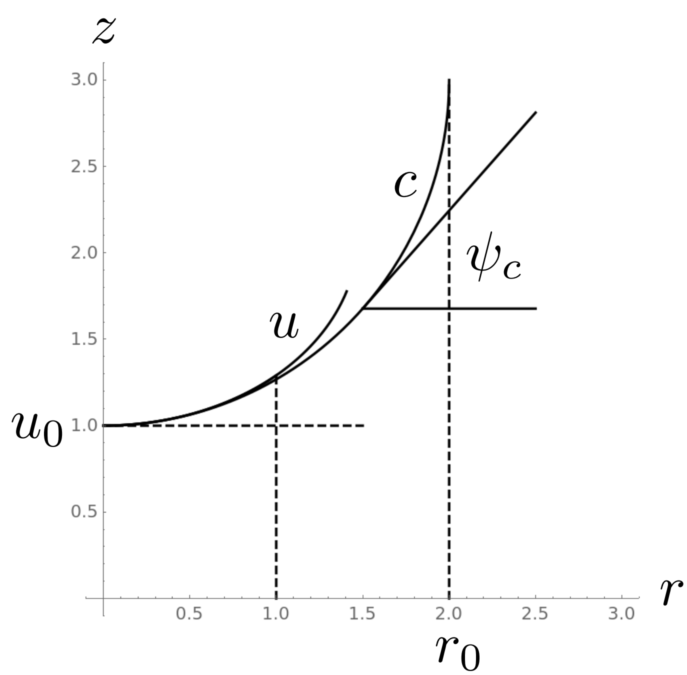

Figure 2.2: Comparison of a capillary meridian given by the graph of a function u and a circular arc given by the graph of a function c with $c(0) = u_0$, $c'(0) = 0$, and $c''(0) = u''(0) = u_0/2$. We obtain by comparison that the inclination of the meridian is greater than that of the circular arc at corresponding radii $r > 0$. Consequently, the capilary meridian is always above the circular arc, that is, $u(r) > c(r)$ for $0 < r < R$, and the capillary meridian must become vertical at some positive radius $R \leq r_0 = 2/u_0$ where r_0 is the radius at which the circular arc becomes vertical (at finite height $u_0 + 2/u_0$). I have drawn an extended portion of the capillary meridian so the meridian can be clearly distinguished from the circular arc, though I have not drawn it up to the vertical point at $r = R$ nor indicated the location of the actual value of R. It remains unclear at this point in the discussion if the height of the capillary meridian tends to infinity as $r \nearrow R$ (which, it turns out, it does not) and if the strict inequality $R < r_0 = 2/u_0$ holds (which, it turns out, does hold).

Before we attempt to discuss the answers to these new questions about the capillary meridian, let me extend the comments concerning the comparison inequality (2.19). Each sine-quotient of the form

$$\frac{u'}{\sqrt{1+u'^2}}$$

can be expressed as $\sigma(u')$ where $\sigma : \mathbb{R} \to (-1,1)$ by

$$\sigma(p) = \frac{p}{\sqrt{1+p^2}}.$$

The function σ is a real analytic increasing bijection with inverse

$$\sigma^{-1} : (-1,1) \to \mathbb{R} \qquad \text{by} \qquad \sigma^{-1}(s) = \frac{s}{\sqrt{1-s^2}}.$$

Since σ^{-1} is also increasing, applying σ^{-1} to both sides of (2.19) yeilds

$$u'(r) > c'(r) = \frac{r}{\sqrt{r_0^2 - r^2}} \qquad \text{for} \qquad 0 < r < R$$

as expected, and integration gives

$$u(r) > c(r) \qquad \text{for} \qquad 0 < r < R$$

as well. Thus we see comparison of inclination is a useful and relatively powerful technique. See also Exercise 2.11.

