Mathematical Capillarity

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Contents

Chapter 1

The circular capillary tube problems

1.1 Introduction

Leonardo da Vinci, around 1490, observed carefully a thin column of liquid rising up in a cylindrical glass tube when the end of the tube was inserted in the liquid. The thin column of liquid in the tube resembled a hair, and Leonardo referred to the phenomenon as a "hair-like" action of the liquid.[1] Jacob Bernoulli around 1683 in his paper *Dissertatio de Gravitate Ætheris,* in Latin, suggested that it is more difficult for air to enter a narrow tube than it is for (the) liquid. It is interesting to note that the theory of æther seems to be featured in Bernoulli's paper. Introduction of the mysterious element æther was a popular approach for treating inexplicable phenomena at the time.

Perhaps it can be said that Young and Laplace, in 1805 and 1806, offered the first mathematical framework in which to pose and study specific problems associated with the rise of liquid in a capillary tube. We can say they started with the mathematical assumption that there is a surface separating the liquid and the air inside the tube and that this surface is the graph of a function of two variables. I have tried to draw a picture representing such a surface in Figure 1.1. More precisely, taking specific coordinates, we let

[1]The word "hair" is Capelli in Italian, and this is the origin of the terms capillary action and capillarity.

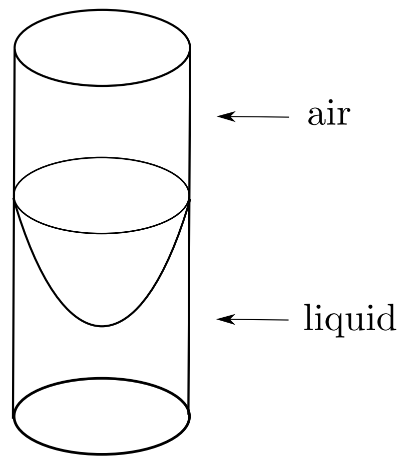

Figure 1.1: drawing of liquid in a capillary tube

$B_r(0)$ be the open disk (or ball) centered at the origin in $\mathbb{R}^2$:

$$B_r(0) = \{(x, y) : x^2 + y^2 < r^2\}.$$

This set, which we take to model the cross-section of the tube is illustrated in Figure 1.2. Thus, the separating surface is the graph of a function

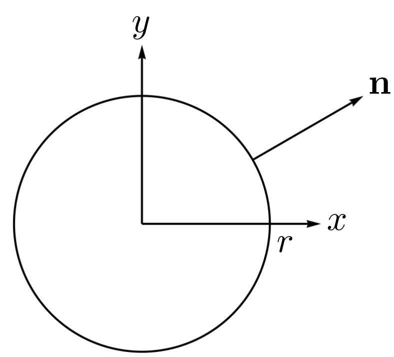

Figure 1.2: The unit disk ($r = 1$) in $\mathbb{R}^2$ with its outward normal at a point

$$u : B_r(0) \to \mathbb{R}.$$

This allows us to use calculus. Young and Laplace proposed the following equations for the function u:

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \kappa u \qquad \text{on } B_r(0) \tag{1.1}$$

$$\frac{Du}{\sqrt{1+|Du|^2}}\cdot \mathbf{n} = \cos\gamma \qquad \text{on } \partial B_r(0). \tag{1.2}$$

In these equations Du denotes the gradient, or vector of first partial derivatives,

$$Du = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

from calculus. The first equation is a partial differential equation, known as the Young-Laplace equation or the capillary equation, with

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{\partial}{\partial x}\left(\frac{\frac{\partial u}{\partial x}}{\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}}\right) + \frac{\partial}{\partial y}\left(\frac{\frac{\partial u}{\partial y}}{\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}}\right)$$

involving second (and first) partial derivatives of the function u. The quantity denoted by κ is assumed constant and is called the capillary constant.

The second equation is a boundary condition known as Young's law. The angle γ is called the contact angle and is also assumed constant. This expression of Young's law assumes the function u extends to the boundary circle of $B_r(0)$,

$$\partial B_r(0) = \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\},$$

with first order partial derivatives defined there. The vector quantity $\mathbf{n}$ is called the outward unit normal to $B_r(0)$ along $\partial B_r(0)$ and is given by

$$\mathbf{n} = \frac{1}{r}(x,y).$$

There are a number of modeling questions we need to address. Among these are the following:

1. What are the physical significance of the capillary constant κ and the contact angle γ?

2. What is the physical significance of the zero level $u = 0$, that is, how are coordinates for the third or z-axis chosen?

3. Why do these equations hold?

It is not difficult to make good and satisfying progress in answering these questions, but both in the spirit of the manner in which these equations were offered by Young and Laplace, and because we will address these topics in more detail in a unified way later, we postpone further discussion at the moment. Instead, let us note that we can now pose (and have some hope of answering) quantitative questions da Vinci or Bernoulli would have appreciated:

1. What is the shape of the surface between the liquid and the air?

2. What is the height that the liquid rises above certain points in the cross-section of the tube?

It will take some time and effort to understand and appreciate this boundary value problem of Laplace and Young (and to obtain some answers from it), but to give an indication that we are on the right track, let us consider a simpler version of the mathematical problem already considered by Euler in 1744.

Exercise 1.2 *Expand the derivatives in the partial differential operator*

$$\mathcal{M}u = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right),$$

and simplify the result.

1.3 Euler's elastica and 2-D capillary surfaces

A two-dimensional version of the equations of Young and Laplace may be considered as follows: We seek $u : (-r, r) \to \mathbb{R}$ with

$$\begin{cases} \dfrac{d}{dx}\left(\dfrac{u'(x)}{\sqrt{1+u'(x)^2}}\right) = \kappa u, \quad |x| < r \\ \\ \dfrac{u'(\pm r)}{\sqrt{1+u'(\pm r)^2}} = \pm\cos\gamma. \end{cases} \tag{1.3}$$

The equations appearing in (1.3) include, first, an ordinary differential equation (ODE) for the function u and, second, a **two point boundary value condition**, which is not exactly the standard initial value one encounters in the study of elementary ODEs but is one which does arise, especially in applications. These equations may be compared to the equations (1.1) and (1.2), and at least a superficial resemblance is obvious. We will make the relation/comparison quite precise, but our first objective is to explain their geometric significance.

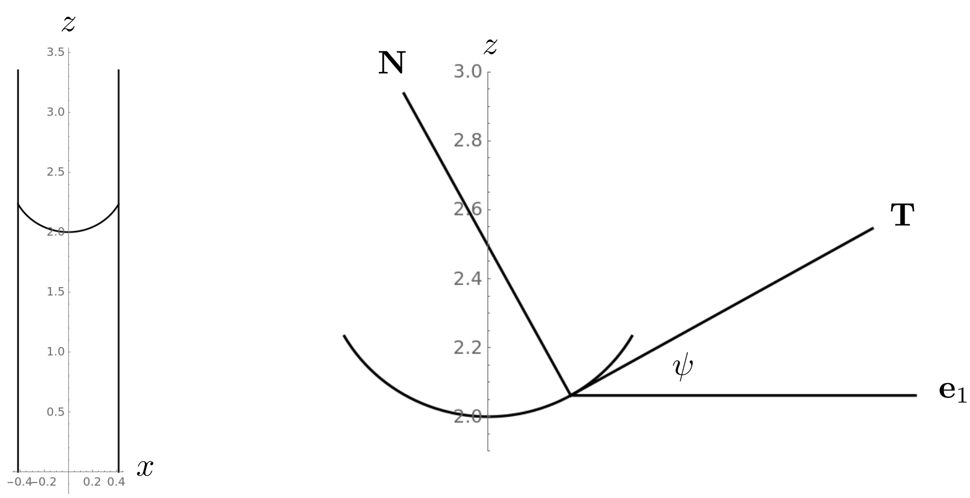

Figure 1.3: A solution of the 2-D capillary equation

In this 2-D capillary problem the interface is a curve instead of a surface, and this simplifies the geometry rather considerably.

1.3.1 Signed curvature of a plane curve

Given a plane curve as indicated on the right in Figure 1.3, the **signed curvature** is the rate at which the inclination angle ψ, between the positive x-axis and the indicated tangent $\mathbf{T}$ increases with respect to arclength along the curve. That is, the signed curvature k is given by

$$k = \frac{d\psi}{ds} \tag{1.4}$$

where the arclength along the curve measured from $(0, u(0))$ is given by

$$s = \int_0^x \sqrt{1 + [u'(t)]^2}\, dt. \tag{1.5}$$

Exercise 1.4 *Compute the signed curvature of the graph of the function* $u : (-r, r) \to \mathbb{R}$ *given by*

$$u(x) = -\sqrt{r^2 - x^2}.$$

Hint: You may wish to parameterize this curve using a different parameter.

Exercise 1.5 *If the graph of a function* $u : (-r, r) \to \mathbb{R}$ *is parameterized by* $\gamma(x) = (x, u(x))$*, what is the velocity vector* $\gamma'(x)$ *associated with the parameterization and what is its length* $|\gamma'(x)|$*.*

Exercise 1.6 *Use a Riemann sum and the relation*

$$\text{rate} \;\times\; \text{times} \;=\; \text{distance}$$

to explain the formula (1.5).

Exercise 1.7 *Draw a triangle on the picture of the graph of* u *on the right in Figure 1.3 illustrating the following relations:*

$$u' = \frac{du}{dx} = \frac{\text{rise}}{\text{run}}, \quad \text{and} \quad \sin\psi = \frac{u'}{\sqrt{1 + u'^2}}.$$

Exercise 1.8 *Find expressions for the unit tangent vector* $\mathbf{T}$ *and the unit upward normal vector* $\mathbf{N}$ *to the graph of a function* u *of one real variable as illustrated in Figure 1.2.*

Notice the arclength relation (1.5) may be used to define two functions. On the one hand, we can think of the arclength $s : (-r, r) \to \mathbb{R}$ by

$$s(x) = \int_0^x \sqrt{1 + [u'(t)]^2}\, dt. \tag{1.6}$$

But on the other hand, the fundamental theorem of calculus tells us

$$\frac{ds}{dx} = \sqrt{1 + [u'(x)]^2} > 0,$$

so $s : (-r, r) \to \mathbb{R}$ has an inverse $x : (-s_0, s_0) \to \mathbb{R}$ where

$$s_0 = \int_0^r \sqrt{1 + [u'(t)]^2}\, dt,$$

defined by the same formula:

$$s = \int_0^{x(s)} \sqrt{1 + [u'(t)]^2}\, dt.$$

Notice the derivative of the function $x = x(s)$ satisfies

$$\frac{dx}{ds} = \frac{1}{\sqrt{1 + [u'(x)]^2}} = \cos\psi.$$

We could write $1/\sqrt{1 + [u'(x(s))]^2}$ in the middle expression and $\psi(s)$ in the last one, but the arclength argument is left out. Note carefully the ambiguity between the use of the symbols x and s here. Each is used in two different ways as an independent parameter and as a dependent parameter, or function. For the arclength s, in particular, we often emphasize the distinction (according to context) by referring to the **arclength parameter** of (1.4) as opposed to the **arclength function** of (1.6).

1.8.1 Interpretation of the 2-D capillary equation

We should now be in a position to understand the geometric significance of the ODE in (1.3). Our definition of the signed curvature (1.4) assumes the inclination angle ψ is a function of arclength, that is, $\psi : (-s_0, s_0) \to \mathbb{R}$. Let us think, momentarily,[2] of the inclination angle ψ as a function of x. Then we can write the left side of the ODE as

$$\frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right) = \frac{d}{dx}(\sin\psi).$$

The chain rule then tells us

$$\frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right) = \cos\psi\,\frac{d\psi}{dx} = \cos\psi\,\frac{d\psi}{ds}\,\frac{ds}{dx} = k.$$

That is, the 2-D capillary equation says *the signed curvature of the interface is a linear function of height* or, as a physicist might say it, the signed curvature is proportional to the height at every point on the interface.

I should like to leave the interpretation of the boundary condition to you.

Exercise 1.9 *Interpret the left side of the boundary condition in terms of the inclination angle ψ. Draw appropriate versions of the tangent and normal on the interface (curve) shown in Figure 1.3 at $x = \pm r$. Identify and label the contact angle γ in your drawings.*

At the end of this chapter, I will state both the 2-D and 3-D capillary problems under consideration as geometric problems. When you have finished the exercises above, you can compare your understanding of the problems to those statements. In the next section I will attempt to describe some simple properties of solutions of the two point boundary value problem (1.3) suggesting we are on the right track.

1.10 Monotonicity and estimates for elastica

Given a solution u of (1.3), if we integrate over the base domain $(-r, r)$ we find

$$\frac{u'}{\sqrt{1+u'^2}}\Big|_{x=-r}^{r} = \kappa \int_{-r}^{r} u(x)\,dx.$$

The boundary condition then implies

$$\int_{-r}^{r} u(x)\,dx = \frac{2\cos\gamma}{\kappa}. \tag{1.7}$$

[2]Technically, we could introduce a new function $\Psi : (-r, r) \to \mathbb{R}$ by $\Psi(x) = \psi(s(x))$ where s is the arclength function (1.6).

The integral quantity on the left here is called the **lifted volume**, and we denote it by

$$V = \int_{-r}^{r} u(x)\, dx.$$

The explicit formula (1.7) we have obtained for the lifted volume has the following qualitative properties or interpretations:

(a) If the contact angle tends to $\pi/2$ (with $\kappa > 0$ fixed), then the lifted volume tends to zero.

(b) If the capillary constant tends to zero (with $0 < \gamma < \pi/2$ fixed), then the lifted volume tends to infinity.

We can also see the following closely related quantitative (monotonicity) properties:

(c) If $0 < \gamma_1 < \gamma_2 < \pi/2$ (with $\kappa > 0$ fixed), then the lifted volumes V_1 and V_2 associated with γ_1 and γ_2 respectively satisfy

$$V_2 < V_1.$$

(d) If two capillary tubes are modeled with capillary constants $0 < \kappa_1 < \kappa_2$ (with $0 < \gamma < 0$ fixed), then the lifted volumes V_1 and V_2 associated with κ_1 and κ_2 respectively satisfy

$$V_2 < V_1.$$

Put another way, a smaller contact angle or a smaller capillary constant results in a greater lifted volume.

We will see in the modeling that the capillary constant κ is naturally assumed positive and one may think of smaller values of κ in any one of the following ways:

(i) The density of the liquid is small.

(ii) The gravitational field is of small magnitude.

(iii) The *surface tension* of the liquid is large.

Density and gravitational strength should have some (at least intuitive) meaning to everyone. Surface tension as a property of a liquid may be unfamiliar, but we will talk more about this later. For now, we can consider the suggestion that

> *The effects of capillarity become more significant when κ becomes smaller,*

and we can make some preliminary attempt to see if what the lifted volume formula implies agrees with our intuition.

Exercise 1.11 *What does the formula (1.7) for lifted volume tell you about the case when $\gamma > \pi/2$? Does this make sense?*

Exercise 1.12 *One might assume that when the contact angle tends to zero, then the lifted volume tends to infinity. Explain both mathematically and intuitively why this is not the case.*

Exercise 1.13 *The interface drawn in Figure 1.3 is a numerically computed, i.e., numerically approximated, solution of (1.3). It is quite accurate, and you can draw such accurate pictures yourself. Use mathematical software like Matlab (`ode45`) or Mathematica (`NDSolve`) to draw/plot (numerically) an interface like that in Figure 1.3. Here are some hints/steps:*

(a) *Expand the outer derivative in the 2-D capillary equation*

$$\frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right) = \kappa u$$

so that you can solve for the second derivative u''.

(b) *Introduce a new dependent variable $v = u'$ so you can express the equation in terms of an equivalent (first order) system of equations, the first of which is $u' = v$.*

(c) *Numerically approximate the solution of the first order system using $\kappa = 1$ and the initial value*

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}.$$

where u_0 is some positive (or negative) central height.

This exercise assumes the interface is symmetric with respect to the vertical line $x = 0$. Can you justify this assumption rigorously?

Exercise 1.14 *Assume u is a* **positive solution** *of the elastica boundary value problem (1.3). Prove the following:*

(a) $u''(x) > 0$ *for all x so that the graph of u is a convex curve.*

(b) *The minimum value of u occurs at $x = 0$.*

(c) *The quantity $u(0)$ is called the* **central rise height**. *It satisfies*

$$u(0) \leq \frac{\cos\gamma}{\kappa r}.$$

We mentioned that it is natural to assume the capillary constant is positive. In fact, the definition arising in or model will be

$$\kappa = \frac{\rho g}{\sigma} \tag{1.8}$$

where ρ is the density of the liquid, g is the gravitational acceleration of the field, and σ is the **surface tension** parameter mentioned above (which is also positive).

Exercise 1.15 *Under the assumption (1.8) write down as many limiting (qualitative) and monotonicity (quantitative) relations as you can between the raised volume (or the central rise height) and the three parameters ρ, g, and σ. See properties (a-d) above. For example,*

> *If γ, r, g, and σ are fixed with $0 < \gamma < \pi/2$ and the density of the liquid tends to zero, then the lifted volume tends to infinity.*

1.16 Summary

The 2-D capillary problem presented above is an essential geometric one:

> *Find an interface curve whose* **curvature** *is a* **linear function of height** *and which meets the vertical lines $x = \pm r$ at an angle γ measured within the liquid.*

The equations of Young and Laplace pose a very similar geometric problem, though the geometry is somewhat more complicated, and the mathematical analysis is much more difficult. We will address both in later chapters. For now, let us attempt to see the geometric problem at least in broad terms. The quantity

$$\mathcal{M}u = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$$

also measures **curvature,** but the curvature measured is not exactly the curvature of a curve. This is a kind of curvature associated with a **surface** called **mean curvature,**[3] and we will discuss the details of the associated geometry later. With this in mind, the capillary equation $\mathcal{M}u = \kappa u$ in (1.1) is a **prescribed curvature equation**.

The boundary condition (1.2) also has an interpretation like that of the 2-D capillary boundary value problem. Notice the vector operator

$$Tu = \frac{Du}{\sqrt{1+|Du|^2}}$$

can be interpreted as the first two components of the unit downward normal to the graph of u:

$$-\mathbf{N} = \left(\frac{u_x}{\sqrt{1+|Du|^2}}, \frac{u_y}{\sqrt{1+|Du|^2}}, -1\right). \tag{1.9}$$

Here we have written $Du = (u_x, u_y)$. Note[4] also that in the case of a curve (when u is a function of one variable), as indicated on the right in Figure 1.3, the upward unit normal is given by

$$\mathbf{N} = \left(-\frac{u'}{\sqrt{1+u'^2}}, 1\right),$$

so we have in (1.9) a direct generalization to graphs of functions of two variables.

The cross-sectional domain $B_r(0)$ may be embedded in $\mathbb{R}^3$ in, for example, the x, y-plane, so that the outward unit normal $\mathbf{n}$ to $\partial B_r(0)$ takes the form

$$\mathbf{n} = \left(\frac{x}{r}, \frac{y}{r}, 0\right)$$

[3]More precisely, $\mathcal{M}u$ is twice the mean curvature of the graph of u at each point.

[4]Remember Exercise 1.8.

and is (by interpretation) an outward unit normal to the surface of the circular cylindrical capillary tube.

With these extensions, the expression $Tu \cdot \mathbf{n}$ in Young's law (1.2) becomes

$$-\mathbf{N} \cdot \mathbf{n} = \cos\gamma.$$

That is to say, the angle between the downward unit normal to the interface and the outward unit normal to the tube is prescribed to be γ.

Exercise 1.17 *Draw a vertical cross-section of the 3-D capillary configuration illustrated in Figure 1.1 and in Figure 1.4 in a plane through* $\sqrt{x^2+y^2} = 0$*. Label the vectors* $\mathbf{N}$ *and* $\mathbf{n}$ *and convince yourself that* γ *is the angle between the interface and the surface of the tube measured within the liquid.*

The geometric problem[5] may now be understood as the following:

> *Find an interface surface whose* **mean curvature** *is a* **linear function of height** *and which meets the vertical wall of the tube* $\sqrt{x^2+y^2} = r$ *at an angle* γ *measured within the liquid.*

Specific problems falling under this general description (and of keen interest to Young and Laplace) would be

1. What is the raised volume
$$V = \int_{B_r(0)} u \ ?$$
2. What is the central rise height $u(0,0)$?
3. What can be said about the shape of the graph of u. Compare the sketch[6] in Figure 1.1 to the numerical plot in Figure 1.4.

[5]Finn may have been the first person to formally identify the capillary problem as a geometric one. Certainly Young and Laplace only offered text and equations.

[6]Figure 1.1 is just a "freehand" sketch in which the curve indicating the bottom of the interface happens to be quadratic (part of an ellipse) because such curves are convenient to draw in the drawing program used.

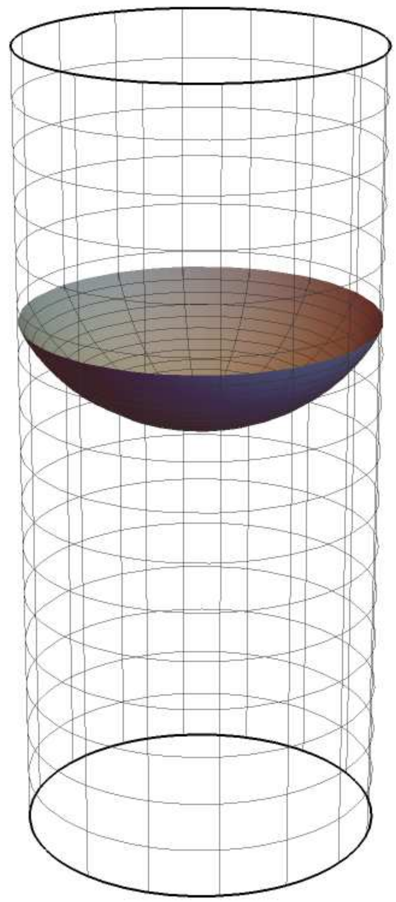

Figure 1.4: A solution of the 3-D capillary equation

1.18 Extras

Notice the "lifted" volume can be negative if $\gamma > \pi/2$.

You may also ask the question: Is it possible to have a solution of (1.1) and (1.2) which is **not** axially symmetric.

Actually, you might not think this is a very good question. You might think the answer is obvious. (Young and Laplace probably agreed with you.)

1.19 Outline of the Course

1. The (circular) capillary tube problem(s)
2. Calculus of Variations and Mean Curvature
3. 2-D capillary surfaces
4. Floating Objects
5. Additional Topics
    (a) axially symmetric capillary equation
    (b) sessile drops
    (c) vertical tubes of other shapes; corners
    (d) pendent drops
    (e) cells and partitioning; grain boundaries
    (f) experimental verification for circular capillary tube and sessile drops
    (g) numerics

I'm also going to include here an (older) previous outline of the course (just for my reference):

I The circular capillary tube (model assumptions)

II 2-D capillary surfaces (1-D calculus of variations)

III The capillary equation (symmetric solutions in the circular tube)

IV Sessile drops

V Floating objects

The (somewhat ambitious) goal in both outlines is to discuss the mathematics of floating objects—and the variational theory of floating objects in particular. There seems to be no complete discussion of this topic in the literature, and the material in mind here is drawn mainly from my own papers. Many of these (especically the later and better ones) are joint papers with Ray Treinen.

Chapter 2

Calculus of Variations and Mean Curvature

2.1 Calculus of Variations

The calculus of variations is, roughly speaking, a theory of minimization.

In the broadest sense, if A is any set and f is a real valued function with domain A, i.e.,

$$f : A \to \mathbb{R},$$

then we can define what it means to minimize the function f in the following terms:

> *An element $a \in A$ is a* **minimizer** *if*
>
> $$f(a) \leq f(x) \qquad \text{for all } x \in A.$$
>
> *Given a minimizer $a \in A$, the real number $f(a)$ is called the* **minimum value** *of f.*

It is pretty obvious that a minimum value is unique while there may be many minimizers. Also, it is not difficult to see that it is quite possible for no minimizer to exist.

In order to proceed further with any kind of theory of minimization, we need more structure on the domain set A. It is also usual to introduce some kind of structure on the function f. If the set A is an interval in the real line, and the function f is differentiable, then the minimization of f is considered in a first course in calculus, or what is often called 1-D (one dimensional) calculus. This simple case is rather important for us, so let's review it.

Theorem 1 *If $x_0 \in (a,b)$ is a minimizer of $f : (a,b) \to \mathbb{R}$ where f is a differentiable function, then*

$$f'(x_0) = 0. \tag{2.1}$$

The condition (2.1) is called a **necessary condition** for a minimizer because any minimizer x_0 (of this sort) must satisfy this condition.

Exercise 2.2 *Give an example of a minimizer $x_0 \in [a,b]$ of a differentiable function $f : [a,b] \to \mathbb{R}$ for which (2.1) fails to hold. Note: When we say a function $f : [a,b] \to \mathbb{R}$, defined on a closed interval $[a,b]$, is differentiable we usually mean there is an extension $\bar{f} : (\bar{a},\bar{b}) \to \mathbb{R}$ for some $\bar{a} < \bar{b}$ with $\bar{a} < a \le b < \bar{b}$, and*

$$\bar{f}\big|_{[a,b]} = f. \tag{2.2}$$

The function $\bar{f}\big|_{[a,b]} : [a,b] \to \mathbb{R}$ is called the **restriction** *of $\bar{f}$ to the interval $[a,b]$, and its values are given (of course) by*

$$\bar{f}\big|_{[a,b]}(x) = \bar{f}(x) \qquad \text{for every } x \in [a,b].$$

Exercise 2.3 *Give an example of a differentiable function $f : (a,b) \to \mathbb{R}$ and a point $x_0 \in (a,b)$ with $f'(x_0) = 0$ which illustrates that (2.1) is not sufficient to imply x_0 is a minimizer.*

Exercise 2.4 *What is the definition of the derivative $f'(x)$ at $x \in (a,b)$ for a differentiable function $f : (a,b) \to \mathbb{R}$?*

Exercise 2.5 *Prove the necessary condition (2.1) for an interior minimizer x_0 of $f : (a,b) \to \mathbb{R}$.*

There is also a second order necessary condition for interior minimizers, but it requires more **regularity** for the function f.

Theorem 2 *If $x_0 \in (a,b)$ is a minimizer of $f : (a,b) \to \mathbb{R}$ where f is a* **twice** *differentiable function, then*

$$f''(x_0) \ge 0. \tag{2.3}$$

Exercise 2.6 *The following conditions on a function* $f : (a,b) \to \mathbb{R}$ *are called* **regularity conditions***:*

1. *(continuity) For each* $x \in (a,b)$*, the function* f *is continuous at* x*.*
2. *(differentiability) For each* $x \in (a,b)$*, the derivative* $f'(x)$ *exists (as a well-defined real number).*
3. *(continuous differentiability) For each* $x \in (a,b)$*, the derivative* $f'(x)$ *exists and the function* $f' : (a,b) \to \mathbb{R}$ *is continuous.*
4. *(twice differentiability) For each* $x \in (a,b)$*, the derivative* $f'(x)$ *exists and the function* $f'(a,b) \to \mathbb{R}$ *is differentiable.*

The set of continuous real valued functions on the interval (a,b) *is denoted by* $C^0(a,b)$*. Let us denote the set of differentiable real valued functions on* (a,b) *by* $\text{Diff}(a,b)$*. The set of continuously differentiable real valued functions on* (a,b) *is denoted by* $C^1(a,b)$*. Let us denote the set of twice differentiable real valued functions on* (a,b) *by* $\text{Diff}^2(a,b)$*. Show*

$$\text{Diff}^2(a,b) \subsetneqq C^1(a,b) \subsetneqq \text{Diff}(a,b) \subsetneqq C^0(a,b).$$

Exercise 2.7 *Prove Theorem 2.*

Exercise 2.8 *Give an example showing the conclusions/necessary conditions (2.1) and (2.3) of Theorem 1 and Theorem 2 respectively, taken together, are not sufficient to imply* x_0 *is a minimizer.*

Exercise 2.9 *Give an example showing the conditions*

$$f'(x_0) = 0 \qquad \text{and} \qquad f''(x_0) > 0$$

are also not sufficient to imply $x_0 \in (a,b)$ *is a minimizer of the function* $f \in \text{Diff}^2(a,b)$*.*

If $f : A \to \mathbb{R}$ and the set A is taken to be an open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, then the minimization problem for f is discussed in a course on multivariable calculus. Some understanding of what happens in these cases, and when A is an open subset of $\mathbb{R}^n$ for any natural number n, is important for us too, and we will review that situation below. These cases fall under the heading of finite dimensional calculus.

A minimization problem in the calculus of variations is distinguished, roughly speaking, by the condition that the set A is infinite dimensional. This terminology is a tiny bit misleading because the notion of dimensionality relies on a vector space structure. On the other hand, an open set $\Omega \subset \mathbb{R}^n$ is usually not a vector space, but there is an obvious (finite dimensional) vector space of which Ω is a subset. Perhaps the best way to proceed is with a relatively simple example in which the domain does happen to be an infinite dimensional vector space:

2.9.1 2-D Capillary Surfaces

Consider $\mathcal{E} : C^1[-r, r] \to \mathbb{R}$ by

$$\mathcal{E}[u] = \int_{-r}^{r} \left[\sqrt{1 + u'(x)^2} + \kappa \frac{u(x)^2}{2} \right] dx - \beta[u(-r) + u(r)]. \tag{2.4}$$

Notice that $\mathcal{E}$ assigns to each continuously differentiable function $u \in C^1[-r, r]$ a real number. Such a function is called a **functional**, and minimizing such functionals is the main objective in the calculus of variations. Put another way, the calculus of variations is the theory of minimizing functionals, more or less, like the functional $\mathcal{E}$ above. Generally speaking, this is a very difficult problem.

It is always a good idea, with a problem like this, to have some understanding of what your functional is computing—or what is the meaning of its value. With this in mind, let us take a somewhat careful look at $\mathcal{E}$ before we proceed to look for minimizers directly.[1]

As in the previous chapter, the graph of the function u represents a possible interface separating the liquid in a capillary tube from the vapor exterior to that liquid. The idea is that the observed interface should, for some reason, be the one minimizing the functional $\mathcal{E}$. In particular, $\mathcal{E}$ should, roughly speaking, measure the energy associated with any proposed interface, and the one that is observed is (the one) minimizing that energy.

[1]This is a little bit of an obscure math joke because we are actually only going to consider what are called the indirect methods in the calculus of variations. Thus, we will actually look for minimizers “indirectly.” There are also what are called the “direct methods in the calculus of variations,” but we won’t really consider those methods in this course.

We can recognize three terms that make up $\mathcal{E}$. The first one might be

$$\int_{-r}^{r} \sqrt{1 + u'(x)^2}\, dx.$$

Hopefully, you recognize this as the length of the graph of u. The idea is that a certain amount of energy is required to maintain an interface between a liquid and the vapor exterior to that liquid. There are a few different ways to look at this. First of all, it is almost surely true that on some microscopic level the separation between the liquid and vapor is much more messy and complicated than the simple C^1 curve we are using to model it. There are molecules of liquid moving around near the separation region. Some are evaporating into the vapor where there is probably a region of higher density near the bulk liquid; some are condensing back into the bulk liquid. In the liquid itself molecules near the separation experience an attraction to more molecules located deeper in the liquid than those closer to the separation. It is assumed this results in a net force pulling those molecules deeper into the liquid. On the other hand, the overall volume of the liquid does not appear to change position appreciably. Thus, it must be assumed other molecules of liquid are either condensing to replace those near the surface which are sinking deeper or deeper molecules are moving (being pushed) outward.

The bottom line of this point of view is that there is kinetic energy associated with the separation region called **free surface** (or interface) **energy**, and two assumptions are made about this energy (in this 2-D case):

1. The free surface energy is proportional to the length of the interface.

2. The observed interface "prefers" to minimize this energy.

The first assumption is probably a relatively reasonable one if the identification of the energy is with the kinetic energy of moving molecules near the separation region—modeled by the interface curve. The units we have are not quite correct since energy is force times length, and in fact, a more physically accurate expression for the free surface energy is

$$\sigma \int_{-r}^{r} \sqrt{1 + u'(x)^2}\, dx$$

where σ is a constant with units of force called the **surface tension**. One can simply think of this as a tension inherent to the particular liquid and vapor (subject to ambient—temperature and pressure—conditions) along the interface. We have simply divided the entire energy by this surface tension constant to obtain a simpler form for $\mathcal{E}$.

The second assumption is quite a bit more mysterious. The word "prefers" is not intended to suggest that the liquid (and/or vapor, molecules, etc.) are sentient. Probably the best interpretation is the following:

> *If a competitor interface were somehow constructed or achieved near the observed equilibrium interface, then the motion of molecules would result in a redistribution of the liquid so as to minimize the length of the interface—subject to other constraints in the problem, including those imposed by the other terms in the energy.*

We are really not saying anything more than that we assume the energy is minimized. However, in practice, this kind of interpretation can be important. Let's consider the next term in the energy, and I will try to explain how and why.

The second term in the energy $\mathcal{E}$ is proportional to

$$\int_{-r}^{r} u(x)\,dx.$$

This term is much easier to understand. The idea is that there is a potential field associated with gravity having the form $-g(0,1)$. If a point mass m is located at the height z in this field, say at the point $(0,z)$, then we imagine it has been moved there from some reference level, say $z=0$, and the potential energy associated with the point mass is given by the force times the distance

$$\int_0^z mg(0,1)\cdot(0,1) = mgz.$$

Similarly, each liquid element ΔV in the area

$$\mathcal{V} = \{(x,z) : |x| < r \text{ and } 0 < z < u(x)\}$$

has associated with it a potential energy

$$\rho \Delta V g z^*$$

where ρ is an areal density and z^* is some representative height for the area element ΔV. Of course, the gravitational potential field $-g(0,1)$ we have taken is not really representative of the inverse square gravitational field of the earth, but it is a reasonable (and usual) approximation near the surface of the earth, where we expect most interesting everyday capillary surfaces will be observed. We are also assuming in the definition of $\mathcal{V}$ that u is positive. With these assumptions the total energy associated with a particular interface is approximated by a sum

$$\sum_j \rho g \Delta V_j z_j^*.$$

This is a Riemann sum for an area integral over $\mathcal{V}$, and (under appropriate regularity assumptions) we can say the gravitational potential energy associated with an interface determined by u should be

$$\lim \sum_j \rho g \Delta V_j z_j^* = \int_{\mathcal{V}} \rho g z = \rho g \int_{\mathcal{V}} z.$$

For the integral we can also write

$$\int_{\mathcal{V}} z = \int_{-r}^{r} \int_0^{u(x)} z \, dz \, dx = \int_{-r}^{r} \frac{u(x)^2}{2} \, dx.$$

The factor in front of this integral for a physical energy is ρg. It will be recalled that we have divided by the surface tension σ to obtain (2.4), and

$$\kappa = \frac{\rho g}{\sigma}$$

is called the capillary constant. The previous principle we attempted to delineate for free surface energy can be, in a sense, easily illustrated for gravitational energy:

> *If a competitor interface were somehow constructed or achieved near the observed equilibrium interface, then the motion of molecules would result in a redistribution of the liquid so as to minimize the gravitational potential energy of the interface—subject to other constraints in the problem, including those imposed by the other terms in the energy.*

Imagine the observed interface modeled in the upper left of Figure 2.1 where the liquid is assumed to be below the interface curve. The suggestion is that were the modification of the observed interface indicated in the upper right constructed and “let go,” then the liquid in the bulge would fall in order to lower the value of $\mathcal{E}$.

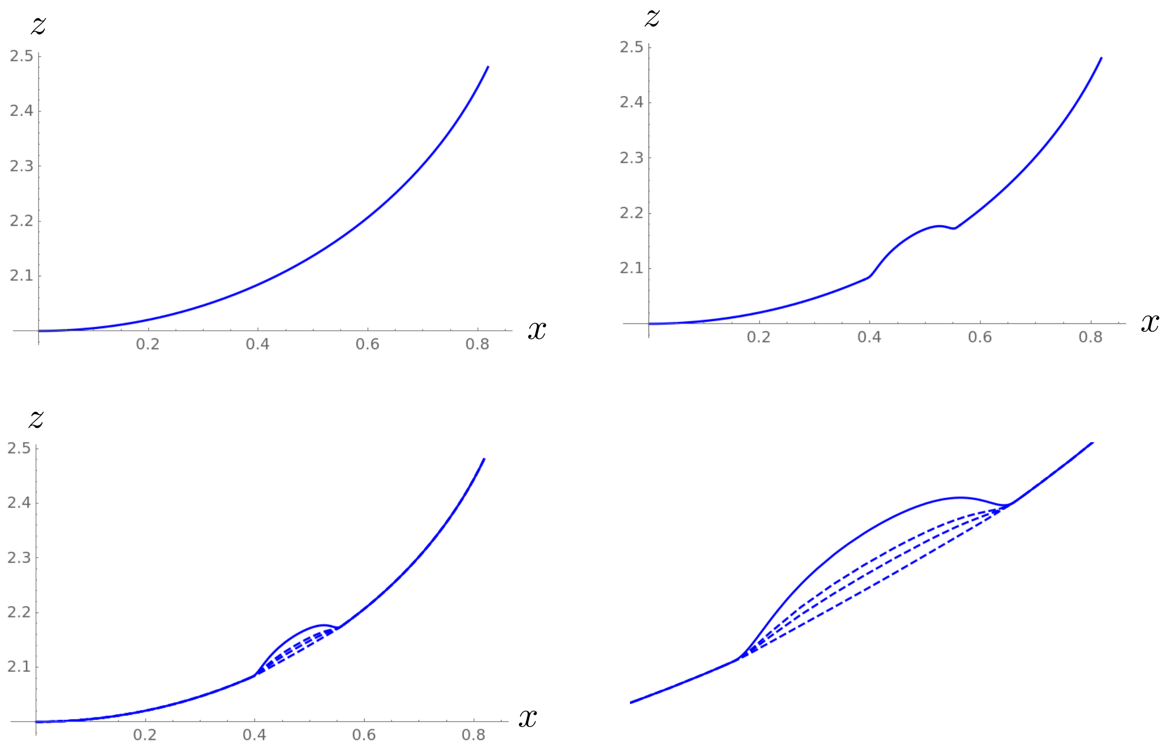

Figure 2.1: A modification of an observed interface (upper row). Clearly elimination of the bulge reduces the gravitational potential energy associated with the modified interface. Notice replacing the bulged portion with a straight line would both reduce the gravitational potential energy and the free surface length. In fact, this would reduce the free surface length to a minimum with respect to possible modification on the bulge region. It is at least plausible, however, that the energy can be reduced further by lowering the interface a little more (making it convex). This reduces the gravitational energy a relatively large amount while increasing the free surface length only slightly. See Exercise 2.10. The suggestion is that the observed interface is precisely the one obtaining the optimal balance to minimize the total energy. The lower row indicates (very roughly) how the liquid might "move" or migrate in a way that lowers energy. You should ask yourself the question: Is that what the liquid would actually do?

Exercise 2.10 *Consider a modification of the unit square*

$$\{(x,z) : 0 \le x, z \le 1\} = [0,1] \times [0,1]$$

obtained by replacing the top edge

$$\{(x,u(x)) : 0 \le x \le 1,\ u(x) \equiv 0\}$$

with the graph of $v(x) = 1 + \epsilon x(x-1)$. *How does the energy*

$$\mathcal{E}[u] = \int_0^1 \left[\sqrt{1 + u'(x)^2} + \kappa \, \frac{u(x)^2}{2} \right] dx$$

change under this modification. There are several possible approaches you can take here. You can plot the value $f(\epsilon) = \mathcal{E}[u]$ *numerically. You can also compute the derivative* $f'(0)$. *In the end, you should try to obtain an understanding of the order to which the length term changes compared to the order to which the gravitational energy term changes.*

Finally, we consider the third term $-\beta[u(-r) + u(r)]$. This term is called the **wetting energy**. Technically, in order to have physically correct units the wetting energy is $\sigma\beta[u(-r)+u(r)]$. Nevertheless, β is a physical constant measuring the differential attraction between the molecules of the liquid and those of the container, or 2-D “tube” consisting of two vertical walls that are straight lines. The constant β is called the **adhesion coefficient**. If $\beta > 0$, then the molecules of the liquid are attracted to those of the walls so that the energy is lower when the wetted portions of the wall

$$\{(-r,z) : 0 < z < u(-r)\} \qquad \text{and} \qquad \{(r,z) : 0 < z < u(r)\}$$

are as long as possible. Notice again, the balance: If $\beta > 0$, then making these segments long tends to increase both the free surface energy and the gravitational potential energy. If $\beta = 0$, then the molecules of the liquid are indifferent toward those of the wall, and if $\beta < 0$, then the molecules of the liquid and those of the wall experience a mutual repelling force.

The discussion of the energy functional above is vague and inadequate. If you can think more deeply about why and how liquid interfaces minimize such a functional, many people will be interested to hear your thoughts. We have merely attempted to make the minimization of $\mathcal{E}$ by observed interfaces seem plausible. What we can do is say more precise things about the model interfaces that do minimize $\mathcal{E}$. At the current time, the ultimate motivation for this description is that it leads to the equations of Young and Laplace and the resulting minimizing interfaces match experimental observations.

Returning to our illustration of the general subject of calculus of variations, we have a specific functional $\mathcal{E} : C^1[-r, r] \to \mathbb{R}$.

Exercise 2.11 *Recall that a vector space E over a field F is a set with a binary operation of* **addition** $+ : E \times E \to E$ *and a* **scaling** *operation* $\cdot : F \times E \to E$ *and having the following properties:*

1. *Addition is commutative: $v + w = w + v$ for all $w, v \in E$.*

2. *Addition is associative: $(v + w) + z = v + (w + z)$ for all $w, v, z \in E$.*

3. *There exists a* **zero vector 0** *with*

$$v + \mathbf{0} = \mathbf{0} + v = v \qquad \text{for all } v \in E.$$

4. *For each vector $v \in E$, there exists an* **additive inverse***, which is another vector $w \in E$ for which*

$$v + w = w + v = \mathbf{0}.$$

The additive inverse vector w of a vector v is denoted by $-v$. See Exercise 2.12 below.

5. *Scaling is associative: $(ab)v = a(bv)$ for all $a, b \in F$ and $v \in E$.*

6. *$0v = \mathbf{0}$ and $1v = v$ for any $v \in E$ where 0 is the additive identity in the field F and 1 is the multiplicative identity in the field F.*

7. *There are two distributive laws for scaling.*

   (a) *Scalars distribute across a sum of vectors:*

$$a(v + w) = av + aw \qquad \text{for all } a \in F \text{ and } v, w \in E.$$

   (b) *A vector distributes across a sum of scalars:*

$$(a + b)v = av + bv \qquad \text{for all } a, b \in F \text{ and } v \in E.$$

Definition 1 *Given two vector spaces X and E over the same field F, a function $L : X \to E$ is* **linear** *if*

$$L(av + bw) = aL(v) + bL(w) \qquad \text{for all } a, b \in F \text{ and } v, w \in X.$$

Show $C^1[-r, r]$ and $\mathbb{R}$ are both vector spaces over $\mathbb{R}$, but $\mathcal{E} : C^1[-r, r] \to \mathbb{R}$ is not linear.

Exercise 2.12 *The following are some basic exercises concerning the notion of a vector space.*

(a) *Show that the zero vector in a vector space is unique.*

(b) *Show that the additive inverse of any vector v in a vector space is unique.*

(c) *Show that the compatibility properties for scaling involving the additive and multiplicative identities in the field given in condition 6 of the definition of a vector space follow independently from the other properties defining a vector space. Thus condition 6 may be omitted from the definition.*

(d) *Look up and write down carefully the definition of a field.*

(e) *Explain how the integers mod 3*

$$\mathbb{Z}_3 = \{0, 1, 2\}$$

is a field.

Exercise 2.13 *Let V be a vector space over a field F.*

Definition 2 *A set $B \subset V$ is a* **basis** *for V if the following conditions hold:*

1. *Given any vector $v \in V$ there exist (finitely many) vectors $v_1, v_2, \ldots, v_k \in V$ and there exist scalars $c_1, c_2, \ldots, c_k \in F$ for which*

$$v = \sum_{j=1}^{k} c_j v_j.$$

2. *Given elements* $w_1, w_2, \dots, w_\ell \in B$ *and* $a_1, a_2, \dots, a_\ell \in F$ *with the elements of* B *distinct, if*

$$\mathbf{0} = \sum_{j=1}^{\ell} a_j w_j, \qquad \textit{then } a_1 = a_2 = \dots a_\ell = 0.$$

The following are basic exercises concerning the notion of a basis:

(a) *Given any subset* $A \subset V$*, the* **span** *of* A *is defined to be the set of all linear combinations of elements from* A*, that is,*

$$\operatorname{span}(A) = \left\{ \sum_{j=1}^{k} c_j v_j : v_1, v_2, \dots, v_k \in A \text{ and } c_1, c_2, \dots, c_k \in F \right\}.$$

Show $\operatorname{span}(A)$ *is a vector field over the same field* F*. Thus, the first condition defining a basis* B *may be written simply as* $\operatorname{span}(B) = V$*, i.e.,* B *is a spanning set.*

(b) *Any subset* $A \subset V$ *satisfying the second condition defining a basis for* V*, that is, given elements* $w_1, w_2, \dots, w_\ell \in A$ *and* $a_1, a_2, \dots, a_\ell \in F$ *with the elements of* A *distinct, if*

$$\mathbf{0} = \sum_{j=1}^{\ell} a_j w_j, \qquad \textit{then } a_1 = a_2 = \dots a_\ell = 0,$$

is said to be **linearly independent**. *Show any vector* v *in the span of a linearly independent set* A *can be written uniquely as a linear combination of distinct elements of* A*, i.e, if*

$$\sum_{j=1}^{\ell} a_j w_j = \sum_{j=1}^{k} c_j v_j$$

for some distinct $w_1, w_2, \dots, w_\ell \in A$*, some distinct* $v_1, v_2, \dots, v_k \in A$ *and some* $a_1, a_2, \dots, a_\ell, c_1, c_2, \dots, c_k \in F$*, then*

$$\{v_1, v_2, \dots, v_k\} = \{w_1, w_2, \dots, w_\ell\},$$

and in particular $k = \ell$*, and there exists a* **permutation***, i.e., a bijection* $\phi : \{1, 2, \dots, k\} \to \{1, 2, \dots, \ell = k\}$*, such that*

$$v_j = w_{\phi(j)} \qquad \text{and} \qquad c_j = a_{\phi(j)} \qquad \textit{for } j = 1, 2, \dots, k.$$

Thus, an alternative definition of a basis B for a vector space V is a subset $B \subset V$ for which each element $v \in V$ can be written as a unique linear combination of distinct elements in B.

Exercise 2.14 *A vector space V is said to be* **finite dimensional** *if there exists a basis B for V with finitely many elements. A vector space V is said to be* **infinite dimensional** *if it is not finite dimensional, i.e., if no basis with finitely many elements exists.*

(a) *Two vector spaces E and V over the same field are said to be* **isomorphic** *(as vector spaces) if there is a linear bijection $L : E \to V$. Show any finite dimensional vector space V is isomorphic to F^n for some n.*

(b) *Show $C^1[-r, r]$ is infinite dimensional.*

According to the preceding exercises, we have a real valued nonlinear functional $\mathcal{E} : C^1[-r, r] \to \mathbb{R}$ defined on an infinite dimensional vector space $C^1[-r, r]$, and we can attempt to minimize $\mathcal{E}$. It turns out that, in the grand scheme of things, the fact that $C^1[-r, r]$ is an infinite dimensional vector space is not directly representative of the infinite dimensionality inherent to the problems of the calculus of variations, but the review of vector spaces, and infinite dimensional vector spaces in particular, will be useful and necessary. The functional $\mathcal{E} : C^1[-r, r] \to \mathbb{R}$ is, in fact, a rather typical example of the kinds of functionals considered in the calculus of variations.

The object corresponding to a first derivative of a functional like $\mathcal{E}$ is called a **first variation** or **Gateaux differential**. Here is the construction: Let $u, \phi \in C^1[-r, r]$ and consider $v = u + \epsilon\phi$. The quantity

$$\delta\mathcal{E}[\phi] = \delta_u\mathcal{E}[\phi] = \left[\frac{d}{d\epsilon}\,\mathcal{E}[u + \epsilon\phi]\right]_{\big|_{\epsilon=0}}$$

is called the first variation of $\mathcal{E}$ at u in the direction ϕ. In this definition, we are thinking of u and ϕ fixed. After the value of $\delta\mathcal{E}$ is computed, we may think of u and/or ϕ as arguments of the first variation. If u is a minimizer of $\mathcal{E}$, then $\mathcal{E}[v] = \mathcal{E}[u + \epsilon\phi] \geq \mathcal{E}[u]$, and $f(\epsilon) = \mathcal{E}[u + \epsilon\phi]$ (with ϕ fixed) is a real valued function of one variable, ϵ, with a minimum at $\epsilon = 0$. Therefore, if u is a minimizer of $\mathcal{E}$, then

$$\delta_u\mathcal{E}[\phi] \equiv 0 \qquad \text{for every } \phi \in C^1[-r, r].$$

We might be worried about whether or not the derivative with respect to ϵ exists and, if so, if the limit as ϵ tends to zero exists as well. Let's see if we can make a computation to determine if concerns about this are valid. Writing out $\mathcal{E}[v]$ from (2.4) we have

$$\begin{aligned}\mathcal{E}[v] &= \int_{-r}^{r} \left[\sqrt{1+v'(x)^2} + \kappa\,\frac{v(x)^2}{2}\right] dx - \beta[v(-r)+v(r)] \\ &= \int_{-r}^{r} \left[\sqrt{1+[u'(x)+\epsilon\phi'(x)]^2} + \kappa\,\frac{[u(x)+\epsilon\phi(x)]^2}{2}\right] dx \\ &\qquad - \beta[u(-r)+\epsilon\phi(-r)+u(r)+\epsilon\phi(r)].\end{aligned}$$

Thus, forming the difference quotient

$$\frac{\mathcal{E}[u+\epsilon\phi] - \mathcal{E}[u+(\epsilon+h)\phi]}{h}$$

we obtain

$$\begin{aligned}&\frac{1}{h}\int_{-r}^{r} \left[\sqrt{1+[u'(x)+(\epsilon+h)\phi'(x)]^2} - \sqrt{1+[u'(x)+\epsilon\phi'(x)]^2}\right] dx \\ &\qquad + \kappa\int_{-r}^{r} \left[[u(x)+\epsilon\phi(x)]\phi(x) + \frac{h\phi(x)^2}{2}\right] dx - \beta[\phi(-r)+\phi(r)].\end{aligned}$$

The First term can be written as

$$\int_{-r}^{r} \frac{2[u'(x)+\epsilon\phi'(x)]\phi'(x) + h\phi'(x)}{\sqrt{1+[u'(x)+(\epsilon+h)\phi'(x)]^2} + \sqrt{1+[u'(x)+\epsilon\phi'(x)]^2}}\, dx.$$

From these expressions, it is clear the limit as h tends to zero exists and

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{E}[v] &= \int_{-r}^{r} \frac{[u'(x)+\epsilon\phi'(x)]\phi'(x)}{\sqrt{1+[u'(x)+\epsilon\phi'(x)]^2}}\, dx \\ &\qquad + \kappa\int_{-r}^{r} [u(x)+\epsilon\phi(x)]\;\phi(x)\, dx - \beta[\phi(-r)+\phi(r)].\end{aligned}$$

The derivative with respect to ϵ does exist, and evaluation at $\epsilon = 0$ is also immediate:

$$\delta\mathcal{E}[\phi] = \int_{-r}^{r} \left[\frac{u'(x)}{\sqrt{1+u'(x)^2}}\,\phi'(x) + \kappa u(x)\,\phi(x)\right] dx - \beta[\phi(-r)+\phi(r)].$$

We pause to remark/recall that if $u \in C^1[-r,r]$ is a minimizer of $\mathcal{E}$, then $\delta_u \mathcal{E}[\phi] = 0$ for all $\phi \in C^1[-r,r]$. Our computation allows us to write this condition as

$$\int_{-r}^{r} \left[\frac{u'(x)}{\sqrt{1+u'(x)^2}}\, \phi'(x) + \kappa u(x)\, \phi(x) \right] dx = \beta[\phi(-r)+\phi(r)]$$
$$\text{for all } \phi \in C^1[-r,r]. \tag{2.5}$$

It is not entirely clear what this (integral) condition implies about the minimizer u. We can say, more generally, however that **any** function $u \in C^1[-r,r]$ for which (2.5) holds is called a **weak extremal** for the functional $\mathcal{E}$. A weak extremal is an analogue of a (1-D calculus) *critical point* in the calculus of variations; a weak extremal need not be a minimum; it might be a maximum or neither a minimum nor maximum.