2.3 Coefficients and comparison

The singularity in the equation at $r = 0$ is an interesting problematic aspect of any analysis of the axially symmetric solutions. One of our initial objectives will be to say something about the coefficients in the power series expansion

$$u(r) = \sum_{k=0}^{\infty} \frac{u^{(2k)}(0)}{(2k)!}\, r^{2k}$$

and attempt to obtain estimates from above and below for the solution u. We can rewrite the equation in the form

$$u'' = u(1+u'^2)^{3/2} - \frac{1}{r}\, u'(1+u'^2)$$

and set

$$f_2(z,p_1) = z\ (1+p_1^2)^{3/2}$$
$$g_2(z,p_1) = -p_1^3 - p_1$$

so that

$$u'' = f_2(u,u') + \frac{1}{r} g_2(u,u'). \tag{2.21}$$

Notice that under our assumption that u is a real analytic solution we must have

$$\lim_{r\searrow 0} f_2(u,u') = \lim_{r\searrow 0} u(1+u'^2)^{3/2} = u_0$$

and

$$\lim_{r\searrow 0} \frac{1}{r}\ f_2(u,u') = \lim_{r\searrow 0} \frac{u'(1+u'^2)}{r} = u''(0).$$

It follows from (2.21) that $2u''(0) = u_0$ and

$$u''(0) = \frac{u_0}{2}.$$

More generally, if we assume (inductively) that

$$u^{(j)} = f_j(u,u',\dots,u^{(j-1)}) + \frac{1}{r}\ g(u,u',\dots,u^{(j-1)})$$

for some smooth functions $f_j = f_j(p_0,p_1,\dots,p_{j-1})$ and $g_j = g_j(p_0,p_1,\dots,p_{j-1})$, then

$$r\ u^{(j)} = r f_j(u,u',\dots,u^{(j-1)}) + g_j(u,u',\dots,u^{(j-1)})$$

and

$$r\ u^{(j+1)} = f_j(u,u',\dots,u^{(j-1)}) + r\ \sum_{m=0}^{j-1} \frac{\partial f_j}{\partial p_m}(u,u',\dots,u^{(j-1)})\ u^{(m+1)} + \sum_{m=0}^{j-1} \frac{\partial g_j}{\partial p_m}(u,u',\dots,u^{(j-1)})\ u^{(m+1)}.$$

Thus $u^{(j+1)}$ has the same inductive form

$$u^{(j+1)} = f_{j+1}(u,u',\dots,u^{(j)}) + \frac{1}{r}\ g_{j+1}(u,u',\dots,u^{(j)})$$

with

$$\begin{aligned} f_{j+1} &= f_{j+1}(p_0, p_1, \ldots, p_j) \\ &= \sum_{m=0}^{j-1} \frac{\partial f_j}{\partial p_m}\, p_{m+1} \quad \text{and} \\ g_{j+1} &= g_{j+1}(p_0, p_1, \ldots, p_j) \\ &= f_j + \sum_{m=0}^{j-1} \frac{\partial g_j}{\partial p_m}\, p_{m+1}. \end{aligned}$$

2.4 Exercises

Exercise 2.1 (continuous differentiability) Let a and b be extended real numbers with $a < b$. Recall that a real valued function $u : (a,b) \to \mathbb{R}$ is **differentiable at** $x \in (a,b)$ if

$$\lim_{t \to 0} \frac{u(x+t) - u(x)}{t} = u'(x)$$

exists. The function $u : (a,b) \to \mathbb{R}$ is **differentiable** (on all of (a,b)) if u is differentiable at each $x \in (a,b)$. A differentiable function determines a real valued function $u' : (a,b) \to \mathbb{R}$, and is said to be **continuously differentiable** if the function $u' \in C^0(a,b)$. In this case, we write $u \in C^1(a,b)$, that is, the collection of all continuously differentiable functions on an open interval (a,b) is denoted $C^1(a,b)$. These definitions all apply to a real valued function with domain an open interval (a,b).

(a) Assume $b < \infty$ and show the following are equivalent:

(i) The limit

$$u'(b^-) = \lim_{t \nearrow b} \frac{u(b+t) - u(b)}{t}$$

exists as a real number.

(ii) There exists a function $\overline{u} \in C^1(a,\infty)$ with

$$\overline{u}\big|_{(a,b)} \equiv u.$$

If either of the equivalent conditions **(i)** and **(ii)** hold, we say u is **continuously differentiable** on $(a, b]$. The collection of all continuously differentiable functions on the half closed interval $(a, b]$ is denoted $C^1(a, b]$.

(b) Formulate a notion of continuous differentiability for a real valued function $u : [a, b) \to \mathbb{R}$ when $-\infty < a$.