Theorem 3 *(first necessary condition in the calculus of variations) A minimizer $u \in C^1[-r,r]$ of $\mathcal{E} : C^1[-r,r] \to \mathbb{R}$ given by (2.4) is a weak extremal for $\mathcal{E}$.*

In order to proceed further, we assume additional regularity on a minimizer (or extremal) u.

Theorem 4 *(C^2 weak extremals) If $u \in C^2[-r,r]$ is a weak extremal for $\mathcal{E}$ given by (2.4), then*

$$\frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}} \right) = \kappa u(x) \qquad \textit{for } x \in (-r,r), \tag{2.6}$$

and

$$\frac{u'(\pm r)}{\sqrt{1+u'(\pm r)^2}} = \pm\beta. \tag{2.7}$$

Proof: If $u \in C^2[-r,r]$, then the curvature of the graph

$$\frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}} \right)$$

makes sense, and we may integrate the first term in (2.5) by parts to obtain

$$\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\,\phi(x)\right)\Big|_{x=-r}^{r} - \int_{-r}^{r}\left[\frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) - \kappa u(x)\right]\phi(x)\,dx$$

$$= \beta[\phi(-r)+\phi(r)].$$

That is,

$$\begin{aligned}\int_{-r}^{r}&\left[\frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) - \kappa u(x)\right]\phi(x)\,dx\\ &= \left[\frac{u'(r)}{\sqrt{1+u'(r)^2}} - \beta\right]\phi(r) - \left[\frac{u'(-r)}{\sqrt{1+u'(-r)^2}} + \beta\right]\phi(-r)\end{aligned} \tag{2.8}$$

$$\text{for all } \phi \in C^1[-r,r].$$

Let us assume, by way of contradiction that the factor

$$\frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) - \kappa u(x)$$

in the integral in (2.8) is nonzero at some point $x = x_0 \in (-r,r)$. By continuity, then, there is some $\epsilon > 0$ for which

$$\frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) - \kappa u(x) \neq 0 \quad \text{for } |x-x_0| < \epsilon.$$

Notice that this assertion assumes ϵ is small enough so that $x \in [-r,r]$ for every x with $|x-x_0| < \epsilon$. We know, furthermore, by the intermediate value theorem that this integrand assumes a single sign on the entire interval $x_0 - \epsilon < x < x_0 + \epsilon$.

Exercise 2.15 *Explain why (i.e., give explicit estimates showing) we can assume* $\{x : |x-x_0| < \epsilon\} \subset (-r,r)$. *Also, explain the application of the intermediate value theorem in detail.*

Exercise 2.16 *There exists a function $\phi \in C^1[-r, r]$ satisfying the following:*

(a) $\phi(x) \equiv 0$ *for* $|x - x_0| \geq \epsilon$.

(b) $\phi(x) > 0$ *for* $|x - x_0| < \epsilon$.

Substituting the function ϕ from Exercise 2.16, for which $\phi(-r) = \phi(r) = 0$, into (2.8) we conclude

$$\int_{-r}^{r} \left[\frac{d}{dx} \left(\frac{u'(x)}{\sqrt{1 + u'(x)^2}} \right) - \kappa u(x) \right] \phi(x)\, dx = 0.$$

That is,

$$\int_{x_0-\epsilon}^{x_0+\epsilon} \left[\frac{d}{dx} \left(\frac{u'(x)}{\sqrt{1 + u'(x)^2}} \right) - \kappa u(x) \right] \phi(x)\, dx = 0.$$

The factors in the integrand here are both nonzero and neither changes sign on the interior interval $(x_0 - \epsilon, x_0 + \epsilon)$. This is a contradiction implying (2.6) must hold identically.

In view of what we have just shown (2.8) simplifies to

$$\left[\frac{u'(r)}{\sqrt{1 + u'(r)^2}} - \beta \right] \phi(r) = \left[\frac{u'(-r)}{\sqrt{1 + u'(-r)^2}} + \beta \right] \phi(-r)$$

for all $\phi \in C^1[-r, r]$.

Taking any $\phi \in C^1[-r, r]$ for which $\phi(r) = 0$ but $\phi(-r) \neq 0$, we get

$$\frac{u'(-r)}{\sqrt{1 + u'(-r)^2}} = -\beta.$$

Similarly, when $\phi(-r) = 0$ but $\phi(r) = 0$, we conclude

$$\frac{u'(r)}{\sqrt{1 + u'(r)^2}} = \beta. \qquad \square$$

Notice that we have obtained in Theorem 4 the 2-D capillary surface equation (and ordinary differential equation) and the boundary condition subject to the assumption that the adhesion coefficient β satisfies

$$|\beta| < 1$$

so that the equation $\cos\gamma = \beta$ defines a unique contact angle γ strictly between 0 and π. It may be observed that there was no particular physical restriction suggesting $|\beta| < 1$, and it can be fairly asked: *What if we consider the functional $\mathcal{E}$ with $|\beta| \geq 1$?* Let us postpone consideration of this question until the next chapter where we discuss solutions of Euler's equation for elastic curves.

2.16.1 Calculus of Variations

With at least one example of the process (typical to the calculus of variations) by which one begins with a functional and, in an effort to minimize its value or find a minimizer, arrives at a differential equation, let us consider the process in a somewhat more general framework.

Problems in the calculus of variations **always** involve two important sets, which are usually sets of functions. These two sets are the **admissible class** $\mathcal{A}$ and the set of **perturbations** $\mathcal{V}$. The admissible class is the domain of the functional under consideration. Thus, we consider

$$\mathcal{F} : \mathcal{A} \to \mathbb{R},$$

and we (typically) seek to minimize $\mathcal{F}$. The set $\mathcal{A}$ is very often *not* a vector space, though it was in our example above. The set of perturbations $\mathcal{V}$ is almost always a vector space—and an infinite dimensional vector space in the case of the calculus of variations. The perturbations can be thought of roughly as differences of admissible functions. In particular, given an admissible function $u \in \mathcal{A}$ and a perturbation $\phi \in \mathcal{V}$, we require

$$v = u + \epsilon\phi \in \mathcal{A} \qquad \text{for } \epsilon \in \mathbb{R} \text{ with } |\epsilon| \text{ small.}$$

We need this in order to compute the first derivative

$$\frac{d}{d\epsilon}\mathcal{F}[u + \epsilon\phi]$$

and hence the first variation

$$\delta_u\mathcal{F}[\phi] = \left(\frac{d}{d\epsilon}\mathcal{F}[u + \epsilon\phi]\right)_{\big|_{\epsilon=0}}.$$

If you think of $\mathcal{V}$ as differences admissible functions with

$$v = u + a\,\phi \in \mathcal{A} \qquad \text{and} \qquad w = u - b\,\psi \in \mathcal{A}$$

both admissible with $\phi, \psi \in \mathcal{V}$, then $v - w = a\,\phi + b\,\psi$ should also be in $\mathcal{V}$ (at least for small a and b). This means $\mathcal{V}$ is closed under "small" linear combinations making $\mathcal{V}$ a vector space at least on a small scale.

Also by assuming the set of perturbations $\mathcal{V}$ is a vector space, the first variation (Gateaux derivative or functional derivative) is a functional defined on a vector space:

$$\delta_u \mathcal{F} : \mathcal{V} \to \mathbb{R}.$$

This makes it possible to understand the first variation as a **linear** functional. In fact, under the most common structural assumption for $\mathcal{F} : \mathcal{A} \to \mathbb{R}$, the first variation will always be a linear functional. Namely, if $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ is an **integral functional** of the form

$$\mathcal{F}[u] = \int_a^b F(x, u(x), u'(x))\, dx$$

where $\mathcal{A}$ is some subset (not necessarily a subspace) of $C^1[a, b]$ and $F : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $F = F(x, z, p)$ is continuously differentiable, then $\delta_u \mathcal{F} : \mathcal{V} \to \mathbb{R}$ by

$$\delta_u \mathcal{F}[\phi] = \int_a^b \left(\frac{\partial F}{\partial z}(x, u(x), u'(x))\, \phi(x) + \frac{\partial F}{\partial p}(x, u(x), u'(x))\, \phi'(x) \right) dx$$

where $\mathcal{V}$ is a subspace of $C^1[a, b]$. The first variational formula is often written in a shorter form obtained by suppressing the arguments of the functions in the integrand:

$$\delta_u \mathcal{F}[\phi] = \int_a^b \left(\frac{\partial F}{\partial z}\, \phi + \frac{\partial F}{\partial p}\, \phi' \right) dx \qquad (2.9)$$

Exercise 2.17 *Compute the first variation formula (2.9) for an integral functional and verify $\delta_u \mathcal{F}$ is linear (assuming $\mathcal{V}$ is a vector subspace of $C^1[a, b]$).*

At this point consideration of several other examples is in order. In particular, we should like to see an example where the admissible class is (naturally) not a vector space. This is easy to illustrate.

Exercise 2.18 *Consider the length of the graph of a function $u : [a, b] \to \mathbb{R}$ with $u \in C^1[a, b]$. Write down the formula for the length functional $\mathcal{L} : \mathcal{A} \to \mathbb{R}$ where*

$$\mathcal{A} = \{u \in C^1[a, b] : u(a) = y_a \text{ and } u(b) = y_b\}.$$

Explain why $\mathcal{A}$ is **not** *a vector space, but the set of perturbations*

$$\mathcal{V} = \{u - v : u, v \in \mathcal{A}\} \qquad \text{is a vector space.}$$

What can you say about minimizers for this problem?

The shortest graph problem above is a very simple and popular example of a problem in the calculus of variations. The next problem is very similar to it in several ways.

Exercise 2.19 *Again let us take*

$$\mathcal{A} = \{u \in C^1[a, b] : u(a) = y_a \text{ and } u(b) = y_b\}$$

and

$$\mathcal{V} = C_0^1[a, b] = \{\phi \in C^1[a, b] : \phi(a) = \phi(b) = 0\}.$$

This time consider $\mathcal{D} : \mathcal{A} \to \mathbb{R}$ by

$$\mathcal{D}[u] = \int_a^b u'(x)^2\, dx.$$

This is called the **Dirichlet energy** *of a function $u \in C^1[a, b]$.*

(a) *Find the unique minimizer in the case $y_a = y_b$. Prove your answer is the only possible minimizer in this case.*

(b) *Explain why the minimizer when $y_a = y_b$ is inadmissible when $y_a \neq y_b$ and that the minimum value of $\mathcal{D}$ (if it exists when $y_a \neq y_b$) is positive. Hint: Use the mean value theorem. Can you find an explicit lower bound for the minimum value of $\mathcal{D}$ (in terms of a, b, y_a, and y_b)?*

(c) *Compute the first variation of $\mathcal{D}$.*

(d) *Assume a minimizer u_0 exists and is in $C^2[a, b]$. Integrate by parts to find an ordinary differential equation satisfied by the minimizer u_0. State and solve the natural boundary value problem for this ordinary differential equation for the minimizer.*

(e) *Again assuming $y_a \neq y_b$, compare the values of $\mathcal{D}[u_c]$ where c is fixed with $a < c < b$ and*

$$u_c(x) = \begin{cases} y_a, & 0 \leq x \leq c, \\ (y_b - y_a)(x - c)/(b - c) + y_a, & c \leq x \leq b. \end{cases}$$

There is something quite interesting about the functions u_c in the last part of Exercise 2.19. Do you see what it is?

Here are two (much harder) but still quite popular calculus of variations problems:

Exercise 2.20 *(Brachistochrone) Consider the points $A = (0, H)$ and $B = (1, h)$ in the plane with $0 < h < H$. If we consider the path*

$$\{(x, -(H - h)x + H) : 0 \le x \le 1\}$$

connecting A to B, we can imagine a point mass (or frictionless bead) that starts from rest at A and slides down to B (under the influence of a downward gravitational field $-g(0, 1)$). Assuming the mass is constrained to the specified path, notice that the gravitational force can be decomposed in components parallel and orthogonal to the path as

$$-mg(0, 1) = -mg \sin\psi(\cos\psi, \sin\psi) + mg \cos\psi(\sin\psi, -\cos\psi)$$

where $\psi = \tan^{-1}(h - H) < 0$. The component orthogonal to the path must be absorbed by a reaction force and, according to Newton's second law the other component gives acceleration to the mass according to

$$\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right) = -g \sin\psi(\cos\psi, \sin\psi).$$

In this case we can find an explicit expression for the motion, and then essentially everything is known.

(a) *How long does it take for this mass to move from A to B?*

In other cases, computing the time of travel is not so easy. Let us say a path is given by the graph of a function in the set $u \in C^1[0, 1]$ with $u(0) = H$ and $u(1) = h$.

Let us also assume, as with the straight line path given above, a frictionless bead starting from rest at A will move to B along the graph of this function (under the influence of gravity) and arrive at B in finite time T. Denote the motion of this mass by

$$\mathbf{r}(t) = (x(t), y(t)) = (x(t), u(x(t))) \qquad \text{for} \qquad 0 \le t \le T.$$

(b) *Recall the arclength relation*

$$s = \int_0^x \sqrt{1 + u'(\xi)^2}\, d\xi.$$

Differentiate this expression twice with respect to time to obtain

$$\frac{d^2 s}{dt^2} = -g \sin \psi$$

where the inclination angle ψ *is defined by*

$$(\cos \psi, \sin \psi) = \left(\frac{1}{\sqrt{1 + u'(x)^2}}, \frac{u'(x)}{\sqrt{1 + u'(x)^2}} \right)$$

as usual. Hint(s): Take the component of force along the path as we did for the straight line path to conclude

$$\frac{d^2 \mathbf{r}}{dt^2} = -g \sin \psi (\cos \psi, \sin \psi)$$

in general. Then compute $d^2\mathbf{r}/dt^2$ *directly and compare what you get to your expression for* $d^2 s/dt^2$.

(c) *Show the quantity*

$$\mathcal{C} = \frac{1}{2} m \left(\frac{ds}{dt} \right)^2 + mgu(x(t))$$

is constant. Hint: Differentiate $\mathcal{C}$ *with respect to* t *and use the previous part.*

(d) *Assume*

$$\frac{dx}{dt} > 0$$

so that $x : [0, T] \to [0, 1]$ *has in inverse* $\tau : [0, 1] \to [0, T]$ *giving the time* $\tau = \tau(\xi)$ *at which the mass has* x*-coordinate* ξ. *Show*

$$T[u] = \frac{1}{\sqrt{2g}} \int_0^1 \sqrt{\frac{1 + u'(x)^2}{H - u(x)}}\, dx.$$

Hint: Use the fundamental theorem of calculus to write T as an integral of $d\tau/d\xi$. Then use the chain rule to show

$$\frac{d\tau}{d\xi} = \frac{\frac{ds}{dx}}{\frac{ds}{dt}}.$$

Finally, use the conserved quantity to express ds/dt in terms of u.

A minimizer of the time of travel functional $T : \mathcal{A} \to \mathbb{R}$ where

$$\mathcal{A} = \left\{ u \in C^1[0,1] : u(0) = H,\ u(1) = h,\ \text{and } \int_0^1 \sqrt{\frac{1+u'(x)^2}{H-u(x)}}\, dx < \infty \right\}$$

is called a brachistochrone or "shortest time" function. This is an example where the perturbation space

$$\mathcal{V} = \{\phi \in C^1[0,1] : \phi(0) = 0 = \phi(1)\}$$

cannot be interpreted as the set of differences of admissible functions. Nevertheless, one has for each $u \in \mathcal{A}$ the crucial condition

$$u + \epsilon\phi \in \mathcal{A} \qquad \text{when } \phi \in \mathcal{V} \text{ and } |\epsilon| \text{ is small enough.}$$

This is enough to compute the first variation and determine minimizers.

Incidentally, this problem was posed publicly (and somewhat flamboyantly) by Johann Bernoulli in 1696:

> *I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.*

It can certainly be argued that while the shortest path and minimum Dirichlet energy problems have (at least) obvious candidates for minimizers, this problem illustrates the fact that the calculus of variations can be used to obtain very non-obvious information.

Exercise 2.21 *Consider $A : \mathcal{A} \to \mathbb{R}$ by*

$$A[u] = 2\pi \int_0^1 u(x)\sqrt{1+u'(x)^2}\,dx$$

where

$$\mathcal{A} = \{u \in C^1[0,1] : u(0) = z_0,\ u(1) = z_1,\ u > 0\}.$$

This functional gives the area of a surface of rotation generated by rotating the graph of u around the x-axis. Find the first variation and the differential equation satisfied by C^2 minimizers. What you will obtain is called the axially symmetric **minimal surface equation***; it is in fact the equation of meridian curves for axially symmetric surfaces with zero mean curvature. (We will discuss mean curvature in the next section.)*

Finding the actual minimizers for the functionals in the last two problems is relatively difficult.

Note on regularity of function classes

For reasons that should become clear later—and would also become clear if the brachistochrone and axially symmetric minimal surface problems were studied further—it is natural to require less regularity than we have required above for admissible functions and more regularity for perturbations. In fact, we already considered functions whose regularity was less than the nominal regularity of the admissible class in part (e) of Exercise 2.19 concerning Dirichlet energy. Taking the set

$$\{u \in C^1[a,b] : u(a) = y_a \text{ and } u(b) = y_b\}$$

used in some of the examples above, we usually replace this with the larger admissible class

$$\{u \in \sqsubset^1[a,b] : u(a) = y_a \text{ and } u(b) = y_b\}$$

where $\sqsubset^1[a,b]$ denotes the subspace of $C^0[a,b]$ consisting of **piecewise** C^1 **functions**. For each function $u \in \sqsubset^1[a,b]$ there exists a partition $a = x_0 < x_1 < x_2 < \cdots < x_m = b$ such that for $j = 1, 2, \ldots, m$

$$u\Big|_{[x_{j-1},x_j]} \in C^1[x_{j-1}, x_j].$$

Here are three reasons to consider admissible classes with lower regularity:

1. It is easier to find minimizers and prove minimizers exist—because you are allowing more possibilities. (This is particularly important in the direct methods of the calculus of variations which we will not really cover, and certainly wont emphasize, in this course.)

2. It is easier to make modifications/variations of a given admissible function and remain in the admissible class.

3. Sometimes minimizers do not have the regularity you would expect. For example sometimes minimizers turn out to be piecewise C^1 instead of C^1. See Exercise 2.22 below.

Alternatives to the space of piecewise C^1 functions may be found among the spaces of functions with **weak derivatives**. These spaces are considered, for example, in the text *One-dimensional Variational Problems* by Buttazzo, Giaquinta, and Hildebrandt. The piecewise C^1 functions, however, are a quite traditional choice found, for example, in the classic text *Introduction to the Calculus of Variations* by Hans Sagan.

Exercise 2.22 *(Newton's profile of minimal drag) Isaac Newton modeled the drag on an axially symmetric object of maximum radius R as proportional to*

$$N[u] = \int_0^R \frac{x}{1+u'(x)^2}\,dx$$

where the graph of $u \in \sqsubset[0,R]$ gives the rigid profile meeting the opposing fluid medium. For example, if $u(x) \equiv 0$, then one is considering a flat cylinder $\{(x,y,z) : x^2+y^2 \le R^2 \text{ and } z \le 0\}$ or

$$\{(x,y,z) : x^2+y^2 \le R^2 \text{ and } -L \le z \le 0\}$$

moving vertically upward and

$$N[u] = \frac{R^2}{2}$$

can be viewed as giving a measure of the resistance encountered.

(a) *If one caps the cylinder with a hemisphere, what does Newton's resistance measurement give? Newton mentioned the comparison (of the value for the hemisphere to that for the cylinder) specifically and apparently viewed it as an encouraging sign that his functional was measuring the quantity he had in mind.*

(b) *Compute the Newtonian resistance $N[u]$ for the conical cap determined by*

$$u(x) = \frac{H}{R}\,(R - x).$$

In practice, it may be impractical to construct a nose cone of arbitrarily large height H. Thus, we introduce the admissible class

$$\mathcal{A} = \{u \in \sqsubset^1[0, R] : u(0) = H,\ u(R) = 0,\ \text{and } u' \le 0\}$$

for $H > 0$ fixed. The next part gives some indication about the origin of the monotonicity requirement $u'(x) \le 0$ for $0 \le x \le R$.

(c) *Plot the profile determined by $u(x) = H\sin^2(2\pi nx/R)$ and compute $N[u]$.*

(d) *We may assume every function $u \in \mathcal{A}$ has $u(x) \equiv H$ on some interval $0 \le x \le R_0 < R$. Among the admissible functions*

$$u(x) = \begin{cases} H, & 0 \le x \le R_0, \\ H(R - x)/(R - R_0), & R_0 \le x \le R, \end{cases}$$

which has the least Newtonian resistance $N[u]$?

There are two more exercises at the end of this section on Newton's resistance functional. The first suggests a kind of justification/derivation for the functional itself, and the second gives a start at finding some actual minimizers and proving that every minimizer satisfies $u(x) \equiv H$ for $0 \le x \le R_0$ and some $R_0 > 0$ and that

$$\lim_{x \searrow R_0} u'(x) < 0$$

so that a minimizer satisfies $u \in \sqsubset^1[0, R]\backslash C^1[0, R]$.

Let us now turn our attention to the vector space of perturbations. A typical collection of perturbations is $C_c^\infty(a, b)$ which is a relatively much smaller vector space than

$$C_0^1[a, b] = \{\phi \in C^1[a, b] : \phi(a) = \phi(b) = 0\}.$$

The functions in $C_c^\infty(a, b)$ are infinitely differentiable and have support compactly contained in the interior interval (a, b). This requires a little explanation.

Open and Closed Sets; Support

We have mentioned the **open interval** $(a, b) \subset \mathbb{R}^1$ and the **open disk**

$$B_r(x_0, y_0) = \{(x, y) : (x - x_0)^2 + (y - y_0)^2 < r^2\} \subset \mathbb{R}^2.$$

These are both examples of **open balls**. Note in particular, that the open interval can be expressed as the set of all points in $\mathbb{R}^1$ whose distance from the center $(a + b)/2$ is less than the radius $(b - a)/2$. In fact open balls are prototypical open sets in any **metric space** which is a set with a notion of distance between pairs of points. More precisely, a set X is a metric space if there is a function $d : X \times X \to [0, \infty)$ satisfying

1. $d(x, y) = d(y, x)$ for all $x, y \in X$. (symmetry)
2. $d(x, y) = 0$ if and only if $x = y$. (positive definite)
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (triangle inequality)

The function d is called a distance function or metric (distance). Every finite dimensional Euclidean space $\mathbb{R}^n$ is a metric space with

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{j=1}^{n} (y_j - x_j)^2}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. This is called the Euclidean metric and the value of the Euclidean metric is also denoted $|\mathbf{y} - \mathbf{x}|$ or sometimes $\|\mathbf{y} - \mathbf{x}\|$ if one is worried about confusion with the absolute value function on $\mathbb{R}$.

Using any metric (on a metric space X) one defines the open ball of radius $r > 0$ and center $p \in X$ by

$$B_r(p) = \{x \in X : d(x, p) < r\}.$$

Also a subset $U \subset X$ of any metric space X is said to be **open** if for each $p \in U$, there is some $r > 0$ such that

$$B_r(p) \subset U.$$

Exercise 2.23 *Show that an open ball in any Euclidean space is open.*

Technically, there can be other notions of open sets and we should be a little more careful and say a set is *open with respect to the metric topology* if the condition above holds. For our purposes at the moment, however, we can assume the only topologies of interest are metric topologies of the sort just described.