(c) Formulate a notion of continuous differentiability for a real valued function $u : [a, b] \to \mathbb{R}$ when $-\infty < a < b < \infty$.

(d) Show $C^1[a, b] \subset C^0[a, b]$.

Exercise 2.2 Show that for $a > 0$ and $u_0' \neq 0$, there is no solution of the initial value problem

$$\begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} + \dfrac{1}{r}\dfrac{u'}{\sqrt{1+u'^2}} = u, \quad 0 < r < a \\ \\ u(0) = u_0 \\ u'(0) = u_0'. \end{cases}$$

Exercise 2.3 Assume $a > 0$ and $u \in C^2(0, a) \cap C^1[0, a)$ is a solution of (2.5) as described above. Let u denote also the even extension of u to the open interval $(-a, a)$ as usual, and assume $u \in C^2(-a, a)$. Show

$$u''(0) = \frac{u_0}{2}.$$

Exercise 2.4 Let u_1 be a solution of (2.7) satisfying the existence, uniqueness, and regularity assumptions described in section 2.2. Specifically,

(i) $u_1 \in C^2(0, a_1) \cap C^1[0, a_1)$ is a solution of (2.7).

(ii) If $u \in C^2(0, a) \cap C^1[0, a)$ is a solution of (2.5) for some $a > 0$, then $u(r) \equiv u_1(r)$ for $0 < r < \min\{a, a_1\}$.

(iii) The even extension of u_1, also called u_1, satisfies $u_1 \in C^\omega(-a_1, a_1)$.

Let $u \in C^2(0, a) \cap C^1[0, a)$ be a solution of (2.5) for some $a > 0$ with $a > a_1$. Show the following:

(a) If $u_2 \in C^2(0, a_2) \cap C^1[0, a_2)$ is a solution of

$$\begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} + \dfrac{1}{r}\,\dfrac{u'}{\sqrt{1+u'^2}} = u, \quad 0 < r < a_2 \\ \\ u(0) = u_0 \\ u'(0) = 0 \end{cases} \tag{2.22}$$

for some $a_2 > 0$, then $u_2(r) \equiv u(r)$ for $0 < r < \min\{a, a_2\}$.

(iii) The even extension of u, also called u, satisfies $u \in C^\omega(-a, a)$.

Hint: Use the properties of the initial solution u_1 and Theorem 7.

Exercise 2.5 Let ϵ and a be positive real numbers. Assume $u \in C^3(-a - \epsilon, a + \epsilon)$ is a solution of the singular initial value problem

$$\begin{cases} u'' = f(r, u, u'), \quad -a < r < a \\ u(0) = u_0 \\ u'(0) = 0 \end{cases}$$

for the capillary equation with

$$f(r, z, p) = z\ (1 + p^2)^{3/2} - \frac{1}{r}\ p(1 + p^2).$$

If $u''(r) > 0$ for $0 < r < a$, then show $u''(a) > 0$. Hint: Argue by contradiction.

Exercise 2.6 Show the suprema in (2.11), (2.12), and (2.13) define the same number R associated with the singular initial value problem (2.5). Hint: By integration a third condition

$$u'(r) = \int_0^r u''(\rho)\, d\rho \qquad \text{and} \qquad u(r) = u_0 + \int_0^r u'(\rho)\, d\rho.$$

Exercise 2.7 Assume u is a solution of the singular initial value problem

$$\begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} + \dfrac{1}{r}\,\dfrac{u'}{\sqrt{1+u'^2}} = u, \quad 0 < r < R \\ \\ u(0) = u_0 \\ u'(0) = 0 \end{cases}$$

given in (2.14) where

$$R = \sup\{\alpha \in \mathbb{R} : u''(r) > 0 \text{ for } 0 < r < \alpha\}$$

as in (2.11). Assume $R < \infty$ and show directly that

$$\lim_{r \nearrow R} u'(r) = +\infty.$$

Conclude as an immediate corollary that there is no solution u_1 of the singular initial value problem

$$\begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} + \dfrac{1}{r}\,\dfrac{u'}{\sqrt{1+u'^2}} = u, \quad 0 < r < a_1 \\ \\ u(0) = u_0 \\ u'(0) = 0 \end{cases}$$

for any $a_1 > R$, that is $a_1 \le R$. Hint(s): Argue by contradiction, and complete the following steps.