Exercise 2.24 *Show a finite intersection of open sets is open and any possible union of open sets is open.*

Definition 3 *The* **interior** *of any set (in a metric space) is the union of all open balls inside that set. If $A \subset X$ and X is a metric space, we denote the interior of A by* $\text{int}(A)$ *and*

$$\text{int}(A) = \bigcup_{\substack{x \in X,\, r>0 \\ B_r(x) \subset A}} B_r(x).$$

Exercise 2.25 *Show the interior of a set is always open.*

A set $A \subset X$ is defined to be **closed** if the complement

$$A^c = X \backslash A = \{x \in X : x \notin A\} \qquad \text{is open.}$$

Exercise 2.26 *Show any intersection of closed sets is closed.*

This brings us to a crucial construction: The **closure** of any subset A of a metric space X is defined to be the smallest closed set containing A. That is, the closure of A is

$$\text{clos}(A) = \overline{A} = \bigcap_{\substack{C \supset A \\ C\text{:closed}}} C.$$

Exercise 2.27 *Show a set is closed if and only if the set is its own closure.*

Definition 4 *(support) Given a function $u : A \to \mathbb{R}$ defined on a subset A of the Euclidean space $\mathbb{R}^n$, the* **support** *of u, denoted by* $\text{supp}(u)$*, is the closure of the set of points where u is nonzero. That is,*

$$\text{supp}(u) = \overline{\{\mathbf{x} \in A : u(\mathbf{x}) \neq 0\}}.$$

Definition 5 *A set $A \subset X$ where X is a metric space is* **bounded** *if there is some $p \in X$ and some $r > 0$ such that*

$$A \subset B_r(p).$$

In the case where $X = \mathbb{R}^n$ is Euclidean space, we may take the center of the bounding ball to be the origin $\mathbf{0}$. Then a set is bounded if there is some $r > 0$ such that

$$|\mathbf{x}| < r \qquad \text{for all } \mathbf{x} \in A.$$

Definition 6 *A set $K \subset \mathbb{R}^n$ is* **compact** *if K is closed and bounded.*

Definition 7 *A function $u : A \to \mathbb{R}$ defined on a set $A \subset \mathbb{R}^n$ is said to have* **compact support** *in A if $\text{supp}(u)$ is compact and*

$$\text{supp}(u) \subset \text{int}(A).$$

This condition is often written as $\text{supp}(u) \subset\subset A$, which is read "the function u has support compactly contained in A" or "the function u is compactly supported in A" for short.

We are now (almost) in a position to discuss $C_c^\infty(a, b)$. We have mentioned that the set of continuous real valued functions on $[a, b]$ is denoted by $C^0[a, b]$, and the set of continuously differentiable real valued functions on $[a, b]$ is denoted by $C^1[a, b]$. These are vector spaces over $\mathbb{R}$ and $C^0[a, b] \supset C^1[a, b]$. Naturally, we can also require continuity or differentiability only at interior points of (a, b), and the corresponding vector spaces are denoted by $C^0(a, b) \supset C^1(a, b)$. We can also require the existence of more continuous derivatives: The functions in $C^k(a, b)$ have derivatives of order k which are continuous at each point in (a, b), and we have an infinite collection of nested vector subspaces:

$$C^0(a, b) \supset C^1(a, b) \supset C^2(a, b) \supset \cdots$$

Showing strict inequality in each of these inclusions is one way to show each of these vector subspaces is infinite dimensional.

$$C^\infty(a, b) = \bigcap_{k=0}^{\infty} C^k(a, b).$$

In some sense, most familiar functions are in this (kind of) space. Most familiar functions have derivatives of all orders: polynomials, exponentials, sine and cosine. The tangent function is in $C^\infty(-\pi/2, \pi/2)$.

$$C_c^\infty(a,b) = \{u \in C^\infty(a,b) : \text{supp}(u) \subset\subset (a,b)\}.$$

If you haven't been shown a function in $C_c^\infty(a,b)$, or thought carefully about it for a long time, then you probably do not know any nonzero functions in this set.

Exercise 2.28 *Show there exists a nonzero C^∞ function with compact support.*

2.28.1 Calculus of Variations—second pass

One advantage of using a very small perturbation space is that the theorems above hold under less restrictive hypotheses. Notice that to require

$$\mathcal{F}[u] \le \mathcal{F}[u + h\phi] \qquad \text{for every } \phi \in C_0^1[a,b]$$

is much more than requiring

$$\mathcal{F}[u] \le \mathcal{F}[u + h\phi] \qquad \text{for every } \phi \in C_c^\infty(a,b)$$

simply because $C_c^\infty(a,b)$ is effectively a subset of $C_0^1[a,b]$.

Here is a somewhat more standard treatment of some of the results above for an integral functional $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ defined on an admissible class in $\sqsubset^1[a,b]$. If $\mathcal{F}$ is given by

$$\mathcal{F}[u] = \int_a^b F(x, u, u')\, dx,$$

then the function $F = F(x,z,p)$ is called the **Lagrangian** for the variational problem. The first variation of $\mathcal{F}$ at u in the direction $\phi \in C_c^\infty(a,b)$ is defined by

$$\delta_u \mathcal{F}[\phi] = \left[\frac{d}{d\epsilon} \int_a^b F(x, u + \epsilon\phi, u' + \epsilon\phi')\, dx\right]\Big|_{\epsilon=0}.$$

Theorem 5 *A function $u \in \mathcal{A}$ for which*

$$\delta_u \mathcal{F}[\phi] \equiv 0 \qquad \textit{for all } \phi \in C_c^\infty(a,b)$$

is called a **weak extremal** *of $\mathcal{F}$, and one has the first necessary condition*

$$\int_a^b \left[\frac{\partial F}{\partial z}\phi + \frac{\partial F}{\partial p}\phi'\right] dx = 0 \qquad \textit{for all } \phi \in C_c^\infty(a,b).$$

The key tool for the proof of the next result is called the **fundamental lemma of the calculus of variations**:

Lemma 6 *If* $f : (a, b) \to \mathbb{R}$ *is a continuous function and*

$$\int_a^b f(x)\phi(x)\,dx = 0 \qquad \textit{for every } \phi \in C_c^\infty(a, b),$$

then $f(x) = 0$ *for* $x \in (a, b)$.

Theorem 7 *A weak extremal for* $\mathcal{F}$ *which is* C^2 *on any open subinterval* $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ *satisfies the ordinary differential equation*

$$\frac{d}{dx}\left(\frac{\partial F}{\partial p}\right) = \frac{\partial F}{\partial z} \tag{2.10}$$

on the interval $(x_0 - \delta, x_0 + \delta)$.

The second order ordinary differential equation (2.10) is called the **Euler-Lagrange** equation for the functional $\mathcal{F}$.

There are various generalizations of these results, but understanding the simple concept that minimization problems for integral functionals lead to differential equations is a good start.

Exercise 2.29 *Prove the fundamental lemma.*

Exercise 2.30 *Use the fundamental lemma (and integration by parts) to prove the Euler-Lagrange equation holds for* C^2 *weak extremals.*

Local Minimizers

Exercise 2.9 illustrates that critical points can be local minimizers in finite dimensional calculus without being global minimizers. The same thing can happen in the calculus of variations, but up until this point we have not introduced enough structure to make sense of the notion of local versus global minimizers. The key is the introduction of a distance between elements in the admissible class $\mathcal{A}$. In particular, we already have discussed the notion of a metric distance and we certainly want to have such a distance on $\mathcal{A}$. Most commonly, however, the metric distance we will use comes from an additional abstract structure which it is well worth discussing:

Definition 8 *Given a vector space V over the field $\mathbb{R}$, i.e., a real vector space, a function $\| \cdot \| : V \to [0, \infty)$ is called a* **norm***, and the vector space V is called a* **normed vector space***, if the following conditions hold:*

1. $\|cv\| = |c|\|v\|$ *for every $c \in \mathbb{R}$ and every $v \in V$ (non-negative homogeneity)*
2. $\|v\| = 0$ *if and only if $v = \mathbf{0}$ (positive definite)*
3. $\|v + w\| \leq \|v\| + \|w\|$ *(triangle inequality)*

Exercise 2.31 *Show that every normed vector space is a metric space with metric distance $d(v, w) = \|v - w\|$.*

Exercise 2.32 *Show that $\|u\| = \max\{|u(x)| : a \leq x \leq b\}$ defines a norm on $C^0[a, b]$. (You'll need some theorems from 1-D calculus for this.) This is called the "C zero" norm, the L^∞ norm, the "sup" norm, and the uniform norm; it goes by many names.*

Exercise 2.33 *Show $C^0_B(a, b) = \{u \in C^0(a, b) : \sup\{|u(x)| : a < x < b\}\}$ is a vector subspace of $C^0(a, b)$ and*

$$\|u\|_{C^0} = \sup\{|u(x)| : a < x < b\}$$

is a norm on $C^0_B(a, b)$ (the subspace of bounded continuous functions on (a, b)).

Exercise 2.34 *There are many important continuous functions which are not in $C^0_B(a, b)$, and the sup norm is not a norm on $C^0(a, b)$. Consider $d : C^0(a, b) \times C^0(a, b) \to [0, \infty)$ by*

$$d(f, g) = \min\{1, \sup\{|f(x) - g(x)| : a < x < b\}.$$

Is d a metric on $C^0(a, b)$?

There are a good many important vector spaces, like $C^0(a, b)$, which are not (at least in any natural way) normed spaces. Please note/recall that normed spaces are required to be vector spaces but metric spaces, in general, are not required to be vector spaces. If the notion of a metric, however, is coupled with the condition of being a vector space by the introduction of certain axioms one is led to (or may stumble upon) the theory of **topological vector spaces**. Doing analysis in the framework of topological vector spaces can become somewhat complicated, so we will try to avoid that, but it's perhaps worth knowing such a thing/structure is out there.

Exercise 2.35 *Consider* $[\,\cdot\,] : C^1[a,b] \to [0,\infty)$ *by*

$$[u] = \max\{|u'(x)| : a \le x \le b\}.$$

The function $[\,\cdot\,]$ *is called the* C^1 **seminorm**.

(a) *Determine which properties of a norm* $[\,\cdot\,]$ *satisfies. Those are the defining properties of a seminorm.*

(b) *Show that* $\|\ \cdot\ \|_1 : X \to [0,\infty)$ *given by*

$$\|v\|_1 = \|v\| + [v]$$

where $\|\ \cdot\ \|$ *is any norm on a vector space* X *and* $[\,\cdot\,]$ *is any seminorm on* X *is a norm on* X.

The sum of the C^0 *"sup" norm and the* C^1 *seminorm is called the* C^1 *norm on* $C^1[a,b]$.

(c) *Define a* C^1 *seminorm and a* C^1 *norm on a suitable subspace* $C^1_B(a,b)$ *of* $C^1(a,b)$.

Definition 9 *Let* $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ *be a functional defined on a admissible class of functions* $\mathcal{A}$ *which is a subset of a normed vector space* X *containing the subspace of variations* $\mathcal{V}$. *An admissible function* $u \in \mathcal{A}$ *is said to be a* **local minimizer** *of* $\mathcal{F}$ *relative to the norm on* X *if there exists some* $\delta > 0$ *such that the following holds:*

If $v \in \mathcal{A}$ *and* $\|u - v\|_X \le \delta$, *then* $u - v \in \mathcal{V}$ *and*

$$\mathcal{F}[u] \le \mathcal{F}[v].$$

This definition gives rise to the notion of local C^0 minimizers (if one takes the C^0 norm on $C^0[a,b]$) and of local C^1 minimizers (if one happens to have $\mathcal{A} \subset C^1[a,b]$ and takes the C^1 norm).

Theorem 8 *(first order necessary conditions in the calculus of variations) A local* C^0 *minimizer* $u \in \mathcal{A} \subset C^1(a,b) \cap C^0(a,b)$ *of the integral functional* $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ *given by*

$$\mathcal{F}[u] = \int_a^b F(x,u,u')\,dx$$

is a weak extremal:

$$\int_a^b \left[\frac{\partial F}{\partial z}\,\phi + \frac{\partial F}{\partial p}\,\phi'\right] dx = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b).$$

If the local C^0 minimizer satisfies $u \in C^2(a,b)$, then u is a solution of the Euler-Lagrange equation in the interior of the interval (a,b):

$$\frac{d}{dx}\left(\frac{\partial F}{\partial p}\right) = \frac{\partial F}{\partial z} \qquad a < x < b.$$

Additional Exercises

Exercise 2.36 *(Newton's drag functional) One can heuristically motivate the interpretation of the quantity*

$$N[u] = \int_0^R \frac{x}{\sqrt{1+u'(x)^2}}\,dx$$

as a measure of the resistance against a moving profile along the following lines: To a moving point mass m having velocity $\mathbf{v}$ one can associate a **momentum vector** *$m\mathbf{v}$ and a potential energy $m|\mathbf{v}|^2/2$. Assume we are given a very small mass m at rest that encounters a large moving profile, associated presumably to a large mass. We can shift reference frame and consider the profile at rest and the small mass as moving and striking the profile with a particular orientation. In particular, if we assume the initial momentum vector of the mass is $-m|\mathbf{v}|(0,1)$ and the profile is given by $\{(x,u(x)) : 0 \le x \le R\}$, then the component of the momentum vector orthogonal to the profile at impact is*

$$[-m|\mathbf{v}|(0,1)\cdot(\sin\psi, -\cos\psi)]\,(\sin\psi, -\cos\psi) = m|\mathbf{v}|\cos\psi\,(\sin\psi, -\cos\psi)$$

where the inclination angle ψ is defined by

$$(\cos\psi, \sin\psi) = \left(\frac{1}{\sqrt{1+u'(x)^2}}, \frac{u'(x)}{\sqrt{1+u'(x)^2}}\right)$$

as usual. Assume this component of the momentum is completely absorbed by the profile. Accordingly, we assume the profile absorbs the kinetic energy associated with this component of momentum.

(a) *What is the absorbed kinetic energy from the mass m?*

(b) *Instead of a finite point mass m, approximate the absorbed energy with a mass of the form*

$$m_{ij} = \rho\, x_j^* \,(\theta_i - \theta_{i-1})\,(x_j - x_{j-1})$$

where ρ is a constant areal mass density and $x_j^*\,(\theta_i - \theta_{i-1})\,(x_j - x_{j-1})$ is a local area element given in polar coordinates. Summing over i and j write an approximation for the total absorbed energy as a Riemann sum converging to an integral over the disk $B_R(0)$.

(c) *Show the integral expression from the last part is proportional to Newton's functional.*

Hint:

$$\sum_{i,j} \frac{1}{2}\,\rho\, x_j^*\,|\mathbf{v}|^2\,\cos^2\psi\,(\theta_i - \theta_{i-1})\,(x_j - x_{j-1})$$
$$\sim \frac{\rho}{2}|\mathbf{v}|^2 \sum_{i,j} \frac{x_j^*}{1 + u'(x_j^*)^2}\,(\theta_i - \theta_{i-1})\,(x_j - x_{j-1}).$$

Exercise 2.37 *(flat tipped minimizers) We consider Newton's profile of minimal drag problem with $R = H = 1$. Consider the function $f : [1,\infty) \to \mathbb{R}$ by*

$$f(t) = \frac{t}{(1+t^2)^2}\left(\frac{3}{4}\,t^4 + t^2 - \frac{7}{4} - \log t\right).$$

Set $t_0 = f^{-1}(1)$ and $r_0 = 4T/(1+T^2)^2$.

(a) *Find t_0 and r_0 numerically.*

(b) *Use mathematical software to plot the profile $\{(x, u_0(x)) : 0 \le x \le 1\}$ satisfying $u_0(x) = 1$ for $0 \le x \le r_0$ with the remainder of the graph given parametrically by*

$$(x(t), z(t)) = (0, 1) + \frac{r_0(1+t^2)^2}{4t}\,(1, -f(t)), \qquad 1 \le t \le t_0.$$

(c) *Show $\lim_{x \searrow r_0} u'(x) < 0$.*

(d) *Let R_0 be fixed with $0 \le R_0 < 1$. Consider a function $u : [0,1] \to [0,1]$ with*

(i) $u \in C^0[0,1]$,

(ii) $u(x) \equiv 1$ *for* $0 \le x \le R_0$,

(iii) $u'(x) < 0$ *for* $R_0 < x \le 1$, *and*

(iv) $u(1) = 0$.

Note that the restriction

$$u\big|_{[R_0,1]} : [R_0, 1] \to [0,1]$$

has an inverse $w : [0,1] \to [R_0, 1]$. Assume $w \in C^1[0,1]$ and set $v(t) = w(1-t)$. Show

$$N[u] = \frac{v(0)^2}{2} + \int_0^1 \frac{v(t)v'(t)^3}{1+v'(t)^2}\,dt.$$

(e) *Consider $M : \mathcal{M} \to \mathbb{R}$ by*

$$M[v] = \frac{v(0)^2}{2} + \int_0^1 \frac{v(t)v'(t)^3}{1+v'(t)^2}\,dt$$

on

$$\mathcal{M} = \{v \in C^2[0,1] : v(0) \ge 0,\ v(1) = 1,\ \text{and } v' \ge 0\}.$$

Compute the Euler-Lagrange equation and show the solution v leads to the function u_0 defined in part (b). (This is somewhat tricky, but at least you should be able to show the function v_0 obtained from u_0 solves the Euler-Lagrange equation.)

2.38 Partial Differential Equations

One generalization we do want to consider is exemplified by deriving the equations of Laplace and Young for a capillary surface in a vertical tube. Let us, in this instance, assume the tube has general cross-section $\mathcal{U}$ where $\mathcal{U}$ is a bounded open subset of $\mathbb{R}^2$ having boundary (which is a topological term we need to define) a smooth simple closed curve. What we mean by this is the following: The **boundary** of any set (in a metric space, e.g., $\mathbb{R}^n$) is the intersection of the closure of the set with the closure of the complement of the set. That is,

$$\partial\mathcal{U} = \overline{\mathcal{U}} \bigcap \overline{\mathbb{R}^2 \backslash \mathcal{U}}.$$

Exercise 2.39 *Show a x is in the boundary ∂A of any set A if and only if for every $r > 0$*

$$B_r(x) \cap A \neq \phi \qquad \text{and} \qquad B_r(x) \cap A^c \neq \phi.$$

There are a couple equivalent ways we can say what it means for an open bounded set $\mathcal{U} \subset \mathbb{R}^2$ to have a smooth simple closed curve as boundary. It is no easy task to show they are equivalent, but we can state the conditions.

There exists a surjective[2] twice continuously differentiable vector valued function $\alpha : \mathbb{R} \to \partial\Omega \subset \mathbb{R}^2$ with the following properties

1. For some $L > 0$, the restriction
$$\alpha\Big|_{[0,L)} : [0, L) \to \partial\Omega \qquad \text{is one-to-one and onto,}$$

2. $\alpha(L) = \alpha(0)$, and

3. $\alpha(t + L) = \alpha(t)$ for all $t \in \mathbb{R}$.

Being twice continuously differentiable here means $x, y \in C^2(\mathbb{R})$ where $\alpha(t) = (x(t), y(t))$.

Alternatively, we can define a **homotopy** of a loop as follows: Given a loop, which is just a continuous function $\alpha : [0, L] \to \mathcal{U}$ with $\alpha(L) = \alpha(0)$, a homotopy of α (relative to $\mathcal{U}$) is a continuous function $h : [0, 1] \times [0, 1] \to \mathcal{U}$ satisfying the following

1. $\alpha(t) = h(t, 0)$ for $0 \leq t \leq 1$ and

2. $h(0, s) = h(1, s)$ for $0 \leq s \leq 1$.

A homotopy h of the loop α is a **fixed point homotopy** if $h(0, s) = \alpha(0)$ for $0 \leq s \leq 1$. A homotopy if a **contraction to a point** if there is a point $\mathbf{p} \in \mathcal{U}$ for which $h(t, 1) \equiv \mathbf{p}$ for $0 \leq t \leq 1$.

[2] Surjective means "onto" in the sense that for each $\mathbf{p} \in \Omega$, there is some $t \in \mathbb{R}$ with $\alpha(t) = \mathbf{p}$.

The open set $\mathcal{U} \subset \mathbb{R}^2$ is **simply connected** if for every loop $\alpha : [0, 1] \to \mathcal{U}$ there exists a fixed point homotopy (relative to $\mathcal{U}$) which is a contraction of α to $\alpha(0)$.

A bounded open set $\mathcal{U} \subset \mathbb{R}^2$ has boundary a simple closed curve if (and only if) the following hold:

1. $\mathcal{U}$ is simply connected, and

2. for each $\mathbf{p} \in \partial\mathcal{U}$, there exists some $a > 0$, a unit vector $\mathbf{u} = (u_1, u_2)$, and a function $g \in C^2[-a, a]$ with $g'(0) = 0$ such that

$$\begin{aligned}\mathcal{U} \cap \{\mathbf{p} + s\mathbf{u} + t\mathbf{u}^\perp : s, t \in [-a, a]\} \\ = \{\mathbf{p} + s\mathbf{u} + t\mathbf{u}^\perp : t \geq g(s) \text{ and } -a \leq s \leq a\}.\end{aligned}$$

In these sets $\mathbf{u}^\perp = (-u_2, u_1)$.

Exercise 2.40 *A collection of open sets $\{U_\alpha\}_{\alpha\in\Gamma}$ where Γ is any indexing set is called an* **open cover** *of a set A if*

$$A \subset \bigcup_{\alpha\in\Gamma} U_\alpha.$$

A subset $A \subset \mathbb{R}^n$ is compact if and only if it has the following property: Given any open cover $\{U_\alpha\}_{\alpha\in\Gamma}$ of A, there exist finitely many sets $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_k}$ in the open cover such that

$$\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_k}\} \qquad \text{is still an open cover of } A.$$

This is called the Hiene-Borel Theorem. The finite subcollection of open sets in this result is called a finite subcover.

Exercise 2.41 *Show that the boundary of an open bounded subset of $\mathbb{R}^n$ is compact.*

Returning to the capillary tube problem: Let $\mathcal{U}$ be a bounded open subset of $\mathbb{R}^2$ with boundary a simple closed curve. Let σ be a surface tension constant with units (force)/distance. Let $\beta \in (0, 1)$ be a dimensionless constant adhesion coefficient, i.e., β has units 1. Let g be the usual gravitational constant. Given $u \in C^1(\overline{\mathcal{U}})$, which means there exists an open set $U \supset \overline{\mathcal{U}}$ and an extension $\overline{u} : U \to \mathbb{R}$ with continuous partial derivatives

$$\frac{\partial \overline{u}}{\partial x} \quad \text{and} \quad \frac{\partial \overline{u}}{\partial y},$$

both in $C^0(U)$, such that

$$\overline{u}\Big|_{\mathcal{U}} = u,$$

we define the **capillary energy** of u to be

$$\mathcal{E}[u] = \sigma \int_{\mathcal{U}} \sqrt{1 + |Du|^2} - \sigma\beta \int_{\partial\mathcal{U}} u + \frac{\rho g}{2} \int_{\mathcal{U}} u^2.$$

As in the 2-D case, the first term is called the free surface energy, the second term is called the wetting energy, and the third term is called the gravitational potential energy.

Exercise 2.42 *Explain why*

$$A[u] = \int_{\mathcal{U}} \sqrt{1 + |Du|^2}$$

is the area functional.

Exercise 2.43 *Obtain the gravitational energy as a limit of a Riemann sum approximating an integral over the volume*

$$\{(x, y, z) : (x, y) \in \mathcal{U} \text{ and } 0 < z < u(x, y)\}$$

(assuming $u > 0$).

2.43.1 The first variation of area

We wish to compute a variation

$$\left[\frac{d}{d\epsilon} \int_{\mathcal{U}} \sqrt{1 + |D(u + \epsilon\phi)|^2}\right]_{\epsilon=0}.$$

Let us first recall that the vector function $Du : \overline{\mathcal{U}} \to \mathbb{R}^2$ is the **gradient field** or total derivative of u given by the vector of first partials:

$$Du = \left(\frac{\partial \overline{u}}{\partial x}, \frac{\partial \overline{u}}{\partial y}\right),$$

and when we write $|Du|^2$ we are indicating the use of the Euclidean norm:

$$|Du|^2 = \left(\frac{\partial \overline{u}}{\partial x}\right)^2 + \left(\frac{\partial \overline{u}}{\partial y}\right)^2.$$

Thus, the value of the area functional

$$\int_{\mathcal{U}} \sqrt{1 + |Du|^2}$$

is an example of an integral of a real valued function of two variables x and y over an open subset $\mathcal{U} \subset \mathbb{R}^2$. Certainly such integrals are considered in a course on multivariable calculus. It is likely that we will need to understand such integrals, and what can be done with them, a bit better than they are understood by most students who have taken such a course. In view of this, I have typed up in the next section an exposition of certain aspects of integration. It might be worth looking at before reading further. I have also included a review of differentiation and verious kinds of derivatives which may be consulted if desired or necessary.

Let us write $A : C^1(\overline{\mathcal{U}}) \to \mathbb{R}$ to denote the area functional and calculate the **first variation of area** δA. The area of a perturbed graph given by $u + \epsilon\phi$ where $\phi \in C_c^\infty(\mathcal{U})$ is given by

$$\int_{\mathcal{U}} \sqrt{1 + |Du + \epsilon D\phi|^2}.$$

Thus, by the chain rule

$$\frac{d}{d\epsilon} \int_{\mathcal{U}} \sqrt{1 + |Du + \epsilon D\phi|^2} = \int_{\mathcal{U}} \frac{(Du + \epsilon D\phi) \cdot D\phi}{\sqrt{1 + |Du + \epsilon D\phi|^2}},$$

and

$$\delta A_u[\phi] = \int_{\mathcal{U}} \frac{Du \cdot D\phi}{\sqrt{1 + |Du + \epsilon D\phi|^2}} = \int_{\mathcal{U}} Tu \cdot D\phi$$

where $Tu = Du/\sqrt{1 + |Du|^2}$ is the projection projection of the downward unit normal field $(u_x, u_y, -1)/\sqrt{1 + |Du|^2}$ encountered in Chapter 1. To the real scaling ϕTu of this projection field and use the divergence theorem to write

$$\int_{\mathcal{U}} \operatorname{div}(\phi Tu) = \int_{\partial\mathcal{U}} \phi Tu = 0.$$

owing to the fact that ϕ has support compactly contained in $\mathcal{U}$. There is a general product formula for the divergence applying to a real scaling of a vector field, namely,

$$\operatorname{div}(w\mathbf{v}) = Dw \cdot \mathbf{v} + w \operatorname{div} \mathbf{v}.$$

Exercise 2.44 *If $\mathcal{U}$ is an open subset of $\mathbb{R}^n$ with $w : \mathcal{U} \to \mathbb{R}$ satisfying $w \in C^1(\mathcal{U})$ and $\mathbf{v} \in C^1(\mathcal{U} \to \mathbb{R}^n)$, then*

$$\operatorname{div}(w\mathbf{v}) = Dw \cdot \mathbf{v} + w \operatorname{div} \mathbf{v}.$$

Prove this identity two different ways

(a) *Use the definition of the divergence as a limit of flux density.*

(b) *Verify the formula in terms of standard rectangular coordinates.*

This is a good time to pause and note that when we restrict to perturbations $\phi \in C_c^\infty(\mathcal{U})$, we are considering what are called **interior variations**. As pointed out in the previous section, this is a commonly considered and convenient vector space of perturbations. In the capillary tube problem, however, it is also important to consider more general variations.

Theorem 9 *If $u \in C^2(\mathcal{U})$, then the interior variation of area at u is given by*

$$\delta A_u[\phi] = -\int_{\mathcal{U}} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\phi \qquad \text{for } \phi \in C_c^\infty(\mathcal{U}).$$

Thus, we encounter the mean curvature operator

$$\mathcal{M}u = \operatorname{div} Tu = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$$

of the previous chapter.

Exercise 2.45 *Compute the interior variation of the full capillary energy $\mathcal{E}$ to show that for $u \in C^2(\mathcal{U})$*

$$\delta \mathcal{E}_u[\phi] = \int_{\mathcal{U}} \left[-\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + f(u)\right]\phi$$

for an appropriate function $f : \mathbb{R} \to \mathbb{R}$ and all $\phi \in C_c^\infty(\mathcal{U})$.

Mean Curvature

Now let us consider a little surface geometry in coordinates. Say $u \in C^2(\mathcal{U})$ has graph $\mathcal{G} = \{(x, y, u(x, y)) : (x, y) \in \mathcal{U}\}$. We have discussed the signed curvature of a plane curve with

$$k = \left(\frac{u'}{\sqrt{1 + u'^2}} \right)' = \frac{u''}{(1 + u'^2)^{3/2}}$$

when the curve is given as the graph of a function $\{(x, u(x)) : x \in (a, b)\}$. In this context, the signed curvature can also be realized as the derivative, with respect to arclength, of the inclination angle ψ with respect to the horizontal; see Exercise 2.20. In fact,

$$\sin \psi = \frac{u'}{\sqrt{1 + u'^2}}$$

so that

$$k = \frac{d}{dx}[\sin \psi] = \frac{d}{ds}[\sin \psi] \frac{ds}{dx} = cos\psi \ \frac{d\psi}{ds} \ \frac{ds}{dx} = \frac{d\psi}{ds}.$$

Exercise 2.46 *Reparameterize the graph* $\{(x, u(x)) : x \in (a, b)\}$ *by arclength to show*

$$\frac{ds}{dx} = \sqrt{1 + u'^2}.$$

More generally, the **curvature vector** $\vec{k}$ of a space curve $\alpha : (a, b) \to \mathbb{R}^n$ at a point $\alpha(t_0)$ with $t_0 \in (a, b)$ is defined as follows: Reparameterize α by arclength obtaining, for some $\epsilon > 0$, a parameterization of (perhaps a portion of) the same curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ defined on $\{s : |s| < \epsilon\}$ and satisfying $\gamma(0) = \alpha(t_0)$. Then,

$$\vec{k} = \frac{d^2\gamma}{ds^2}(0).$$

This definition assumes reparameterization by arclength is possible and that the derivatives to be computed exist. Sufficient conditions for this to be the case are the following:

1. $\alpha \in C^2((a, b) \to \mathbb{R}^n)$ and

2. $\alpha'(t_0) \neq \mathbf{0}$.

Exercise 2.47 *The arclength of a curve* $\alpha \in C^1((a,b) \to \mathbb{R}^n)$ *is*

$$s = \int_{t_0}^{t} |\alpha'(\tau)|\, d\tau.$$

Assuming $\alpha'(t_0) \neq \mathbf{0}$, *reparameterize* α *to obtain an arclength parameterization* $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ *as in the definition above and compute*

$$\frac{d\gamma}{ds}.$$

Exercise 2.48 *Given* $u \in C^2(a,b)$ *and* $\alpha(x) = (x, u(x))$, *find the curvature vector* $\vec{k}$ *of the graph* $\mathcal{G} = \{(x, u(x)) : x \in (a,b)\}$ *at each point, and find an expression for the signed curvature of the graph with respect to the upward normal.*

Exercise 2.49 *Assuming* $\alpha \in C^2((a,b) \to \mathbb{R}^n)$ *with* $\alpha'(t_0) \neq \mathbf{0}$, *compute the curvature vector* $\vec{k}$ *at* $\alpha(t_0)$ *in terms of* γ.

In physics, it is often convenient to express derivatives with respect to time using a "dot" instead of a prime so that velocity v is given by the derivative of position $\dot{x}$ with respect to time and acceleration a is given by the derivative of velocity $\ddot{x}$ with respect to time. This seems to be a tradition started by Newton and it leaves open the prime notation for derivatives with respect to space. A similar tradition is convenient when one makes curvature calculations like those above: We denote derivatives with respect to the parameter t (or whatever parameter is used to define α) with a prime and derivatives with respect to arclength with a "dot." Thus, $\dot{\gamma} = \alpha'/|\alpha'|$ and $\vec{k} = \ddot{\gamma}$.

It will be noted that there is no immediately obvious notion of signed curvature for a space curve. There are certain situations, however, where such a notion does make sense. If that curve happens to lie in a two-dimensional plane and a particular unit normal N (to the curve within that plane) is specified at a point $\alpha(t_0)$, then we may define the signed curvature of α at $\alpha(t_0)$ with respect to N by

$$k = \vec{k} \cdot N$$

where $\vec{k}$ is the curvature vector to the curve at $\alpha(t_0)$. The value of the signed curvature in this context is sometimes denoted by k_N.

Exercise 2.50 *Show the new notion of signed curvature for a graph* $\{(x, u(x)) : x \in (a, b)\}$ *agrees with the previous definition if we take as the specified normal*

$$N = \frac{(-u', 1)}{\sqrt{1 + u'^2}},$$

that is, the upward unit normal to the graph.

Perhaps this is a good start to understanding the curvature of curves.

Exercise 2.51 *Find the curvature of the graph of the function* $u(x) = \sqrt{r^2 - x^2}$ *for* $|x| < r$*. Find the curvature of the graph of the function* $u(x) = -\sqrt{r^2 - x^2}$ *for* $|x| < r$*.*

Let us return to our simple surface which is the graph of a function u:

$$\mathcal{G} = \{(x, y, u(x, y)) : (x, y) \in \mathcal{U}\}.$$

If we want to talk about the curvature of this surface, things are somewhat (more) complicated. We note that there are many curves passing through each point $(x_0, y_0, u(x_0, y_0))$ on the surface, and it is reasonable to imagine that the curvatures of these curves are somehow related to the curvature of the surface at this point. There are a several nominally different ways to think about (and compute) the kind of curvature (mean curvature) that is prescribed by the capillary equation. Probably we should think about at least a couple of them.

Take the upward unit normal to the surface $\mathcal{G}$ is given by

$$N = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |Du|^2}}.$$

Exercise 2.52 *Explain why the vectors* $X_x = (1, 0, u_x)$ *and* $X_y = (0, 1, u_y)$ *are linearly independent tangent vectors to* $\mathcal{G}$ *and compute* N *using the cross product* $X_x \times X_y$*.*

We denote the tangent plane to $\mathcal{G}$ at X by $T_X\mathcal{G}$. Thus,

$$T_X\mathcal{G} = \{aX_x + bX_y : (a, b) \in \mathbb{R}^2\}.$$

Any nonzero tangent vector $\mathbf{v} = (v_1, v_2, v_3) \in T_X\mathcal{G}$ determines a unique plane $\Pi = \Pi(\mathbf{v})$ orthogonal to $\mathbf{v} \times N$. Such a **normal plane** intersects the surface

$\mathcal{G}$ in a curve, and we would like to compute the signed curvature of this curve in the plane Π with respect to N at the point $X = (x, y, u(x, y))$.

To illustrate how this computation works, we make a specific choice of unit tangent vector

$$\mathbf{v} = \frac{X_x}{|X_x|} = \frac{(1, 0, u_x)}{\sqrt{1 + u_x^2}}.$$

Let us denote the associated plane by Π_α where we imagine $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^3$ parameterizes the intersection curve on some interval $(-\epsilon, \epsilon)$ with $\alpha(0) = X$. A unit normal to Π is

$$\mathbf{w} = N \times \mathbf{v}.$$

Computing and writing this vector as a column vector we have

$$\mathbf{w} = \frac{1}{\sqrt{1 + |Du|^2}\,\sqrt{1 + u_x^2}} \begin{pmatrix} -u_x u_y \\ 1 + u_x^2 \\ u_y \end{pmatrix}.$$

We also write

$$\Pi_\alpha = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 : [(\xi, \eta, \zeta) - (x, y, u)] \cdot \mathbf{w} = 0\}.$$

The intersection of Π_α with the graph of u

$$\mathcal{G} = \{(\xi, \eta, u(\xi, \eta)) : (\xi, \eta) \in \mathcal{U}\}$$

in some small neighborhood of $X = (x, y, u(x, y))$ is a C^2 curve. This follows from the **implicit function theorem**. This is a touch tricky, so let's see if we can give the details of how it works: Consider the function $\Psi : \mathcal{U} \to \mathbb{R}^2$ by

$$\Psi \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ [(\xi, \eta, u(\xi, \eta)) - (x, y, u)] \cdot \mathbf{w} \end{pmatrix}.$$

I've written the arguments as columns here because they are (I think) a little easier to visualize and compute with in that form. Notice $\Psi(x, y) = (x, 0)$. Also, the transformation Ψ has total derivative

$$D\Psi = \begin{pmatrix} 1 & 0 \\ & \\ (1, 0, u_x(\xi, \eta)) \cdot \mathbf{w} & (0, 1, u_y(\xi, \eta)) \cdot \mathbf{w} \end{pmatrix}.$$

In particular, at $(\xi, \eta) = (x, y)$, we have $\det D\Psi \neq 0$. These are the hypotheses of the **inverse function theorem**, which then tells us there is an open ball $B_\delta(x,y) \subset \mathcal{U}$ such that Ψ restricted to $B_\delta(x,y)$ has a well-defined C^2 inverse with domain $\mathcal{V} = \Psi(B_\delta(x,y)) \subset \mathbb{R}^2$ and $(x,0) \in \mathcal{V}$. We write the second component of Psi^{-1} as ϕ, so that

$$\Psi^{-1}\begin{pmatrix} \xi \\ p \end{pmatrix} = \begin{pmatrix} \xi \\ \phi(\xi,p) \end{pmatrix}.$$

Setting $\eta(\xi) = \phi(\xi,0)$ it is easy to check $\alpha : (x-\delta, x+\delta) \to \mathbb{R}^2$ by

$$\alpha(\xi) = (\xi, \eta(\xi), u(\xi, \eta(\xi)))$$

is a parameterization of the intersection curve near X with $\alpha' = (1, \eta', u_x + \eta' u_y) \neq \mathbf{0}$. What we have actually done here is give the proof of the implicit function theorem in this case applied directly would say that if

$$\frac{\partial}{\partial\eta}\left\{ [(\xi,\eta,u(\xi,\eta)) - (x,y,u)] \cdot \mathbf{w} \right\}\Big|_{(x,y)} \neq 0,$$

then there is some $\delta > 0$ for which the equation

$$[(\xi,\eta,u(\xi,\eta)) - (x,y,u)] \cdot \mathbf{w} = 0$$

determines η uniquely as a C^2 function of ξ for $x - \delta < \xi < x + \delta$. We get the same conclusion.

Reparameterizing by arclength, we can assume the intersection curve is given locally by

$$\gamma(s) = (\xi(s), \eta(s), u(\xi(s), \eta(s)))$$

with $\gamma(0) = X = (x,y,u)$. Parameterization by arclength means that the tangent vector $\dot\gamma = (\dot\xi, \dot\eta, \dot\xi u_x + \dot\eta u_y)$ is a unit vector where $u_x = u_x(\xi,\eta)$ and $u_y = u_y(\xi,\eta)$. That is,

$$\dot\xi^2 + \dot\eta^2 + (\dot\xi u_x + \dot\eta u_y))^2 = 1. \tag{2.11}$$

In the particular case under consideration, we are also assuming

$$\dot\gamma(0) = \mathbf{v} = \frac{X_x}{|X_x|} = \frac{(1,0,u_x)}{\sqrt{1+u_x^2}}.$$

Exercise 2.53 *Use the inverse/implicit function theorem to generalize the construction above with* $\mathbf{v}$ *any unit vector in* $T_X\mathcal{G}$.

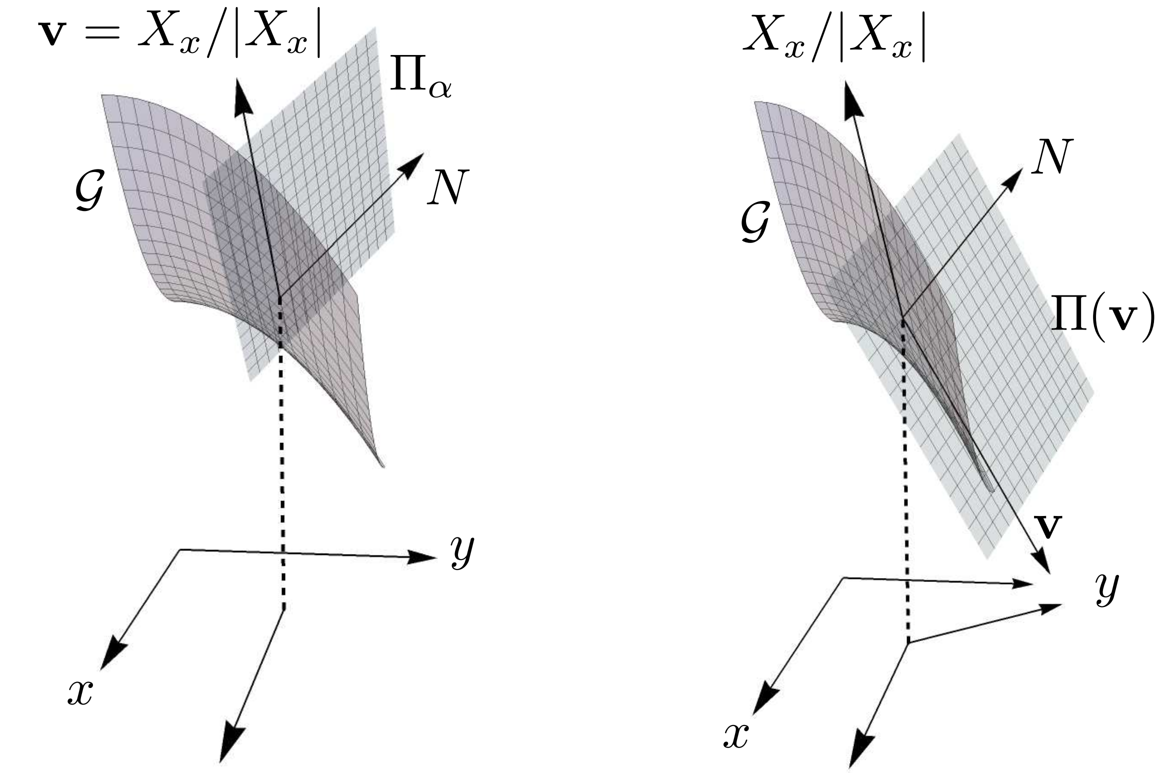

Figure 2.2: Planes normal to the graph of a function of two variables

If we differentiate the relation $[\gamma(s) - (x, y, u)] \cdot \mathbf{w} = 0$, noting that the vectors $X = (x, y, u)$ and $\mathbf{w}$ are independent of the arclength s, we conclude $\dot{\gamma} \cdot \mathbf{w} = 0$. Using the expression for $\mathbf{w}$ computed above, we see this implies

$$-\dot{\xi} u_x u_y + \dot{\eta}(1 + u_x^2) + (\dot{\xi} u_x + \dot{\eta} u_y) u_y = 0. \qquad (2.12)$$

We should be careful to recognize something about this dot product. Notice the three components of $\dot{\gamma}$ appearing here. Each involves dependence on the arclength s with $\dot{\xi} = \dot{\xi}(s)$ and $\dot{\eta} = \dot{\eta}(s)$. Note very carefully, the third component:

$$\dot{\xi} u_x + \dot{\eta} u_y = \dot{\eta}(s) u_x(\xi(s), \eta(s)) + \dot{\eta}(s) u_y(\xi(s), \eta(s)).$$

The remaining first partial derivatives in (2.12) are evaluated at (x, y). Thus, in the first and second terms

$$u_x u_y = u_x(x,y)u_y(x,y) \quad \text{and} \quad 1+u_x^2 = 1+[u_x(x,y)]^2 \quad \text{independent of } s.$$

Similarly, the second factor in the third term is $u_y = u_y(x,y)$, independent of s, and not $u_y(\xi(s),\eta(s))$. If we evaluate (2.12) at $s=0$, however, there is a cancellation, and we obtain the useful relation $\dot\eta(0)(1+|Du|^2)=0$ according to which $\dot\eta(0)=0$. It follows from (2.11) that $\dot\xi(0)=\pm 1/\sqrt{1+u_x^2}$. With a choice according to which $\dot\gamma(0)=\dot\xi(0)X_u=\mathbf{v}$, we have

$$\dot\xi(0)=\frac{1}{\sqrt{1+u_x^2}}.$$

As mentioned above, we would like to compute the curvature of the intersection curve—the signed curvature as a plane curve (graph) in Π_α with respect to N. This value is given by

$$k_\alpha = \ddot\gamma\cdot N = \ddot\gamma(0)\cdot N_X.$$

We find

$$\ddot\gamma = \ddot\xi(1,0,u_x)+\ddot\eta(0,1,u_y)+(0,0,\dot\xi^2u_{xx}+2\dot\xi\dot\eta u_{xy}+\dot\eta^2u_{yy}).$$

Evaluating at $s=0$, this becomes

$$\ddot\gamma(0) = \ddot\xi X_x+\ddot\eta X_y+(0,0,\dot\xi^2u_{xx}+2\dot\xi\dot\eta u_{xy}+\dot\eta^2u_{yy}).$$

Since X_x and X_y are tangent vectors to $\mathcal{G}$, both orthogonal to N at the point $X\in\mathcal{G}$, the dot product is given by

$$k_\alpha = \frac{\dot\xi^2u_{xx}+2\dot\xi\dot\eta u_{xy}+\dot\eta^2u_{yy}}{\sqrt{1+|Du|^2}} = \frac{u_{xx}}{(1+u_x^2)\sqrt{1+|Du|^2}}. \tag{2.13}$$

Let us now pause to think carefully (as carefully as we can) about this value. In particular, let us attempt to compare this value to what we know about the curvature of planar graphs. If $N=(0,0,1)$, that is, if the tangent plane $T_X\mathcal{G}$ is horizontal with $u_x=u_y=0$, then $k_\alpha=u_{xx}$ as we would expect. Now, if u_x is nonzero but $u_y=0$, then $N=(-u_x,0,1)/\sqrt{1+u_x^2}$, and

$$k_\alpha = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \tag{2.14}$$

and this matches precisely what we would expect for a planar graph according to the familiar formula

$$k = \frac{u''}{(1+u'^2)^{3/2}}$$

for the signed curvature. In this case, for the surface, the vector $\mathbf{w}$ is horizontal. In fact according to our formula for $\mathbf{w}$ in this case we will have $\mathbf{w} = (0, 1, 0)$. The normal plane is vertical and parallel to the x, z-plane, and the intersection curve is given by $\alpha(\xi) = (\xi, y, u(\xi, y))$. This is all as it should be: a second derivative reduced/scaled by the reciprocal of the cube of the length scaling factor.

The interesting, geometrically new, phenomenon here is how the curvature of the intersection curve changes with the tilt in the other (y) coordinate direction. First of all, when $u_y \neq 0$ the X_x normal plane is not vertical and parallel to the x, z-coordinate plane. The normal curvature, however, is still (just) a scaling of u_{xx}.

Exercise 2.54 *Perhaps the simplest situation in which the phenomenon captured in (2.14) is operative and evident is when the graph $\mathcal{G}$ is the graph of a circular cylinder. Start with the cylinder $x^2 + z^2 = r^2$, and then express half of this cylinder as a graph $\mathcal{G}$, and tilt $\mathcal{G}$ using a rotation*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Explain why it makes sense that the curvature of the tilted cylinder in the normal planes determined by X_x at each point are given by (2.14). Does it make any difference for, say the bottom half of the cylinder if one considers

$$u(x, y) = -\sqrt{r^2 - x^2} - \tan\theta \; y?$$

Notice the vectors $\mathbf{w}$ and N also determine a unique plane

$$\Pi_\beta = \{(\xi, \eta, \zeta) : [(\xi, \eta, \zeta) - (x, y, u)] \cdot (1, 0, u_x) = 0\}$$

passing through $X = (x, y, u(x, y)) \in \mathcal{G}$ and orthogonal to $\mathbf{v}$, i.e. containing $\mathbf{w}$ and N. The intersection $\Pi_\beta \cap \mathcal{G}$ is also a planar curve that can be parameterized by arclength with

$$\gamma(s) = (\xi, \eta, u(\xi, \eta))$$

as above. Several of the computations above apply, but differentiating the defining relation $(\gamma - X) \cdot X_x = 0$, we find

$$\dot{\gamma} \cdot X_x = (\dot{\xi}, \dot{\eta}, \dot{\xi} u_x + \dot{\eta} u_y) \cdot (1, 0, u_x) = 0$$

where, as above, $u_x = u_x(\xi, \eta)$ and $u_y = u_y(\xi, \eta)$ depend on s in the first vector, but $u_x = u_x(x, y)$ in the second tangent vector is independent of s. Evaluating at $s = 0$ this time, we obtain

$$(1 + u_x^2)\dot{\xi}(0) + u_x u_y \dot{\eta}(0) = 0.$$

It follows that for some nonzero constant c we must have

$$\dot{\xi}(0) = -c u_x u_y \quad \text{and} \quad \dot{\eta}(0) = c(1 + u_x^2).$$

From the condition $|\dot{\gamma}| = 1$ and the choice $\dot{\eta}(0) > 0$, we find after some simplification that

$$c = \frac{1}{\sqrt{1 + |Du|^2}\,\sqrt{1 + u_x^2}}$$

so that

$$\dot{\xi}(0) = -\frac{u_x u_y}{\sqrt{1 + |Du|^2}\,\sqrt{1 + u_x^2}} \quad \text{and} \quad \dot{\eta}(0) = \frac{\sqrt{1 + u_x^2}}{\sqrt{1 + |Du|^2}}.$$

Substituting these values in the expression for $\ddot{\gamma}(0) \cdot N$ from above, we have the signed curvature of this intersection curve with respect to the normal N at the point X satisfies

$$k_\beta = \frac{\dot{\xi}^2 u_{xx} + 2\dot{\xi}\dot{\eta} u_{xy} + \dot{\eta}^2 u_{yy}}{\sqrt{1 + |Du|^2}}$$

$$= \frac{1}{(1 + |Du|^2)^{3/2}} \left(\frac{u_x^2 u_y^2}{1 + u_x^2}\, u_{xx} - 2 u_x u_y\, u_{xy} + (1 + u_x^2)\, u_{yy} \right).$$

It is quite easy to see from this expression that

$$k_\alpha + k_\beta = \mathcal{M} u$$

is the quantity we have called the mean curvature of the graph.

We have shown

> *The* **mean curvature** *given by the expression $\mathcal{M}u$ is the sum of the curvatures of two orthogonal planar curves lying on the graph $\mathcal{G}$ of u, each taken as a signed curvature with respect to the surface normal N which also lies in the (normal) plane containing each curve.*

The derivation above leaves open the possibility that one of the two normal planes determining one of the intersections curves must be the plane $\Pi(\mathbf{v})$ determined by the special tangent vector $\mathbf{v} = X_x/|X_x|$. Thus, we can ask: Is this quantity $\mathcal{M}u$ something of fundamental geometric meaning as curvature, or is it somehow dependent on the particular coordinates we have used, and consequently, the first normal plane we have chosen?

Perhaps the derivation suggests, however, a more general construction:

> *Let $\mathbf{v}$ be any unit length vector in $T_X\mathcal{G}$, and let $\mathbf{w} = N \times \mathbf{v}$. Let k_α be the signed curvature of the intersection of the normal plane $\Pi(\mathbf{v})$ orthogonal to $\mathbf{w}$ with respect to N, and let k_β be the signed curvature of the intersection of the normal plane $\Pi(\mathbf{w})$ orthogonal to $\mathbf{v}$ with respect to N. Is the number*
>
> $$k_\alpha + k_\beta$$
>
> *always equal to $\mathcal{M}u$?*

In fact, the suggested construction is correct:

> *If Π_α and Π_β are* **any pair of orthogonal planes intersecting along the normal line** *to a C^2 surface $\mathcal{S}$ at a point $X \in \mathcal{S}$, then each of the two planes intersects $\mathcal{S}$ locally in a planar curve. The two resulting planar curves have some signed curvatures k_α and k_β with respect to a choice of normal N, and the average of these two numbers is called the* **mean curvature** *$\mathcal{H}$ of the surface. The mean curvature is independent of the choice of orthogonal planes and depends only on the surface $\mathcal{S}$ and the (unit) normal N (chosen among two possibilities). According to this construction*
>
> $$\mathcal{H} = \frac{k_\alpha + k_\beta}{2} \quad \text{and} \quad \mathcal{M} = 2\mathcal{H}. \tag{2.15}$$

The last expression relating the mean curvature operator $\mathcal{M}$ and the value of the mean curvature assumes the surface $\mathcal{S}$ is given by the graph of a function.