(a) Show the limit

$$\lim_{r \nearrow R} u(r)$$

also exists as a (finite) real number $u(R)$ with $u_0 < u(R) < \infty$.

(b) Write down a nonsingular initial value problem for the capillary ODE (2.4) with initial value at $r = R$ and apply Theorem 7 to find a solution $v \in C^2(0, a)$ of the singular initial value problem (2.5) on an interval $[0, a)$ with $a > R$.

(c) Use Theorem 7 again to show $v(r) = u(r)$ for $0 \le r < R$ and consequently

$$v''(R) = \lim_{r \nearrow R} u''(r) \ge 0.$$

(d) Show $v''(0) > 0$ and get a contradiction of the definition of R.

Note this argument still leaves open the possibility that $R = +\infty$.

Exercise 2.8 Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Prove carefully that if

(i) $f \in C^1[\alpha, \beta]$,

(ii) $f(x) > 0$ for $\alpha < x < \beta$, and

(iii) $f(\beta) = 0$,

then $f'(\beta) \le 0$. Explain how this assertion is used to obtain (2.15) and step **(d)** in Exercise 2.7.

Exercise 2.9 Assume u is a solution of the singular initial value problem

$$\begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} + \dfrac{1}{r}\,\dfrac{u'}{\sqrt{1+u'^2}} = u, \quad 0 < r < R \\ \\ u(0) = u_0 \\ u'(0) = 0 \end{cases}$$

given in (2.14) where

$$R = \sup\{\alpha \in \mathbb{R} : u''(r) > 0 \text{ for } 0 < r < \alpha\}$$

as in (2.11). Assume $R = +\infty$ and show that in this case there exists a unique solution $u_1 \in C^\omega(\mathbb{R})$ of the singular initial value problem

$$\begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} + \dfrac{1}{x}\,\dfrac{u'}{\sqrt{1+u'^2}} = u, \quad x \in \mathbb{R}\backslash\{0\} \\ \\ u(0) = u_0 \\ u'(0) = 0. \end{cases}$$

Show also that the solution u_1 is even and satisfies $u_1''(x) > 0$ for $x \in \mathbb{R}$, so that u_1' is odd with $u_1'(r) > 0$ for $r > 0$ and $u_1(x) \ge u_0$ for $x \in \mathbb{R}$ with equality only for $x = 0$.

Exercise 2.10 Give a modified version of the proof of Theorem 8 that applies to the case $R = +\infty$.

Exercise 2.11 (inclination angle) Given $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, assume $u, v \in C^2[\alpha, \beta]$ with $u'(\alpha) = v'(\alpha)$.

(a) Plot the graphs of the functions σ and σ^{-1} where

$$\sigma(p) = \frac{p}{\sqrt{1+p^2}}.$$

(b) If $\sigma(u') > \sigma(v')$ for $\alpha < x < \beta$, then is it true that $u''(x) > v''(x)$?

(c) If $\sigma(u') > \sigma(v')$ for $\alpha < x < \beta$, then is it true that

$$\frac{u''(x)}{(1+u'(x)^2)^{3/2}} > \frac{v''(x)}{(1+v'(x)^2)^{3/2}}?$$

These are the curvatures of the graphs of u and v respectively.

Exercise 2.12 (uniqueness for nonsingular ODEs) Note that the ODE for a capillary meridian in the singular initial value problem

$$\begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} + \dfrac{1}{r}\,\dfrac{u'}{\sqrt{1+u'^2}} = u, \quad 0 < r < a_2 \\ \\ u(0) = u_0 \\ u'(0) = 0 \end{cases} \tag{2.23}$$

where $a_2 > 0$ can be written in the form

$$u'' = f(r, u, u')$$

with $f \in C^\omega(I \times \mathbb{R}^2)$ given by

$$f(r, z, p) = z\ (1+p^2)^{3/2} - \frac{1}{r}\ p\ (1+p^2) \tag{2.24}$$

on any interval $I = (\alpha, \beta)$ with $0 < \alpha < \beta$. In short, the capillary equation is nonsingular for positive r, and so the initial value problem is well-behaved.

The following steps give an alternative approach to obtaining the unique solution of the singular initial value problem for the capillary meridian discussed above.