In fact, every **surface** (a concept we have not actually defined carefully but which one can hope[3] is a relatively intuitively clear concept) can be expressed as a union of graphs of functions, so in particular, coordinates ξ and η can be chosen so that all points in the surface $\mathcal{S}$ near a given point $X \in \mathcal{S}$ are congruent to a graph

$$\mathcal{G} = \{(\xi, \eta, u(\xi, \eta)) : (\xi, \eta) \in \mathcal{U}\}$$

for some open set $\mathcal{U} \subset \mathbb{R}^2$. According to the above assertion, it does not matter which graph is chosen to locally represent $\mathcal{S}$.

There are various ways to see the mean curvature $\mathcal{H}$ is a geometric quantity as described above. The following is one way:

Say we take a different direction $\mathbf{v} \in T_X\mathcal{G}$ and an orthogonal direction $\mathbf{w} = N \times \mathbf{v} \in T_X\mathcal{G}$. Rather than try to generalize the computation above for $\mathbf{v} = X_x \| X_x|$ directly, note that this new tangent vector $\mathbf{v} = (v_1, v_2, v_3)$ must have some nonzero projection into the x, y-plane, namely $\mathbf{u}_1 = (v_1, v_2)/\sqrt{v_1^2 + v_2^2}$.

Exercise 2.55 *Explain how we know* $(v_1, v_2) \neq \mathbf{0} \in \mathbb{R}^2$.

We can represent $\mathcal{G}$ as a graph in new coordinates as follows: We first write $\mathbf{u}_1 = (\cos\theta, \sin\theta)$ determining the angle θ uniquely in the interval $[0, 2\pi)$. We then consider the function $\tilde{u} : \tilde{\mathcal{U}} \to \mathbb{R}$ by

$$\tilde{u}(\xi, \eta) = u(x + \xi\cos\theta - \eta\sin\theta, y + \xi\sin\theta + \eta\cos\theta)$$

on an appropriate domain $\tilde{\mathcal{U}} \subset \mathbb{R}^2$.

Exercise 2.56 *Find the "appropriate" domain* $\tilde{\mathcal{U}}$ *in terms of the domain* $\mathcal{U} \subset \mathbb{R}^2$ *for* $u \in C^2(\mathcal{U})$*, and show there exists a rigid motion* $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ *(translation and rotation) such that*

$$\tilde{\mathcal{G}} = \{(\xi, \eta, \tilde{u}(\xi, \eta)) : (\xi, \eta) \in \tilde{\mathcal{U}}\}$$

is the (congruent) image of $\mathcal{G}$ *under* ρ*, that is*

$$\tilde{\mathcal{G}} = \{\rho(X) : X \in \mathcal{G}\}$$

[3]If you do not know the technical definition of a surface and (for some reason) are not interested in looking it up and understanding it at the moment, then you might write down any example you can imagine being a surface and see if you can express every small enough piece of that surface as the graph of a function. For example, one might start with $\partial B_r(\mathbf{0}) = \{(x, y, z) : x^2 + y^2 + z^2 = r^2\}$. Perhaps we will also remedy this deficiency soon.

with $(1, 0, \tilde{u}_\xi) = \rho(\mathbf{v})$*. Thus, the sum of the normal curvatures associated with* $\mathbf{v}$ *and* $\mathbf{w}$ *is the same as calculating* $\mathcal{M}\tilde{u}(0,0)$*, which we know to be*

$$\mathcal{M}\tilde{u} = \frac{(1+\tilde{u}_\eta^2)\tilde{u}_{\xi\xi} - 2\tilde{u}_\xi\tilde{u}_\eta\tilde{u}_{\xi\eta} + (1+\tilde{u}_\xi^2)\tilde{u}_{\eta\eta}}{(1+|D\tilde{u}|^2)^{3/2}}.$$

In view of the above construction/exercise we compute $\mathcal{M}\tilde{u}$:

$$\tilde{u}_\xi = u_x\cos\theta + u_y\sin\theta \quad \text{and} \quad \tilde{u}_\eta = -u_x\sin\theta + u_y\cos\theta.$$

The denominator $(1+|D\tilde{u}|^2)^{3/2}$ in $\mathcal{M}\tilde{u}$ is easily calculated at this point and found to be $(1+|Du|^2)^{3/2}$, which is promising. The coefficients involving first order terms are more complicated, but straightforward to compute:

$$\begin{aligned} 1+\tilde{u}_\xi^2 &= 1 + u_x^2\cos^2\theta + 2u_xu_y\cos\theta\sin\theta + u_y^2\sin^2\theta \\ \tilde{u}_\xi\tilde{u}_\eta &= -u_x^2\cos\theta\sin\theta + u_xu_y(\cos^2\theta - \sin^2\theta) + u_y^2\cos\theta\sin\theta \\ 1+\tilde{u}_\eta^2 &= 1 + u_x^2\sin^2\theta - 2u_xu_y\cos\theta\sin\theta + u_y^2\cos^2\theta. \end{aligned}$$

Finally, for the second order derivatives we have

$$\begin{aligned} \tilde{u}_{\xi\xi} &= u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta \\ \tilde{u}_{\xi\eta} &= -u_{xx}\cos\theta\sin\theta + u_{xy}(\cos^2\theta - \sin^2\theta) + u_{yy}\cos\theta\sin\theta \\ \tilde{u}_{\eta\eta} &= u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta + u_{yy}\cos^2\theta. \end{aligned}$$

Algebraically, the calculation of the expression

$$(1+\tilde{u}_\eta^2)\ \tilde{u}_{\xi\xi} - 2\tilde{u}_\xi\tilde{u}_\eta\ \tilde{u}_{\xi\eta} + (1+\tilde{u}_\xi^2)\ \tilde{u}_{\eta\eta}$$

becomes somewhat long and cumbersome. With this in mind, we compute the products giving the coefficients of the second order terms one by one. The coefficient of u_{xx} is the sum of three terms: The first is from the product $(1+\tilde{u}_\eta^2)\tilde{u}_{\xi\xi}$ and is given by

$$\cos^2\theta(1 + u_x^2\sin^2\theta - 2u_xu_y\cos\theta\sin\theta + u_y^2\cos^2\theta). \tag{2.16}$$

The second is from $-2\tilde{u}_\xi\tilde{u}_\eta\tilde{u}_{\xi\eta}$ and is

$$2\cos\theta\sin\theta(-u_x^2\cos\theta\sin\theta + u_xu_y(\cos^2\theta - \sin^2\theta) + u_y^2\cos\theta\sin\theta). \tag{2.17}$$

The third comes from $(1+\tilde{u}_\xi^2)\tilde{u}_{\eta\eta}$:

$$\sin^2\theta(1 + u_x^2\cos^2\theta + 2u_xu_y\cos\theta\sin\theta + u_y^2\sin^2\theta). \tag{2.18}$$

It is not difficult to see that the sum of (2.16), (2.17), and (2.18) simplifies to

$$1 + u_y^2.$$

Thus, we have established

$$\mathcal{M}\tilde{u} = \frac{1}{(1 + |Du|^2)^{3/2}} \left[(1 + u_y^2)\ u_{xx} + \cdots \right].$$

The coefficient of u_{xy} is similarly the sum of three terms:

$$\begin{aligned}
&2\cos\theta\sin\theta(1 + u_x^2\sin^2\theta - 2u_xu_y\cos\theta\sin\theta + u_y^2\cos^2\theta) \\
&\quad - 2(\cos^2\theta - \sin^2\theta)(-u_x^2\cos\theta\sin\theta + u_xu_y(\cos^2\theta - \sin^2\theta) + u_y^2\cos\theta\sin\theta) \\
&\qquad - 2\cos\theta\sin\theta(1 + u_x^2\cos^2\theta + 2u_xu_y\cos\theta\sin\theta + u_y^2\sin^2\theta).
\end{aligned}$$

This sum, as might be expected, simplifies to $2u_xu_y$. Finally, the coefficient of u_{yy} is

$$\begin{aligned}
&\sin^2\theta(1 + u_x^2\sin^2\theta - 2u_xu_y\cos\theta\sin\theta + u_y^2\cos^2\theta) \\
&\quad - 2\cos\theta\sin\theta(-u_x^2\cos\theta\sin\theta + u_xu_y(\cos^2\theta - \sin^2\theta) + u_y^2\cos\theta\sin\theta) \\
&\qquad + \cos^2\theta(1 + u_x^2\cos^2\theta + 2u_xu_y\cos\theta\sin\theta + u_y^2\sin^2\theta) \\
&= 1 + u_x^2.
\end{aligned}$$

We have shown $\mathcal{M}\tilde{u}(0,0) = \mathcal{M}u(x,y)$ is independent of the choice of orthogonal vectors $\mathbf{v}$ and $\mathbf{w}$ in $T_X\mathcal{G}$.

Geometric meaning and curvature

Let us return to the discussion surrounding (2.13) and attempt to think about this expression a little more carefully and filling out a little more the connection with the curvature of curves and the broader idea of what it means for a quantity to be **geometric**.

If $u : (a,b) \to \mathbb{R}$ has $u \in C^2(a,b)$ and graph a curve $\{(x, u(x)) : x \in (a,b)\}$, then the values of u are made geometric by the consideration of the graph.

When you think of the distance $\xi = \xi(t)$ traveled by someone walking, or someone in a car, or a baseball, or a rocket, then that distance alone (as a function of time) is analytic or physical but not necessarily geometric. It becomes geometric when we plot the curve $\{(t, \xi(t)) \in \mathbb{R}^2 : t \in (a,b)\}$

which is the graph of the distance as a function of time. Once we have this graph, then the value $\xi'(t)$ may be thought of as geometric: The **slope** of the tangent line to the graph. Without the graph, the **rate of change** $\xi'(t)$ of the distance with respect to time is merely analytic or physical.

We are quite accustomed to identify the physical/analytic meaning with the geometric meaning in this instance, and forget there is a difference. The point of this discussion is that the situation changes with the second derivative $u''(x)$ of $u \in C^2(a, b)$. This quantity, **the second derivative** by itself, **has no geometric meaning**.

Though this startling declaration may be obvious, it also may be quite subtle for some people, so I will elaborate. Geometric meaning in relation to the function $u \in C^2(a, b)$ is associated, and only associated, with the graph of the function u, which is a curve. That curve, as a geometric object, may have a relation to a fixed direction, like a direction specified as horizontal, given by a quantity like slope or inclination. If we know which direction is horizontal, and we know $u'(x)$ with respect to this horizontal direction (measured by the quantity x), then we know something about the geometry of the curve—the inclination of the curve at the point $(x, u(x))$ on the curve—just as $u(x)$ tells us something about the orthogonal distance from $(x, u(x))$ to the horizontal. If we know $u''(x)$, however, this tells us (almost) nothing about the geometry of the graph and, since that is the only geometry we have, nothing geometric (period).

As an illustration, consider the specific function $\xi(t) = t^2$. In Figure 2.3 I have plotted the graph of the function ξ along with three small disks focusing on three different portions of the graph and having centers on specific points $(t, \xi(t))$ on the graph. Now, if I were to tell you a particular point $(t, \xi(t))$ at the center of one of these disks is a point where $\xi'' = 2$, could you look at the three disks, and determine the point to which I was referring (from the geometry)? In fact, you cannot determine anything geometric from the information $\xi''(t) = 2$. If you know time is measured in seconds and ξ is measured in meters, you can tell something physical: The rate of change of ξ' with respect to time (or the **acceleration**) at this point (and every point) is 2 meters per second. More generally, you can tell something analytic, namely that the rate of change of ξ' with respect to the quantity t (whatever the appropriate units may happen to be) is 2. This kind of information can be useful both computationally (analytically) and physically, but it is not geometric.

Geometric information comes from the graph and the value of the second

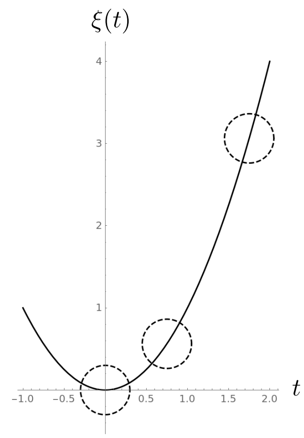

Figure 2.3: The graph of a "position" function $\xi = \xi(t) = t^2$.

derivative is not simply related to the geometry of the graph. As we know, one quantity

$$k = \frac{u''}{(1 + u'^2)^{3/2}},$$

a combination of the first and second derivatives called curvature, does give precise quantitative geometric information. We may qualify our comments by pointing out that $u''(x)$ does give some **qualitative geometric information**: If $u''(x)$ is positive, we can infer the geometric convexity of the graph of u with respect to the horizontal. It is the quantitative measure of that convexity we cannot discern without the curvature. And, as we know, the formula says that if the inclination of the graph is zero (i.e., has slope zero) at a point, then $u''(x)$ gives the curvature, but if the inclination is nonzero, then the number $u''(x)$ is strictly larger in absolute value than the curvature and must be diminished by a factor

$$\frac{1}{(1 + u'^2)^{3/2}} < 1.$$

This is how curvature works in relation to a second derivative. Why that particular scaling factor is the correct one is difficult to see geometrically, but that is what comes out from the computation using the chain rule. Maybe you can find a nice geometric interpretation for the scaling factor.

Exercise 2.57 *Consider the vectors* $u''(-u', 1)/\sqrt{1 + u'^2}$ *and* $u''(1, u')/\sqrt{1 + u'^2}$

normal and tangent to the graph of $u \in C^2(a,b)$. Can you express the curvature vector $\vec{k} = u''(-u',1)/(1+u'^2)^2$ geometrically in terms of one (or both) of these vectors?

The expression

$$k_\alpha = \frac{u_{xx}}{(1+u_x^2)\sqrt{1+|Du|^2}}$$

given in (2.13) is telling us something new (and geometric) about the curvature of curves on a surface—and indirectly about the curvature of a surface. Let a surface $\mathcal{S}$ be given locally as a graph

$$\mathcal{G} = \{(x, y, u(x,y)) : (x,y) \in \mathcal{U}\}$$

where $\mathcal{U}$ is an open subset of $\mathbb{R}^2$ as usual.

> *If we intersect the surface $\mathcal{S}$ with a vertical plane, say a plane parallel to the x,z-plane, then the signed curvature of the intersection curve at a particular point is*
>
> $$\frac{u_{xx}}{(1+u_x^2)^{3/2}}, \tag{2.19}$$
>
> *and this value is* **always larger** *in absolute value than (or possibly equal to) the normal curvature determined by the tangent vector $(1, 0, u_x)$ at that point. If the* **slope in the orthogonal coordinate direction** *at the point, as measured by u_y, is* **zero**, *then the value (2.19) is the normal curvature. If, however, $u_y \neq 0$, then the value given in (2.19) must be diminished by a factor*
>
> $$\frac{\sqrt{1+u_x^2}}{\sqrt{1+|Du|^2}} < 1. \tag{2.20}$$

It will be noted that the factor in (2.20) is the ratio of the length scaling factor for the coordinate intersection

$$\{(\xi, y, u(\xi, y)) : (\xi, y) \in \mathcal{U}\}$$

to the area scaling factor for the surface. This is how the curvature of curves given by the intersection with normal planes works on a surface.

Exercise 2.58 *Find the* **projection** *of the curvature vector*

$$\vec{k}_x = \frac{u_{xx}}{(1+u_x^2)^2}(-u_x, 0, 1)$$

of the intersection of the vertical plane

$$\Pi_x = \{(\xi, y, \zeta) : (\xi, \zeta) \in \mathbb{R}^2\}$$

with the graph

$$\mathcal{G} = \{(x, y, u(x, y)) : (x, y) \in \mathcal{U}\}$$

onto the normal $N = (-u_x, -u_y, 1)/\sqrt{1+|Du|^2}$ *of the surface.*

In 1776 Jeen Baptiste Marie Charles Meusnier de la Place discovered[4] a remarkable generalization of the construction we have given concerning k_α. The result is purely geometric and captures precisely what is happening.

Theorem 10 (Meusnier's theorem) *Let* $\mathcal{S}$ *be a surface containing a point* $X \in \mathcal{S}$ *and having a unit tangent vector* $\mathbf{v} \in T_X\mathcal{S}$ *at* X. *If* $\gamma : (-\epsilon, \epsilon) \to \mathcal{S}$ *is a parameterization by arclength of* **any curve** *on the surface* $\mathcal{S}$ *with* $\gamma(0) = X \in \mathcal{S}$, $\dot\gamma(0) = \mathbf{v}$, *and well-defined curvature vector* $\ddot\gamma(0)$, *then the number*

$$k_N = \ddot\gamma(0) \cdot N,$$

where N *is a choice of unit normal to* $\mathcal{S}$ *at* X, *is independent of the curve* γ. *This is called the* **normal curvature** *of the surface* $\mathcal{S}$ *at the point* X *and depends only on*

1. *the surface* $\mathcal{S}$,
2. *the tangent direction* $\mathbf{v}$, *and*
3. *the choice of unit normal to* $\mathcal{S}$ *(up to a sign).*

Additional Exercises

Exercise 2.59 *Compare the graphs of the functions* $u : \mathbb{R} \to \mathbb{R}$ *by* $u(x) = x^2$ *and* $v : [x_0 - r, x_0 + r] \to \mathbb{R}$ *by*

$$v(x) = y_0 - \sqrt{r^2 - (x - x_0)^2}.$$

[4]The French name Meusnier is pronounced like "moon yay."

(a) *Discuss the regularity of each function.*

(b) *Compute the curvature of the graph of each function.*

(c) *Given a point $(x, u(x))$ on the graph of u, find a center (x_0, y_0) and radius r so that the graph of v "matches the graph of u to second order" at the point $(x, u(x))$. Show the center and radius you have found are unique.*

(d) *Use numerical software to plot the graph of u and some osculating circles determined by the graph of u.*

2.60 Integration

We wish to discuss the integration of real valued functions on (somewhat) general sets. The basic setup is this: You have a function

$$f : X \to \mathbb{R}$$

where X is a metric space with a **measure**. As we know, the metric on a metric space (or distance function) allows one to measure distances between points and diameters of sets with $\text{diam} : \mathcal{P}(X) \to [0, \infty]$ by

$$\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}.$$

Here we have used $\mathcal{P}(X)$ to denote the collection of all subsets of X. This particular set is also called the **power set** of X and is sometimes denoted 2^X.

A **measure** μ is usually different from the diameter associated with a metric, though these functions can agree, for example on intervals in $\mathbb{R}$ where the measure of an integral (and the diameter of an interval) is its length. Ideally, we can also measure all sets with $\mu : \mathcal{P}(X) \to [0, \infty]$, and little harm is done (usually) if we imagine that to be the case. Technically, it is sometimes only possible to define a measure on some proper subset $\mathcal{M}$ of $\mathcal{P}(X)$ called the collection of **measurable sets**. Measurable sets should have the following properties:

1. $\phi, X \in \mathcal{M}$.

2. If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$.

3. If $A_1, A_2, A_3, \ldots$ comprise a (countable) sequence of sets in $\mathcal{M}$, then the union should also be measurable:

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}.$$

The measure $\mu : \mathcal{M} \to [0, \infty]$ should have the following properties:

1. $\mu\phi = 0$.

2. If $A_1, A_2, A_3, \ldots$ comprise a (countable) sequence of **pairwise disjoint** sets in $\mathcal{M}$, then

$$\mu \bigcup_{j=1}^{\infty} A_j = \sum_{j=1}^{\infty} \mu A_j.$$

The second property is called **countable additivity**.

Given a set X (with a metric and a measure) you can think of an integral as a limit

$$\int_X f = \lim_{\|\mathcal{P}\| \to 0} \sum_j f(x_j^*) \mu A_j$$

where $\mathcal{P} = \{A_j\}_{j=1}^{\infty}$ is a (finite) **partition** of X, that is

$$X = \bigcup_{j=1}^{k} A_j \qquad \text{and} \qquad \mu(A_i \cap A_j) = 0 \text{ for } i \neq j,$$

and $\|\mathcal{P}\| = \max_j \operatorname{diam}(A_j)$.

Two important possibilities (for integration) are the following:

1. $X = \mathcal{U}$ is an open subset of $\mathbb{R}^n$ with $\mu = \mathfrak{L}^n$ given by n-dimensional volume measure (or Lebesgue measure) and d the Euclidean metric. We call this **integration on flat space**.

2. $X = \partial\mathcal{U}$ is the smooth boundary of an open subset of $\mathbb{R}^n$ with $\mu = \mathcal{H}^{n-1}$ given by $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$ (and d again the Euclidean metric). These are examples of **integration on manifolds**.

If you do not know what it means for the boundary of an open subset of $\mathbb{R}^n$ to be "smooth," (or what it means to be a "manifold") do not worry. We can give precise definitions later. You can just think of the boundary of the disk $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ in the case $n = 2$ where you should have a pretty good idea of how one-dimensional Hausdorff measure $\mathcal{H}^1$ should work. You can also think of the boundary of the ball $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ in the case $n = 3$ on which one would use two-dimensional Hausdorff measure, that is area measure for surfaces in $\mathbb{R}^3$.

In practice, computation of an integral on a higher dimensional flat space is often reduced to the computation of **iterated integrals** on lower dimensional spaces by some form of **Fubini's theorem**:

Theorem 11 *(Fubini) If $f : X_1 \times X_2 \to \mathbb{R}$ is defined on the product $X_1 \times X_2 = \{(x_1, x_2) : x_j \in X_j, j = 1, 2\}$ of measurable metric spaces X_1 and X_2, then*

$$\int_{X_1 \times X_2} f = \int_{X_1} \left(\int_{X_2} f \right) = \int_{X_2} \left(\int_{X_1} f \right)$$

where the integrand of $\int_{X_2} f$ is taken to mean the function $g : X_2 \to \mathbb{R}$ given by $g(x) = f(x_1, x)$ for each (fixed) $x_1 \in X_1$, and the integrand of $\int_{X_1} f$ is interpreted similarly.

Integrals over manifolds are usually computed using a **parameterization** and a **change of variables formula**. To describe such a computation, in general terms, we change notation slightly: Let $\mathcal{U} \subset \mathbb{R}^n$ be a flat domain of integration and

$$X : \mathcal{U} \to \mathbb{R}^k \text{ an injection onto its image } M = X(\mathcal{U}).$$

Here M is assumed to be a manifold (or a subset of a manifold) and the function X is a parameterization; this is almost the definition of a manifold. Then we seek a **change of variables formula** which looks like this:

$$\int_M f = \int_{\mathcal{U}} (f \circ X)\ \sigma.$$

In this formula:

1. $f : M \to \mathbb{R}$ is a real valued function on M, as expected,

2. $f \circ X : \mathcal{U} \to \mathbb{R}$ is the composition given by

$$f \circ X(\mathbf{p}) = f(X(\mathbf{p})),$$

and

3. σ is a **scaling factor** for the measures involved.

You can think of σ (roughly) according to the following description:

Using the measure μ on M, the measure of a set $A \subset M$ is

$$\mu A = \int_A 1 = \int_{X^{-1}(A)} \sigma \tag{2.21}$$

where $\sigma : \mathcal{U} \to \mathbb{R}$ is a (scaling) function allowing the computation of μA by integration on the corresponding (flat) preimage

$$X^{-1}(A) = \{\mathbf{p} \in \mathcal{U} : X(\mathbf{p}) \in A\}.$$

The area scaling relation (2.21) is required to hold for all (measurable) sets A in such a way that the value of σ can be recovered by taking a limit

$$\sigma(p) = \lim_{A \to \{p\}} \frac{\mu A}{\mu_{\mathcal{U}}[X^{-1}(A)]} \tag{2.22}$$

where $\mu_{\mathcal{U}}$ is the measure on $\mathcal{U}$ and the limit is taken as A tends to $\{p\}$ as a set. You may recognize (2.22) as defining a kind of derivative of the measure μ.

Exercise 2.61 *Consider the polar coordinates map $\Phi : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$. This function is a smooth bijection on any restriction to a half strip $(0, \infty) \times [\theta_0, \theta_0 + 2\pi)$. Let A be the image under Φ of a rectangle $R = [r_0, r_0 + \epsilon] \times [\theta_0, \theta_0 + \delta]$ for some $r_0 > 0$ and any θ_0. That is,*

$$A = \{\Phi(r, \theta) : r_0 \le r \le r_0 + \epsilon,\ \theta_0 \le \theta \le \theta_0 + \delta\}.$$

Compute the area of A and the limit

$$\lim_{\epsilon, \delta \to 0} \frac{\mathfrak{L}^2 A}{\mathfrak{L}^2 R}$$

where $\mathfrak{L}^2$ denotes area measure in the plane, i.e., 2-dimensional Lebesgue measure.

Let's try to illustrate the notions of integration just introduced using an example. Say we want to integrate on the surface $M = \mathcal{S}$ shown in Figure 2.4.

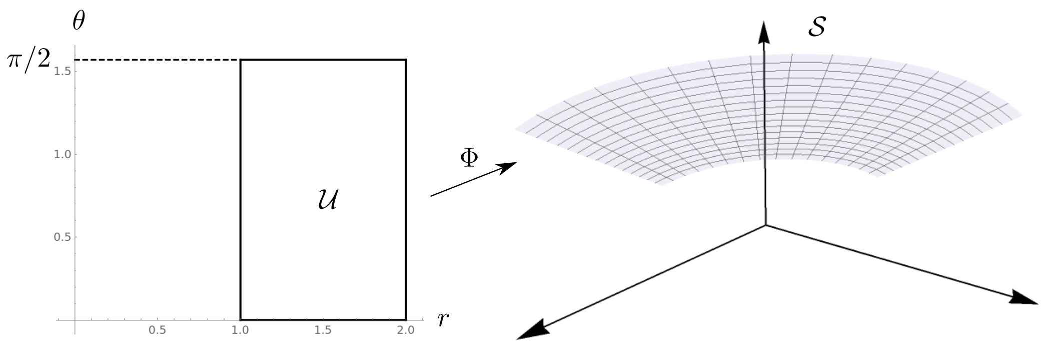

Figure 2.4: A parameterized surface and the associated scaling factor.

This surface is parameterized by

$$\Phi(r,\theta) = \frac{r}{4}\begin{pmatrix} (2+\sqrt{2})\cos\theta - (2-\sqrt{2})\sin\theta \\ (2+\sqrt{2})\sin\theta - (2-\sqrt{2})\cos\theta \\ 2(\cos\theta + \sin\theta) \end{pmatrix}$$

on the rectangle $\mathcal{U} = [1,2] \times [0,\pi/2]$. A small square $[r_0, r_0+\epsilon] \times [\theta_0, \theta_0+\epsilon]$ in the rectangle $\mathcal{U}$ has image approximated by the image of the linearization:

$$\Phi(r,\theta) \sim \Phi(r_0,\theta_0) + d\Phi_{(r_0,\theta_0)}(r-r_0, \theta-\theta_0).$$

The linear part $L = d\Phi_{(r_0,\theta_0)} : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$L\mathbf{v} = D\Phi(r_0,\theta_0)\,\mathbf{v}$$

where $D\Phi$ is the total derivative matrix, in this case

$$D\Phi = \begin{pmatrix} \dfrac{\partial \Phi_1}{\partial r} & \dfrac{\partial \Phi_1}{\partial \theta} \\ \dfrac{\partial \Phi_2}{\partial r} & \dfrac{\partial \Phi_2}{\partial \theta} \\ \dfrac{\partial \Phi_3}{\partial r} & \dfrac{\partial \Phi_3}{\partial \theta} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} (2+\sqrt{2})\cos\theta - (2-\sqrt{2})\sin\theta & -r[(2+\sqrt{2})\sin\theta + (2-\sqrt{2})\cos\theta \\ (2+\sqrt{2})\sin\theta - (2-\sqrt{2})\cos\theta & r[(2+\sqrt{2})\sin\theta + (2-\sqrt{2})\sin\theta] \\ 2(\cos\theta + \sin\theta) & 2r(\cos\theta - \sin\theta) \end{pmatrix}.$$

The image $L([0,\epsilon] \times [0,\epsilon])$ is a parallelogram spanned by the vectors

$$\mathbf{w}_1 = \frac{\epsilon}{4} \begin{pmatrix} (2+\sqrt{2})\cos\theta_0 - (2-\sqrt{2})\sin\theta_0 \\ (2+\sqrt{2})\sin\theta_0 - (2-\sqrt{2})\cos\theta_0 \\ 2(\cos\theta_0 + \sin\theta_0) \end{pmatrix}$$

and

$$\mathbf{w}_2 = \frac{\epsilon r_0}{4} \begin{pmatrix} -[(2+\sqrt{2})\sin\theta_0 + (2-\sqrt{2})\cos\theta_0] \\ (2+\sqrt{2})\sin\theta_0 + (2-\sqrt{2})\sin\theta_0 \\ 2(\cos\theta_0 - \sin\theta_0) \end{pmatrix}.$$

The area of this parallelogram is given by

$$|\mathbf{w}_1||\mathbf{w}_2|\sin A = |\mathbf{w}_1 \times \mathbf{w}_2|$$

where A is the angle between $\mathbf{w}_1$ and $\mathbf{w}_2$ with

$$\sin A = \frac{|\mathbf{w}_1 \times \mathbf{w}_2|}{|\mathbf{w}_1||\mathbf{w}_2|}.$$

Calculating, we find

$$|\mathbf{w}_1||\mathbf{w}_2|\sin A = |\mathbf{w}_1 \times \mathbf{w}_2| = \epsilon^2 r_0.$$

Thus, the linearization takes a square of area ϵ^2 (with corner at $(r_0, \theta_0) \in \mathcal{U}$ precisely onto a parallelogram of area $\epsilon^2 r_0$. Using this relation, we can decompose $\mathcal{U}$ into many small squares $\mathcal{U}_j$ (as indicated for example in Figure 2.5) with images $\Phi(\mathcal{U}_j)$ partitioning $\mathcal{S}$ and observe

$$\begin{aligned} \sum_j f(q_j^*)\, \mu_{\mathcal{S}}[\Phi(\mathcal{U}_j)] &\sim \sum_j f(q_j^*)\, r_j^*\, \mu_{\mathcal{U}}[\mathcal{U}_j] \\ &= \sum_j f(q_j^*)\, r_j^*\, \mathfrak{L}^2[\mathcal{U}_j] \\ &\sim \sum_j f \circ \Phi(p_j^*)\, r_j^*\, \mathfrak{L}^2[\mathcal{U}_j] \end{aligned} \tag{2.23}$$

where $p_j^* = (r_j^*, \theta_j^*) \in \mathcal{U}_j$ and we recall that $\mathfrak{L}^2$ denotes area measure, i.e., two-dimensional Lebesgue measure, in the plane. Taking the limit as the norms of our partitions tend to zero, we obtain the familiar change of variables formula

$$\int_{\mathcal{S}} f = \int_{\mathcal{U}} (f \circ \Phi)\, r.$$

Taking the special case $f \equiv 1$, we obtain (2.21) in the form

$$\mathcal{H}^2(A) = \int_{\Phi^{-1}(A)} r \qquad \text{for subsets } A \text{ of the surface } \mathcal{S}.$$

We may continue with this calculation using Fubini's theorem to write the flat integral on the right in terms of iterated integrals:

$$\int_{\mathcal{S}} f = \int_{r\in[1,2]} r \left(\int_{\theta\in[0,\pi/2]} f \left(\frac{r}{4} \begin{pmatrix} (2+\sqrt{2})\cos\theta - (2-\sqrt{2})\sin\theta \\ (2+\sqrt{2})\sin\theta - (2-\sqrt{2})\cos\theta \\ 2(\cos\theta + \sin\theta) \end{pmatrix} \right) \right)$$

on the rectangle $\mathcal{U} = [1,2] \times [0, \pi/2]$.

Assuming f is a continuous (Riemann integrable) function, we can also write $\int_{\mathcal{S}} f$ in terms of familiar Riemann integrals:

$$\int_{\mathcal{S}} f = \int_1^2 r \left(\int_0^{\pi/2} f \left(\frac{r}{4} \begin{pmatrix} (2+\sqrt{2})\cos\theta - (2-\sqrt{2})\sin\theta \\ (2+\sqrt{2})\sin\theta - (2-\sqrt{2})\cos\theta \\ 2(\cos\theta + \sin\theta) \end{pmatrix} \right) d\theta \right) dr.$$

Exercise 2.62 *Let $X : \mathcal{U} \to \mathbb{R}^k$ be a bijection onto the image manifold $M = X(\mathcal{U})$ where $\mathcal{U}$ is a (flat) domain of integration in $\mathbb{R}^n$.*

(a) *Write down a measure scaling limit relation like (2.22) involving the measure μ on M and the measure $\mathfrak{L}^n$ on $\mathcal{U}$.*

(b) *Carefully justify the change of variables formula*

$$\int_M f = \int_{\mathcal{U}} (f \circ X)\ \sigma$$

along the lines of (2.23).

Here are two change of variables formulas that cover many cases of interest:

Theorem 12 *If $\mathcal{U}$ and $\mathcal{W}$ are open sets of $\mathbb{R}^n$ and $\Phi : \mathcal{U} \to \mathcal{W}$ is a change of variables, i.e., a differentiable bijection (diffeomorphism), then*

$$\int_{\mathcal{W}} f = \int_{\mathcal{U}} f \circ \Phi\ |\det D\Phi|.$$

The total derivative $D\Phi$ is an $n \times n$ matrix, and the scaling factor is

$$\sigma = |\det D\Phi|.$$

Theorem 13 *If $\mathcal{U}$ is an open subset of $\mathbb{R}^n$ and $X : \mathcal{U} \to \mathbb{R}^k$ parameterizes a smooth manifold $M = X(\mathcal{U})$, then*

$$\int_M f = \int_{\mathcal{U}} f \circ X\ \sqrt{\det(DX^T\, DX)}$$

where DX is the $k \times n$ matrix which is the total derivative of X and DX^T is the transpose of DX. The matrix $DX^T\, DX$ is a $n \times n$, square, positive definite matrix with $\det(DX^T\, DX) > 0$. The scaling factor is

$$\sigma = \sqrt{\det(DX^T\, DX)}$$

Exercise 2.63 *Apply Theorem 13 to the parameterization*

$$\Phi(r,\theta) = \frac{r}{4}\begin{pmatrix} (2+\sqrt{2})\cos\theta - (2-\sqrt{2})\sin\theta \\ (2+\sqrt{2})\sin\theta - (2-\sqrt{2})\cos\theta \\ 2(\cos\theta + \sin\theta) \end{pmatrix}$$

to determine the scaling factor for integration on the surface $\mathcal{S}$ given above.

2.63.1 Special Integrands

The general theory of integration as presented above is not really complete in at least two respects. The major omission, perhaps, is that we have not discussed measures and the construction of specific measures in any detail. Closely related to this omission is the fact that we have not discussed conditions under which the limit in the Riemann style integral we have defined converges. It would also be natural to discuss alternatives to the Riemann style limit, but we will not include that discussion here. Hopefully it will be adequate for our purposes to know such questions can be addressed.

Say $\mathcal{U} \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ and $\partial\mathcal{U}$ is a smooth $(n-1)$-dimensional manifold upon which integration is possible and upon which there is a well-defined **outward unit normal field** N. Assume, furthermore, that $\mathbf{v} : \overline{\mathcal{U}} \to \mathbb{R}^n$ is a smooth vector field. Under these circumstances

$$\int_{\partial\mathcal{U}} \mathbf{v} \cdot N$$

is called the **outward flux integral** of $\mathbf{v}$ with respect to $\mathcal{U}$.

Exercise 2.64 *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and consider a "window"*

$$\underline{\mathcal{U}} = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \mathcal{U}\}.$$

Let $\mathbf{v} = (0, 0, v)$ be a smooth vertical field on $\mathbb{R}^3$ with units given by

$$[v] = \frac{\text{mass}}{\text{area time}}$$

where $[\,\cdot\,]$ denotes the units of a quantity. A field in $\mathbb{R}^3$ with these units is called a **mass flow field***. If we take $N = \mathbf{e}_3 = (0, 0, 1)$ what is the physical significance of*

$$\int_{\underline{\mathcal{U}}} \mathbf{v} \cdot N?$$

Note: This integral is also called a flux integral.

Exercise 2.65 *Let $\mathcal{U}$ be a rectangular "window"*

$$\underline{\mathcal{U}} = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [a, b] \times [c, d]\}$$

as described in the previous exercise. Let $\mathbf{v}$ *be a constant constant mass flow field on* $\mathbb{R}^3$ *with third component* $v_3 > 0$. *If we take* $N = \mathbf{e}_3 = (0,0,1)$, *explain the physical significance of*

$$\int_{\underline{\mathcal{U}}} \mathbf{v} \cdot N$$

and draw a picture of this quantity in relation to a picture (you've drawn) of the mass which has passed through $\underline{\mathcal{U}}$ *in one unit of time.*

Exercise 2.66 *If* $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ *denotes the identity (or outward radial) field on* $\mathbb{R}^3$ *given by* $\mathbf{v}(\mathbf{x}) = \mathbf{x}$, *compute the outward flux integral*

$$\int_{\partial B_r(\mathbf{0})} \mathbf{v} \cdot N$$

for $r > 0$.

The Divergence

Let us assume again that $\mathcal{U}$ is an open subset of $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{U}$ and outward unit normal field N. Also, we assume, as before that $\mathbf{v} : \overline{\mathcal{U}} \to \mathbb{R}^n$ is a smooth vector field. Taking a sequence of subdomains $\mathcal{V}$ converging as sets to a singleton $\{\mathbf{p}\}$ with $p \in \mathcal{U}$, we define $\operatorname{div} \mathbf{v} : \mathcal{U} \to \mathbb{R}$ by

$$\operatorname{div} \mathbf{v}(\mathbf{p}) = \lim_{\mathcal{V} \to \{\mathbf{p}\}} \frac{1}{\mathfrak{L}^n} \int_{\partial\mathcal{V}} \mathbf{v} \cdot N.$$

Theorem 14 *(divergence theorem)*

$$\int_{\mathcal{U}} \operatorname{div} \mathbf{v} = \int_{\partial\mathcal{U}} \mathbf{v} \cdot N.$$

Outline of the proof: Partition $\mathcal{U}$ into small pieces $\mathcal{U}_j$ as indicated (in the two-dimensional case) in Figure 2.5. Call the partition $\mathcal{P}$. Then

$$\int_{\partial\mathcal{U}} \mathbf{v} \cdot N = \sum_j \int_{\partial\mathcal{U}_j} \mathbf{v} \cdot N$$

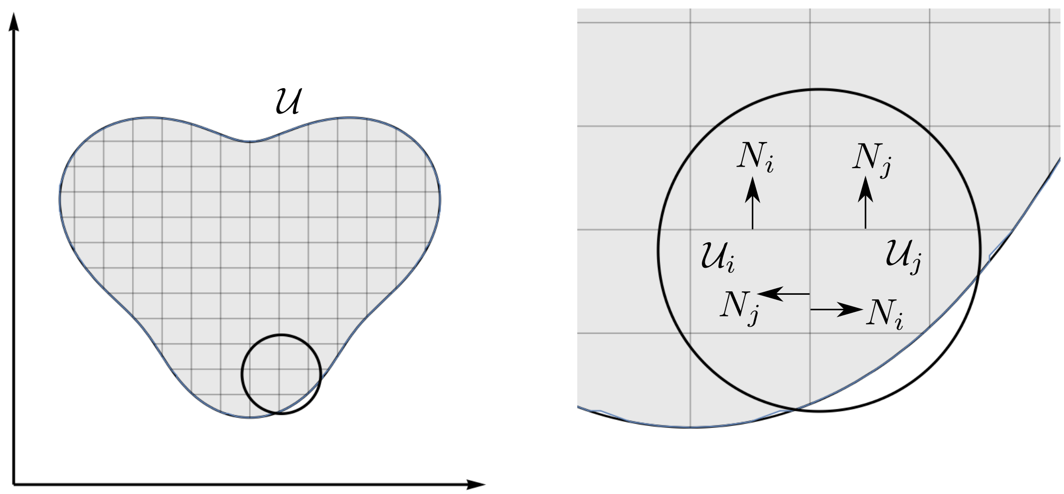

Figure 2.5: Proof of the divergence theorem in the plane; partitioning a region

where $N = N_j$ is the outward unit normal to $\mathcal{U}_j$ in the integrals on the right. Notice how the integrals over the intersections of adjacent pieces cancel one another. We can write this as

$$\int_{\partial\mathcal{U}} \mathbf{v}\cdot N = \sum_j \left(\frac{1}{\mu\mathcal{U}_j}\int_{\partial\mathcal{U}_j} \mathbf{v}\cdot N\right)\mu\mathcal{U}_j = \lim_{\|\mathcal{P}\|\to 0}\sum_j \left(\frac{1}{\mu\mathcal{U}_j}\int_{\partial\mathcal{U}_j} \mathbf{v}\cdot N\right)\mu\mathcal{U}_j.$$

The measure μ, in this case, is $\mathfrak{L}^n$. If $\|\mathcal{P}\|$ is small there is, for each j, some evaluation point $\mathbf{p}_j^* \in \mathcal{U}_j$ such that

$$\frac{1}{\mu\mathcal{U}_j}\int_{\partial\mathcal{U}_j} \mathbf{v}\cdot N \quad \text{is close to} \quad \operatorname{div}\mathbf{v}(\mathbf{p}_j^*).$$

Naturally, we need the differences of these quantities to be uniformly small in j. Given that we can write

$$\int_{\partial\mathcal{U}} \mathbf{v}\cdot N = \lim_{\|\mathcal{P}\|\to 0}\sum_j \left(\operatorname{div}\mathbf{v}(\mathbf{p}_k^*)\right)\mu\mathcal{U}_j = \int_{\mathcal{U}} \operatorname{div}\mathbf{v}. \qquad \square$$

Exercise 2.67 *Let $\mathcal{U} \subset \mathbb{R}^2$ be a set containing a rectangle*

$$R = \{\mathbf{p} + s(1,0) + t(0,1) : (s,t) \in [-\epsilon,\epsilon]\times[-\delta,\delta].$$

Assume $\mathbf{v} : \mathcal{U} \to \mathbb{R}^2$ is a C^1 vector field on $\mathcal{U}$.

(a) *Use the mean value theorem to express the flux integral*

$$\int_{\partial R} \mathbf{v} \cdot N$$

as a sum

$$2\delta \int_{-\epsilon}^{\epsilon} f_1(s)\, ds + 2\epsilon \int_{-\delta}^{\delta} f_2(t)\, dt$$

for appropriate functions f_1 and f_2.

(b) *Use your result to determine the value of*

$$\operatorname{div} \mathbf{v}(\mathbf{p}) = \lim_{\epsilon,\delta \to 0} \frac{1}{\mu R} \int_{\partial R} \mathbf{v} \cdot N$$

in rectangular coordinates.

The divergence theorem is the version of integration by parts we need to find/derive partial differential equations as Euler-Lagrange equations in the calculus of variations. Such partial differential equations are called **variational PDE**.

Derivatives and the Gradient

Our proof of the divergence theorem relies heavily on the convergence of the limit in the definition of the divergence, which we have not shown. In fact, some (not so restrictive) conditions should be satisfied by the vector field $\mathbf{v}$ and the regions $\mathcal{V} \subset \mathcal{U}$ with $\mathcal{V} \to \{\mathbf{p}\}$. We will show the limit exists for several different kinds of regions and for several different kinds of coordinates. It will be noted that the divergence is a differential expression (or a kind of derivative) though it was defined in terms of an integral expression/quantity. As a consequence, filling in the deficiency of showing the divergence exists will benefit from some preliminary discussion of derivatives.

The starting point for essentially all derivatives is the limit of the diffence quotient

$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \tag{2.24}$$

for a real valued function $u : (a, b) \to \mathbb{R}$ of one variable, when this limit exists. This quantity is interpreted (physically) as the (instantaneous) rate of change of the quantity measured by u with respect to the change in the independent variable x and (geometrically) as the slope of the tangent line to the graph of u.

Exercise 2.68 *Assume $x = x(t)$ measures distance (length) and t measures time.*

(a) *Use the formula*

$$\textit{average rate} = \frac{\textit{total net distance}}{\textit{total ellapsed time}}$$

to find and expression for the average rate of change of x over a finite time interval $[a, b]$.

(b) *Interpret your answer geometrically in terms of points on the graph of the function $x : [a, b] \to \mathbb{R}$.*

Given a function $u : \mathcal{U} \to \mathbb{R}$ of two or more variables defined on an open subset $\mathcal{U} \subset \mathbb{R}^n$, a natural generalization of (2.24) is the **directional derivative** given by

$$D_{\mathbf{v}}u(\mathbf{x}) = \lim_{h\to 0} \frac{u(\mathbf{x} + h\mathbf{v}) - u(\mathbf{x})}{h} \tag{2.25}$$

where $\mathbf{v}$ is a (tangent) vector at $\mathbf{x} \in \mathcal{U}$. There are some differences between this kind of difference quotient and (2.24) and several remarks are in order. First of all, it will be remarked that to specialize (2.25) to the one-dimensional case and obtain the same derivative, one must make the particular choice $\mathbf{v} = 1 \in \mathbb{R}$. Thus, our generalization is not only a generalization in dimension but also in the generality of the notion considered. Illustrating this latter generalization, here are two special cases of note:

1. If $\mathbf{v}$ is a unit vector, that is, $|\mathbf{v}| = 1$, then the value of $D_{\mathbf{v}}u(\mathbf{x})$ gives the instantaneous rate of change of the function u in the direction $\mathbf{v}$. Many authors restrict the definition (2.25) to only this case. In particular, in this case one may construct a "graph" over the line $\{\mathbf{x} + t\mathbf{v}\}$ given by taking some (small) $\epsilon > 0$ and considering

$$\mathcal{G} = \{(t, u(\mathbf{x} + t\mathbf{v})) : |t| < \epsilon\}.$$

The difference quotient (2.25) is then recognized as the slope of the secant line to the graph $\mathcal{G}$ determined by the points $(h, u(\mathbf{x} + h\mathbf{v}))$ and $(0, u(\mathbf{x}))$ as indicated in Figure 2.6.

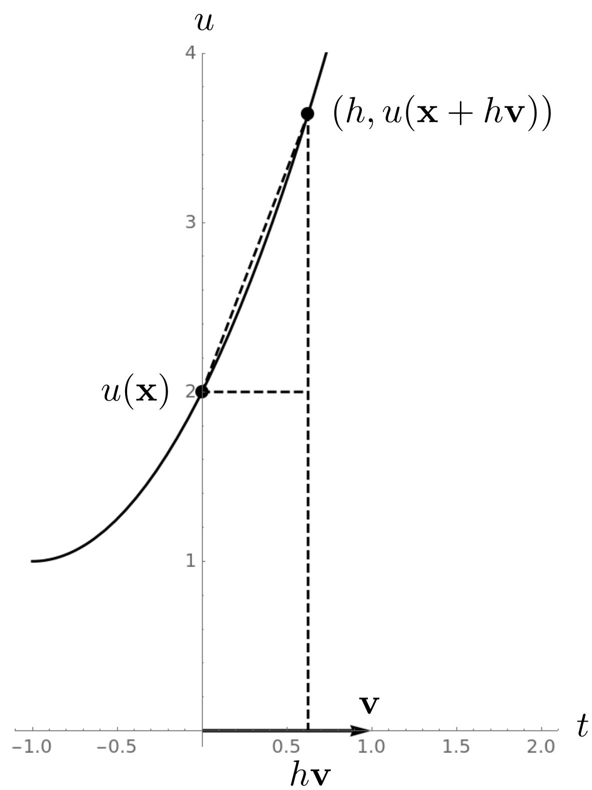

Figure 2.6: Difference quotient.

Exercise 2.69 *In two dimensions, when $\mathcal{U} \subset \mathbb{R}^2$, the graph*

$$\mathcal{G} = \{(x, y, u(x, y)) : (x, y) \in \mathcal{U}\}$$

of the function $u : \mathcal{U} \to \mathbb{R}$ is a surface, and the illustration of Figure 2.6 can be realized in a somewhat different form. Draw such an illustration and interpret the difference quotient in (2.25) in terms of your illustration.

2. If $\mathbf{v}$ is taken to be a **standard unit basis vector** $\mathbf{e}_j$, then the resulting directional derivative has a special name and notation. First of all, recall that the standard unit basis vector $\mathbf{e}_j$ is the vector in $\mathbb{R}^n$ with zeros in all entries except for the j-th entry, which is 1. The vector $\mathbf{e}_j$ is also called the standard coordinate vector (with respect to a choice

of rectangular coordinates). In this case we write

$$\frac{\partial u}{\partial x_j} = D_{\mathbf{e}_j} u$$

and call this quantity a **partial derivative**. The notations

$$D_{x_j} u, \qquad D_j u, \qquad u_{x_j}, \qquad \text{and} \qquad D^{\mathbf{e}_j} u \tag{2.26}$$

are also used to denote this same quantity. In two dimensions $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, and another usual notation is given by

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x} \qquad \text{and} \qquad \frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial y}.$$

Similarly, in three dimensions one finds

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x}, \qquad \frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial x_3} = \frac{\partial u}{\partial z}.$$

Each of the above notations, especially those in (2.26), should be considered carefully and compared to the meaning of this kind of derivative.