(a) Assume there exists some initial solution u_1 of the singular initial value problem

$$\begin{cases} \dfrac{u''}{(1+u'^2)^{3/2}} + \dfrac{1}{r}\,\dfrac{u'}{\sqrt{1+u'^2}} = u, \quad 0 < r < a_1 \\ \\ u(0) = u_0 \\ u'(0) = 0 \end{cases} \tag{2.25}$$

given in (2.5) for some $a_1 > 0$ and that the solution u_1 is unique among solutions $u \in C^2(0, a_1) \cap C^1[0, a_1)$ of the problem.

Show $u_2 : [0, R) \to \mathbb{R}$ given by

$$u_2(r) = u(r),$$

where $u \in C^2(0, a_2) \cap C^1[0, a_2)$ is any solution of the singular initial value problem (2.5) for some $a_2 > r$ and

$$R = \sup\{a \in \mathbb{R} : \text{(2.5) admits a solution } u \in C^2(0, a) \cap C^1[0, a)\},$$

determines a unique real analytic solution $u_2 \in C^\omega(-R, R)$. Hint: Use Theorem 7 and the assumption that R is a well-defined positive extended real number.

(b) Show $R \geq a_1$ and $u_2(r) = u_1(r)$ for $0 \leq r < a_1$.

(c) Show $u_2''(r) > 0$ for $0 \leq r < R$.

Hint: Consider the nonsingular initial value problem

$$\begin{cases} v'' = f(r, v, v'), & a - \epsilon < a < a + \epsilon \\ v(a) = u(a) \\ v'(a) = u'(a) \end{cases} \tag{2.26}$$

where $f \in C^\omega(I \times \mathbb{R}^2)$ has values given in (2.24).

Chapter 3

Maximum and comparison principles

$$Mu = \sum_{i,j=1}^{n} a_{ij}\ D_i D_j u + \sum_{j=1}^{n} b_j\ D_j u$$

$$\langle (a_{ij})\xi, \xi \rangle \geq \lambda |\xi|^2$$

In particular, $a_{ii} \geq \lambda > 0$ for $i = 1, 2, \ldots, n$.

Theorem 10 (weak maximum principle—first version) If M is uniformly elliptic in a bounded open set Ω, the coefficient functions b_j are bounded in Ω, that is $b_j \in C^0(\Omega) \cap L^\infty(\Omega)$, and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $Mu \geq 0$ in Ω, then

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x). \tag{3.1}$$

Proof: First consider the case $Mu > 0$ in Ω. In this case, we obtain a stronger result:

$$u(p) < \max_{x \in \partial\Omega} u(x) \qquad \text{for} \qquad p \in \Omega.$$

In other words the function u cannot obtain an interior maximum. This is a version of the strong maximum principle. To see this, assume there is some $p \in \Omega$ with

$$u(p) = \max_{x \in \partial\Omega} u(x).$$

We have used here the fact that Ω is bounded and $u \in C^0(\overline{\Omega})$. Then we have

$$\frac{\partial u}{\partial x_j}(p) = 0 \qquad \text{for} \qquad j = 1, 2, \ldots, n$$

and

$$D^2u(p) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(p)\right)$$

is a negative semidefinite matrix.[1] Since the coefficient matrix $(a_{ij}(p))$ is positive definite, it follows that

$$\sum_{i,j=1}^{n} a_{ij}(p)\frac{\partial^2 u}{\partial x_i \partial x_j}(p) \le 0.$$

See Lemma 12 below. From this we have

$$\begin{aligned} 0 &< Mu(p) \\ &= \sum_{i,j=1}^{n} a_{ij}(p)\ D_iD_ju(p) + \sum_{j=1}^{n} b_j(p)\ D_ju(p) \\ &= \sum_{i,j=1}^{n} a_{ij}(p)\ D_iD_ju(p) \\ &\le 0, \end{aligned}$$

which is a contradiction.

Next consider $v \in C^\omega(\mathbb{R}^n)$ given by

$$v(x) = \epsilon e^{\gamma x_1}$$

for positive numbers ϵ and γ. Computing we see

$$\begin{aligned} Mv &= \epsilon\ \gamma\ (\gamma\ a_{11} + b_1)\ e^{\gamma x_1} \\ &\ge \epsilon\ \gamma\ (\gamma\ \lambda - B)\ e^{\gamma x_1} \end{aligned}$$

where

$$B = \sup_{x\in\Omega, 1\le j\le n} |b_j(x)| = \sup_{1\le j\le n} \|b_j\|_{L^\infty(\Omega)}.$$

Thus taking

$$\gamma > \frac{B}{\lambda}$$

we have $Mv > 0$ on Ω.

[1] See Lemma 11 below.

Finally, since M is linear we have

$$M[u+v] = Mu + Mv > 0$$

where $v(x) = \epsilon e^{\gamma x_1}$ as above and $\gamma > B/\lambda$. Thus, by the version of the strong maximum principle proved above

$$u(p) + \epsilon\ e^{\gamma p_1} < \max_{x\in\partial\Omega} [u(x) + \epsilon\ e^{\gamma x_1}]$$

for every $p \in \Omega$. Since $u \in C^0(\overline{\Omega})$ and Ω is bounded,

$$\max_{x\in\overline{\Omega}} [u(x) + \epsilon\ e^{\gamma x_1}] = \sup_{p\in\overline{\Omega}} [u(p) + \epsilon\ e^{\gamma p_1}] \leq \max_{x\in\partial\Omega} [u(x) + \epsilon\ e^{\gamma x_1}],$$

that is, the weak maximum principle holds for $u + v$. Since we have imposed no restriction on ϵ up to this point, we can take $\epsilon \searrow 0$ in

$$\max_{x\in\overline{\Omega}} [u(x) + \epsilon\ e^{\gamma x_1}] \leq \max_{x\in\partial\Omega} [u(x) + \epsilon\ e^{\gamma x_1}]$$

and conclude

$$\max_{x\in\overline{\Omega}} u(x) \leq \max_{x\in\partial\Omega} u(x). \qquad \square$$

Lemma 11 (necessary conditions for an interior maximum) If p is a point in an open set U and $u \in C^2(U)$ satisfy

$$u(p) \geq u(x) \qquad \text{for} \qquad x \in U,$$

then $Du(p) = \mathbf{0} \in \mathbb{R}^n$ and $D^2u(p)$ satisfies

$$\langle D^2u(p)\xi, \xi\rangle \leq 0 \qquad \text{for} \qquad \xi \in \mathbb{R}^n.$$

Lemma 12 (definiteness and trace) If $A = (a_{ij})$ is a real symmetric postive definite matrix and $H = (b_{ij})$ is a real symmetric negative semidefinite matrix, then

$$\sum_{i,j} a_{ij} b_{ij} \leq 0.$$

Appendix A

Appendix: Notation

A.1 Sets, functions and regularity

A.1.1 open balls

Given a metric space X, which for us will usually be some Euclidean space $\mathbb{R}^n$, with distance $d : X \times X \to [0, \infty)$, some $r > 0$ and some $p \in X$, the **open ball** of radius r and center p is

$$B_r(p) = \{x \in X : d(x, p) < r\}.$$

A.1.2 functions

Given sets X and Y, a function f with domain X and values taken in the set Y is expressed as $f : X \to Y$.

A.1.3 continuity

In the case X and Y are topological spaces, we write $f \in C^0(X \to Y)$ indicating the function f is continuous. In the case $Y = \mathbb{R}$, this is abbreviated to $f \in C^0(X)$.

A.1.4 C^k and C^∞

In the case X is an open subset of $\mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ the existence and continuity of the first order partials of each coordinate function of $f : X \to$

$\mathbb{R}^m$ is indicated by $f \in C^1(X \to Y)$. Again, when $Y \subset \mathbb{R}$ this is abbreviated to $f \in C^1(X)$.

Some special convention is required to define $C^1(X)$ and $C^1(X \to \mathbb{R}^m)$ when X is some subset of $\mathbb{R}^n$ which is not an open set. The usual meaning we attach to $f \in C^1(X \to \mathbb{R}^m)$ in this case is that there is some open set $U \subset \mathbb{R}^n$ and a function $g : U \to \mathbb{R}^m$ with $X \subset U$ and

$$g\big|_X = f.$$

Notice that in the case when X is open, we can take $g \equiv f$ as the extension. Given $f = (f_1, \ldots, f_m) \in C^1(X \to \mathbb{R}^m)$ furthermore, we define $Df : X \to \mathbb{R}^{mn}$ by

$$Df(\mathbf{x}) = \left(\frac{\partial g_1}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial g_1}{\partial x_n}(\mathbf{x}), \frac{\partial g_2}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial g_2}{\partial x_n}(\mathbf{x}), \ldots, \frac{\partial g_m}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial g_m}{\partial x_n}(\mathbf{x}) \right).$$

With this convention in place, we can define for X an arbitrary subset of $\mathbb{R}^n$ the sets $C^k(X)$ and $C^k(X \to \mathbb{R}^m)$ for $k \geq 2$ as more or less inductively, but some additional notation for (higher order) partial derivatives is helpful to make that process simpler and precise. Given first an open set $X \subset \mathbb{R}^n$, a real valued function $u : X \to \mathbb{R}$, and an element $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}_0^n$ where $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, usually called a **multi-index** we define the order $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$ partial derivative

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial^{\beta_1} x_1 \partial^{\beta_2} x_2 \cdots \partial^{\beta_n} x_n}$$

assuming this partial derivative exists in the usual sense of an iterated single partial derivative at a point in an open set. Finally then for $k \geq 2$ and $X \subset \mathbb{R}^n$ we set

$$C^k(X) = \{ f \in C^{k-1}(X) : D^\beta f \in C^1(X) \text{ for } |\beta| = k - 1 \}.$$

The space $C^k(X \to \mathbb{R}^m)$ is defined similarly; see Exercise A.1.

$$C^\infty(X \to \mathbb{R}^m) = \cap_{k=1}^\infty C^k(X \to \mathbb{R}^m).$$

A.1.5 compact support

Let X be a topological space. If U and V are open subsets of X with $\overline{U} \subset V$ and $\overline{U}$ compact, then we write

$$U \subset\subset V$$

and say U is **compactly contained** in V. More generally, if A and B are any subsets of X, then we write $A \subset\subset B$ to indicate the existence of an open subset U of X for which

$$\overline{A} \subset U \subset\subset \operatorname{int}(B).$$

See Exercise A.2.

Given $f : X \to \mathbb{R}$, the **support** of f is the set

$$\operatorname{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}.$$

Given $f : X \to \mathbb{R}$, we say f is **compactly supported** and write $f \in C_c^0(X)$ if $\operatorname{supp}(f)$ is compact.

If U is an open subset of X and $f : U \to \mathbb{R}$, we say f is **compactly supported in** U if

$$\operatorname{supp}(f) \subset\subset U \qquad \text{and} \qquad \overline{U} \text{ is compact.}$$

In the special case $X \subset \mathbb{R}^n$ and for $k = 1, 2, 3, \ldots$, the space $C_c^k(X)$ is defined by

$$C_c^k(X) = C^k(X) \cap C_c^0(X).$$

Given an open set $U \subset \mathbb{R}^n$ and $f \in C_c^k(U)$, we often let f also denote the extension $g : \mathbb{R}^n \to \mathbb{R}$ with

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \in U \\ 0, & \mathbf{x} \in \mathbb{R}^n \backslash U \end{cases}$$

and write $f \in C_c^k(\mathbb{R}^n)$. Finally,

$$C_c^\infty(X) = \cap_{k=1}^\infty C^k(X) \ \cap \ C_c^0(X).$$

When U is an open subset of $\mathbb{R}^n$, the important space $C_c^\infty(U)$ is sometimes called the **space of test functions**.

A.1.6 Exercises

Exercise A.1 Define $C^k(X \to \mathbb{R}^m)$ for X and arbitrary subset of $\mathbb{R}^n$.

Exercise A.2 If A and B are (any) subsets of a topological space X and $A \subset\subset B$, then show $\overline{A}$ is compact.

Exercise A.3 Given a $f \in C^\infty(X)$ with $X \subset \mathbb{R}^n$, we often let f also denote the extension $g : \mathbb{R}^n \to \mathbb{R}$ with

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \in X \\ 0, & \mathbf{x} \in \mathbb{R}^n \backslash X. \end{cases}$$

If $\text{supp}(f)$ is compact, show $g \in C_c^\infty(\mathbb{R}^n)$.

Exercise A.4 Show

$$C_c^\infty(X) = \cap_{k=1}^\infty C_c^k(X).$$

Bibliography

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