Exercise 2.70 *How does your illustration and explanation from Exercise 2.69 change in the case $\mathbf{v} = \mathbf{e}_j$ is a standard unit basis vector in $\mathbb{R}^2$?*

We have not followed other authors in restricting the directional derivative $D_{\mathbf{v}} u$ to unit vectors $\mathbf{v}$. As a consequence, $D_{\mathbf{v}} u(\mathbf{x})$ does not always give the instantaneous rate of change of the quantity u in the direction $\mathbf{v}$ at $\mathbf{x}$, and we need (perhaps) to be a little careful. First notice $\mathbf{v} = \mathbf{0}$ implies $D_{\mathbf{v}} u = 0$, and this quantity indicates nothing about the local behavior of u near $\mathbf{x}$. Nevertheless, we obtain a well-defined (zero) value in this case. If $\mathbf{v} \neq \mathbf{0}$, then

$$D_{\mathbf{v}} u(\mathbf{x}) = |\mathbf{v}| \lim_{h \to 0} \frac{u(\mathbf{x} + (h|\mathbf{v}|)(\mathbf{v}/|\mathbf{v}|)) - u(\mathbf{x})}{h|\mathbf{v}|} = |\mathbf{v}| D_{\mathbf{v}/|\mathbf{v}|} u$$

is the scaling of the rate of change of u in the unit direction $\mathbf{v}/|\mathbf{v}|$ by the factor $|\mathbf{v}|$. This can be recognized as a familiar form of the **chain rule** which says the derivative $(u \circ v)'(x)$ of a **composition** $u \circ v : (a, b) \to \mathbb{R}$ where

$v : (a, b) \to (c, d)$ and $u : (c, d) \to \mathbb{R}$ is the product of the instantaneous rate of change of u at $v(x)$ and the instantaneous rate of change of v at x:

$$(u \circ v)'(x) = u'(v(x))\ v'(x), \tag{2.27}$$

or (as it is often cryptically expressed)

$$\frac{du}{dx} = \frac{du}{dv}\ \frac{dv}{dx}.$$

In the multivariable case of $D_{\mathbf{v}}u$, the composition is one with u and the vector valued function $\alpha(t) = \mathbf{x} + t\mathbf{v}$ with velocity vector $\alpha' = \mathbf{v}$ is constant. To be explicit

$$\frac{d}{dt}u(\alpha(t))\Big|_{t=0} = D_{\mathbf{v}/|\mathbf{v}|}u(\alpha(0))\ |\alpha'(0)|. \tag{2.28}$$

Exercise 2.71 *Notice that in comparing the one-dimensional chain rule (2.27) with the chain rule we have derived/observed for directional derivatives (2.28) one contains a norm/absolute value which is conspicuously missing in the other.*

(a) *If one applies the definition of a directional derivative (2.25) to a function $u : (a, b) \to \mathbb{R}$ of one variable using only unit vectors $\mathbf{v}$, what is the difference between $D_{\mathbf{v}}u$ and u'? Put another way, how many points are there in the boundary of the one-ball $B_r(0) = \{x \in \mathbb{R} : |x| = 1\}$?*

(b) *Explain why there are no absolute values in (2.27).*

In the context of higher dimensional directional derivatives defined by (2.25) certain additional constructions (often overlooked in 1-D calculus) are of interest. It may first be noted that the expression $D_{\mathbf{v}}u(\mathbf{x})$ has several possible quantities which can be considered as "arguments." Perhaps the simplest way to think about this quantity is with $u : \mathcal{U} \to \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ fixed and argument $\mathbf{x} \in \mathcal{U}$. Thus, $D_{\mathbf{v}}u$ becomes a real valued function on $\mathcal{U}$. This naturally opens the door for repetition of the construction (directional differentiation) and consideration of **higher order** directional derivatives. Naturally, some regularity is required to compute derivatives as limits of difference quotients. The absolute value function is not differentiable at $x = 0$ in this sense, and we have already introduced the continuity/differentiability classes C^0, C^1, C^2, ... in one dimension. We turn to partial derivatives for the analogue in higher dimensions:

Definition 10 *A function* $u : A \to \mathbb{R}$ *defined on* **any** *subset* $A \subset \mathbb{R}^n$ *is* **continuous** *at* $\mathbf{x}_0 \in A$ *if for any* $\epsilon > 0$*, there is some* $\delta > 0$ *such that*

$$|u(\mathbf{x}) - u(\mathbf{x}_0)| < \epsilon \qquad \text{whenever } \mathbf{x} \in A \text{ and } |\mathbf{x} - \mathbf{x}_0| < \delta.$$

The function $u : A \to \mathbb{R}$ *is said to be* **continuous on** A *if* u *is continuous at each point* $\mathbf{x}_0 \in A$*.*

Exercise 2.72 *Show that if the domain of a function* $u : \mathcal{U} \to \mathbb{R}$ *is an open subset* $\mathcal{U} \subset \mathbb{R}^n$*, then the condtion* $\mathbf{x} \in \mathcal{U}$ *may be omitted from the definition. In other words, consider the alternative definition:* u *is continuous at* $\mathbf{x}_0 \in \mathcal{U}$ *if for any* $\epsilon > 0$*, there is some* $\delta > 0$ *such that*

$$|u(\mathbf{x}) - u(\mathbf{x}_0)| < \epsilon \qquad \text{whenever } |\mathbf{x} - \mathbf{x}_0| < \delta.$$

Show a function continuous according to this definition is continuous with respect to the "official" definition above.

Definition 11 *Given an open set* $\mathcal{U} \subset \mathbb{R}^n$*, the set* $C^1(\mathcal{U})$ *consists of the real valued functions* $u : \mathcal{U} \to \mathbb{R}$ *for which the partial derivatives satisfy*

$$\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \in C^0(\mathcal{U}).$$

Exercise 2.73 *Show that* $u \in C^1(\mathcal{U})$ *implies* $u \in C^0(\mathcal{U})$*.*

Definition 12 *Given an open subset* $\mathcal{U} \subset \mathbb{R}^n$ *and a natural number* $k \geq 2$*, the set* $C^k(\mathcal{U})$ *consists of the real valued functions* $u : \mathcal{U} \to \mathbb{R}$ *for which the partial derivatives satisfy*

$$\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \in C^{k-1}(\mathcal{U}).$$

Exercise 2.74 *If* $\mathcal{U}$ *is an open subset of* $\mathbb{R}^n$ *and* $u \in C^2(\mathcal{U})$*, then (show)*

$$D_{\mathbf{v}} D_{\mathbf{w}} u = D_{\mathbf{w}} D_{\mathbf{v}} u \qquad \text{for any vectors } \mathbf{v}, \mathbf{w} \in \mathbb{R}^n.$$

Definition 13 *If* A *is* **any** *subset of* $\mathbb{R}^n$ *and* k *is a natural number with* $k \geq 1$*, then* $C^k(A)$ *consists of those functions* $u : A \to \mathbb{R}$ *for which the following holds: There exists an open set* $\mathcal{U} \subset \mathbb{R}^n$ *and an extension* $\overline{u} \in C^k(\mathcal{U})$ *for which*

1. *$A \subset \mathcal{U}$, and*

2. *the restriction of $\overline{u}$ to A is u:*

$$\overline{u}\big|_A = u.$$

Exercise 2.75 *If $u \in C^0(A)$, then does there (necessarily) exist an extension $\overline{u} : \mathcal{U} \to \mathbb{R}$ with $\mathcal{U}$ some open subset satisfying*

1. *$A \subset \mathcal{U}$, and*

2. *the restriction of $\overline{u}$ to A is u:*

$$\overline{u}\big|_A = u,$$

and $u \in C^0(\mathcal{U})$?

Returning to the possible arguments of $D_\mathbf{v}u(\mathbf{x})$, in addition to $D_\mathbf{v}u : \mathcal{U} \to \mathbb{R}$, we may consider u and $\mathbf{x}$ fixed, so that $D_\mathbf{v}u(\mathbf{x})$ is considered a function of $\mathbf{v}$. This point of view brings to light a distinction which is usually lost (or ignored) in calculus that the collection of vectors $\mathbf{v}$ is ususally distinct from the set of arguments $\mathbf{x} \in \mathcal{U} \subset \mathbb{R}^n$ for the function u. Technically, the directions $\mathbf{v}$ available for computing $D_\mathbf{v}u(\mathbf{x})$ include all vectors in the **tangent space to the domain $\mathcal{U}$ at $\mathbf{x}$** which is $T_\mathbf{x}\mathcal{U} = \mathbb{R}^n$. When the value of the directional derivative is considered as a function of the direction of differentiation $\mathbf{v}$ in this way, the result is called the **differential** of u at $\mathbf{x}$ and is denoted by

$$du_\mathbf{x} : \mathbb{R}^n \to \mathbb{R}.$$

While $u : \mathcal{U} \to \mathbb{R}$, we have a collection of differential functions (one for each $\mathbf{x} \in \mathcal{U}$) with $du_\mathbf{x} : \mathbb{R}^n \to \mathbb{R}$. In addition to having the distinction of having all of $\mathbb{R}^n$ for domain, we have given an argument above showing homogeneity with respect to scaling along the following lines:

Exercise 2.76 *Show that the differential $du_\mathbf{x} : \mathbb{R}^n \to \mathbb{R}$, as we have defined it, satisfies*

$$du_\mathbf{x}(a\mathbf{v}) = a\ du_\mathbf{x}(\mathbf{v}) \qquad \textit{for each } \mathbf{v} \in \mathbb{R}^n \textit{ and } a \in \mathbb{R}.$$

This suggests, at the very least, we should consider the possibility that the functions $du_{\mathbf{x}}$ might be **linear**. It follows from Exercise 2.76, in fact, that in the case $n = 1$ the differential map is linear. At this point it should be confessed that I have again departed, to a certain extent, from standard usage in not making the linearity of a differential an apriori requirement. As I now examine this point in more detail, let me start by recalling that in 1-D calculus the functions in $\text{Diff}(a,b)$, considered as a special case of functions $u : \mathcal{U} \to \mathbb{R}$ where $\mathcal{U}$ might be a higher dimensional domain, are both the functions that are **differentiable** at each point and the functions that have all partial derivatives (of which there is only one) existing at each point, and

$$C^0(a,b) \supsetneqq \text{Diff}(a,b) \supsetneqq C^1(a,b). \tag{2.29}$$

In higher dimensions it is customary to make a distinction so that the **differentiable functions** on a domain $\mathcal{U}$ in a higher dimensional space and those with **all first order partial derivatives existing** at each point $\mathbf{x}$ in $\mathcal{U}$ are **not** the same thing. With this in mind, we introduce the set of functions $u : \mathcal{U} \to \mathbb{R}$ with all first order partial derivatives existing at every point $\mathbf{x}$ in an open subset $\mathcal{U} \subset \mathbb{R}^n$ and call it $\text{pDiff}(\mathcal{U})$. These may be informally called the collection of **partially differentiable functions** on $\mathcal{U}$, and we can also write for (2.29)

$$C^0(a,b) \supsetneqq \text{Diff}(a,b) = \text{pDiff}(a,b) \supsetneqq C^1(a,b).$$

For $\mathcal{U} \subset \mathbb{R}^n$ (any n) and $u \in \text{pDiff}(\mathcal{U})$, there is a linear function $L_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$ associated to u at the point $\mathbf{x}$ with values given by

$$L_{\mathbf{x}}(\mathbf{v}) = Du(\mathbf{x}) \cdot \mathbf{v} = \langle Du(\mathbf{x}), \mathbf{v} \rangle_{\mathbb{R}^n}$$

where $Du : \mathcal{U} \to \mathbb{R}^n$ represents the vector field on $\mathcal{U}$ given in standard coordinates by

$$Du = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right).$$

This vector of first partial derivatives is also called the **total derivative** of u or the **gradient vector**. It should be noted that this is a coordinate dependent expression for the gradient, and we will discuss an important coordinate free version of the gradient vector below. For now, however, we have a linear function $L_{\mathbf{x}}$ at each point $\mathbf{x} \in \mathcal{U}$ associated with each function u in

$$\text{pDiff}(\mathcal{U}) \supsetneqq C^1(\mathcal{U}).$$

The difference between the one dimensional case and the higher dimensional cases starts to become apparent now since there is no simple inclusion relating $\mathrm{pDiff}(\mathcal{U})$ and $C^0(\mathcal{U})$ when $\mathcal{U} \subset \mathbb{R}^n$ and $n > 1$.

Exercise 2.77 *Find a function* $u \in \mathrm{pDiff}(\mathcal{U})\backslash C^0(\mathcal{U})$*. Find a function* $u \in C^0(\mathcal{U})\backslash\, \mathrm{pDiff}(\mathcal{U})$*.*

The notion of differentiability in higher dimensions involves another linear function, potentially different from both $L_{\mathbf{x}}$ and $du_{\mathbf{x}}$ mentioned above, and more explicitly based on first order approximation:

Definition 14 *Given an open subset* $\mathcal{U} \subset \mathbb{R}^n$*, a function* $u : \mathcal{U} \to \mathbb{R}$ *is* **differentiable** *at* $\mathbf{x} \in \mathcal{U}$ *if there exists a linear function* $\ell : \mathbb{R}^n \to \mathbb{R}$ *for which*

$$\lim_{\mathbf{v}\to\mathbf{0}} \frac{u(\mathbf{x}+\mathbf{v}) - u(\mathbf{x}) - \ell(\mathbf{v})}{|\mathbf{v}|} = 0.$$

The collection of functions $u : \mathcal{U} \to \mathbb{R}$ *which are differentiable at each point* $\mathbf{x} \in \mathcal{U}$ *is denoted by* $\mathrm{Diff}(\mathcal{U})$*.*

Exercise 2.78 *Show that the definition of* $\mathrm{Diff}(\mathcal{U})$ *in the special case* $n = 1$*, with* $\mathcal{U}$ *an interval* (a, b)*, is consistent with the previous definition of* $\mathrm{Diff}(a, b)$ *based on the limit of the difference quotient.*

It is, thus, with this notion we obtain a higher dimensional version of (2.29):

$$C^0(\mathcal{U}) \underset{\neq}{\supseteq} \mathrm{Diff}(\mathcal{U}) \underset{\neq}{\supseteq} C^1(\mathcal{U}).$$

In order to verify the inclusion on the right we recall the **mean value theorem** for functions of one variable:

Theorem 15 *Given* $u \in C^0[a, b] \cap C^1(a, b)$*, there exists some* $x_* \in (a, b)$ *with*

$$u'(x_*) = \frac{u(b) - u(a)}{b - a}. \tag{2.30}$$

In the special case $u \in C^1[a, b]$ there is a simple proof of (2.30) using (mainly) the fundamental theorem of calculus and the chain rule. The construction is useful in many contexts, so we present it:

$$u(b) - u(a) = \int_0^1 \frac{d}{dt} u((1-t)a + tb)\, dt = \int_0^1 u'((1-t)b + ta)(b - a)\, dt.$$

The quantity

$$\int_0^1 u'((1-t)b+ta)\,dt$$

is the average value of the integrand $u'((1-t)b+ta)$ which is a continuous function of t, and it follows that for some $t_* \in (0,1)$

$$\int_0^1 u'((1-t)b+ta)\,dt = u((1-t^*)b+t^*a).$$

Now, in the case $u \in C^1(\mathcal{U})$, we claim the linear function $L_{\mathbf{x}}$ given by the Euclidean inner product with the vector of partial derivatives $Du(\mathbf{x})$ gives the linear approximation required by the definition of differentiability. To see this, let $\mathbf{v} = (v_1, v_2, \dots, v_n)$ and note that by the mean value theorem

$$u(\mathbf{x}+v_1\mathbf{e}_1) - u(\mathbf{x}) = D_{\mathbf{e}_1}u(\mathbf{x}_1^*)v_1$$

where $\mathbf{x}_1^* = \mathbf{x} + v_1^*\mathbf{e}_1$ and v_1^* is between 0 and v_1. Similarly,

$$u(\mathbf{x}+v_2\mathbf{e}_2+v_1\mathbf{e}_1) - u(\mathbf{x}+v_1\mathbf{e}_1) = D_{\mathbf{e}_2}u(\mathbf{x}_2^*)v_2$$

where $\mathbf{x}_2^* = \mathbf{x} + v_2^*\mathbf{e}_2 + v_1\mathbf{e}_1$ and v_2^* is between 0 and v_2. Repeating this application of the mean value theorem along the remaining standard coordinate directions, we obtain

$$\begin{aligned} u(\mathbf{x}+\mathbf{v}) - u(\mathbf{x}) &= \sum_{k=2}^n \left[u\left(\mathbf{x} + \sum_{j=1}^k v_j\mathbf{e}_j\right) - u\left(\mathbf{x} + \sum_{j=1}^{k-1} v_j\mathbf{e}_j\right)\right] \\ &\qquad + u(\mathbf{x}+v_1\mathbf{e}_1) - u(\mathbf{x}) \\ &= \sum_{j=1}^n D_{\mathbf{e}_j}u(\mathbf{x}_j^*)v_j \end{aligned}$$

for points $\mathbf{x}_1^*, \mathbf{x}_2^*, \dots, \mathbf{x}_n^*$ tending to $\mathbf{x}$ as $\mathbf{v}$ tends to $\mathbf{0}$. This can be written as

$$u(\mathbf{x}+\mathbf{v}) - u(\mathbf{x}) = \langle D^*u, \mathbf{v}\rangle_{\mathbb{R}^n}$$

where $D^*u = (D_{\mathbf{e}_1}u(\mathbf{x}_1^*), D_{\mathbf{e}_2}u(\mathbf{x}_2^*), \dots, D_{\mathbf{e}_n}u(\mathbf{x}_n^*))$. Therefore,

$$u(\mathbf{x}+\mathbf{v}) - u(\mathbf{x}) - L_{\mathbf{x}}(\mathbf{v}) = \langle (D^*u - Du(\mathbf{x}), \mathbf{v}\rangle_{\mathbb{R}^n}\,.$$

The Cauchy-Schwarz inequality says that for any vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, we have

$$|\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbb{R}^n}| \le |\mathbf{v}||\mathbf{w}|.$$

Applying this inequality we have

$$|u(\mathbf{x}+\mathbf{v}) - u(\mathbf{x}) - L_{\mathbf{x}}(\mathbf{v})| \le |D^* u - Du(\mathbf{x})||\mathbf{v}|$$

and

$$\left| \frac{u(\mathbf{x}+\mathbf{v}) - u(\mathbf{x}) - L_{\mathbf{x}}(\mathbf{v})}{|\mathbf{v}|} \right| = \frac{|u(\mathbf{x}+\mathbf{v}) - u(\mathbf{x}) - L_{\mathbf{x}}(\mathbf{v})|}{|\mathbf{v}|} \le |D^* u - Du(\mathbf{x})|.$$

By the continuity of the partial derivatives

$$lim_{\mathbf{v}\to\mathbf{0}} |D^* u - Du(\mathbf{x})| = 0$$

since

$$D_{\mathbf{e}_k} u(\mathbf{x}_k^*) = \frac{\partial u}{\partial x_k} \left(\mathbf{x} + v_k^* \mathbf{e}_k + \sum_{j \neq k} v_j \mathbf{e}_j \right)$$

and v_k^* is between 0 and v_k. □

The argument above not only shows that $u \in C^1(\mathcal{U})$ is differentiable, but the approximating linear function ℓ in the definition of differentiability can be taken to be the particular linear function $L_{\mathbf{x}}$ obtained using the inner product with the total derivative/gradient vector $Du(\mathbf{x})$.

A multivariable chain rule

Let $u \in C^1(\mathcal{U})$ where $\mathcal{U}$ is an open subset of $\mathbb{R}^2$.

2.78.1 Coordinates and inner product spaces

The relation of vectors and other mathematical constructions to coordinates is often first encountered (and usually not fully appreciated) in a course on linear algebra. The basic idea is also operative in elementary geometry where a circle with a given radius and center can be considered without coordinates and, yet, if one wishes to make certain computations introducing coordinates for the center of a circle is seemingly unavoidable. A similar situation prevails with vectors, linear transformations, and other mathematical constructions.

Two such constructions we wish to consider here are the gradient of a real valued function (of several variables) and the divergence of a vector field. It will be noted that in the first case, we have used coordinates to define the gradient:

$$Du = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right).$$

The divergence, on the other hand, we have defined in a manner that did not use coordinates:

$$\div \mathbf{v}(\mathbf{x}) = \lim_{\mathcal{V} \to \{\mathbf{x}\}} \frac{1}{\mathfrak{L}^n(\mathcal{V})} \int_{\mathcal{V}} \mathbf{v} \cdot \mathbf{n}.$$

Technically, the dot product appearing in this definition involves coordinates if by $\mathbf{v} \cdot \mathbf{n}$ we mean

$$\mathbf{v} \cdot \mathbf{n} = \sum_{j=1}^{n} v_j n_j.$$

The dot product itself, however, can be considered without coordinates, and this distinction may be indicated by the use of a different notation.

Definition 15 *An* **inner product space** *is a vector space* V *equipped with a function* $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ *having the following properties:*

(i) $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$ *for all* $\mathbf{v}, \mathbf{w} \in V$. *(symmetric)*

(ii) $\langle a\mathbf{v} + b\mathbf{w}, \mathbf{z} \rangle = a\langle \mathbf{v}, \mathbf{z} \rangle + b\langle \mathbf{w}, \mathbf{z} \rangle$ *for all* $a, b \in \mathbb{R}$ *and* $\mathbf{v}, \mathbf{w}, \mathbf{z} \in V$. *(bilinear)*

(iii) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ *with equality if and only if* $\mathbf{v} = \mathbf{0}$. *(positive definite)*

Multi-index notation

The last notation for a partial derivative given in (2.26) deserves special notice. Though $D^{\mathbf{e}_j} u$ bears a strong superficial resemblance to the standard notation $D_{\mathbf{e}_j} u$ for a partial derivative, something quite different is in mind. Though not immediately of interest in regard to our present discussion of **first order** partial derivatives, the multi-index notation is quite useful in certain applications, most notably for writing down higher order Taylor approximations in several variables, so let us briefly explain it in passing. The superscript vector in $D^{\beta} u$ can not only be taken as one of the standard unit basis vectors to indicate a single directional derivative in that direction, but β denotes a multi-index which is an element in the set

$$\mathbb{N}_0^n = \{(\beta_1, \beta_2, \ldots, \beta_n) : \beta_j \in \mathbb{N}_0 \text{ for } j = 1, 2, \ldots, n\}$$

and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ denotes the nonnegative integers. In words, $D^\beta u$ indicates the result of taking β_j partial derivatives of u with respect to $\mathbf{e}_j$ for $j = 1, 2, \ldots, n$. It is assumed, when this notation is used, that u is continuously (partial) differentiable $\beta_1+\beta_2+\cdots+\beta_n$ times in any combination of standard directions. In this context, the sum $\sum \beta_j$ is denoted $\|\beta\|$ and called the **norm of the multi-index** β, and a function with this regularity on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is said to be in $C^{\|\beta\|}(\mathcal{U})$. It can then be proved that the order of application of the partial derivatives does not effect the result, so we can write

$$D^\beta u = \frac{\partial^{\|\beta\|} u}{\partial^{\beta_1} x_1 \partial^{\beta_2} x_2 \cdots \partial^{\beta_n} x_n}. \tag{2.31}$$

This may seem complicated, but consider the simplicity and economy of notation obtained in (2.31). Returning to $D_{\mathbf{e}_j} u$ and $D^{\mathbf{e}_j} u$, the former denotes taking a directional derivative in the direction of the vector $\mathbf{e}_j$, which happens to be a partial derivative. Thus, $D_{\mathbf{e}_j} u$ is simply a special case of $D_{\mathbf{v}} u$ as explained above. The expression $D^{\mathbf{e}_j} u$ on the other hand means "one partial derivative with respect to the variable x_j" where $D^\beta u$ has the more general meaning "β_j derivatives with respect to x_j for $j = 1, 2, \ldots, n$."

Definition 16 *Given an open subset $\mathcal{U} \subset \mathbb{R}^n$ and a natural number $k \geq 1$, the set $C^k(\overline{\mathcal{U}})$ consists of all functions $u \in C^k(\mathcal{U})$ such that each partial derivative $D^\beta u$ with $|\beta| \leq k$ has a continuous extension $v_\beta \in C^0(\overline{\mathcal{U}})$ to the closure of $\mathcal{U}$:*

$$v_\beta\big|_{\mathcal{U}} = D^\beta u.$$

Exercise 2.79 *Definition 13 and Definition 16 overlap when $A = \overline{\mathcal{U}}$ is the closure of an open subset of $\mathbb{R}^n$. Are they consistent with one another in this case?*

Chapter 3

2-D Capillary Surfaces

Here we consider first the 2-D version of the Young-Laplace equations which we restate:

$$\begin{cases} \left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \kappa u & \text{for } |x| < r \\ \\ \frac{u'(\pm r)}{\sqrt{1+u'(\pm r)^2}} = \pm\cos\gamma. \end{cases} \tag{3.1}$$

Here $r > 0$ is a given fixed constant. Similarly, $\kappa > 0$ and γ with $|\gamma| < \pi/2$ are assumed given and fixed. For most of our considerations, and exclusively for our initial considerations, we will assume $0 < \gamma < \pi/2$. As mentioned above, (3.1) is a **two point boundary value problem** for the ODE of prescribed signed curvature

$$k = \left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \kappa u.$$

The problem is strongly variational in the sense that solutions turn out to be minimizers of the 2-D capillary energy discussed in the last chapter. Under the assumption $|\beta| < 1$ on the adhesion coefficient β, we showed $\cos\gamma = \beta$. There seems to be no immediate physical restriction leading to the assumption $|\beta| < 1$ or even $|\beta| \leq 1$, and the material of this chapter should put us in a position to consider the possibilities $|\beta| = 1$ and $|\beta| > 1$, each of which has it's own associated peculiarities. Note in particular that some modification of the boundary condition in the Young-Laplace problem (3.1) as stated above is required for the consideration of $\cos\gamma = \pm 1$.

Returning to our comment that (3.1) is a two point boundary value problem for the ODE $k = \kappa u$, we note that two point boundary value problems are not usually contemplated in the standard existence and uniqueness theorem(s) for ODEs, but rather an **initial value problem** is the one traditionally considered. Naturally, we will want to prove an existence and uniqueness theorem for (3.1) though our initial considerations are based on the assumption that a solution exists. These considerations will give us an opportunity to review carefully an existence and uniqueness theorem for initial value problems involving nonlinear systems of ODEs as well as various well-known results from elementary calculus. These will provide indespensible tools in our study, so they should be known and understood "inside and out" so to speak. Our solution $u = u(x)$, which we will initially assume to exist, will be initially assumed to satisfy

$$u \in C^2(-r,r) \cap C^1[-r,r]. \tag{3.2}$$

It may be noted that consideration of $|\cos\gamma| = 1$ might require $u \in C^2(-r,r)\cap C^0[-r,r]$. Under the assumption $|\cos\gamma| < 1$, we will soon be able to show solutions satisfying the regularity condition (3.2) satisfy, in fact, $u \in C^2[-r,r]$ and even $u \in C^\infty[-r,r]$.

It seems to me a strong geometric understanding, as well as some technical/calculational aspects, of the notions of inclination angle and curvature are indespensible at this point. In particular, it should be recalled that the curvature is given by

$$k = \left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \frac{u''}{(1+u'^2)^{3/2}} = \frac{d}{dx}\sin\psi = \frac{d\psi}{ds}$$

where

$$s = \int_0^x \sqrt{1+u'(\xi)^2}\,d\xi$$

is the arclength measured along the interface curve and ψ is the inclination angle defined by

$$(\cos\psi, \sin\psi) = \left(\frac{1}{\sqrt{1+u'^2}}, \frac{u'}{\sqrt{1+u'^2}}\right)$$

as usual.

3.1 Basic Properties

3.1.1 Getting Started

We begin by carefully establishing and emphasizing the assertions surrounding Exercises 1.13 and 1.14 of the introductory chapter. In particular, if we assume $0 < \gamma < \pi/2$, then any solution must be positive, convex, and symmetric. In symbols:

1. $u(x) > 0$ for $|x| \le r$,
2. $u''(x) > 0$ for $|x| \le r$,
3. $u(-x) = u(x)$ for $|x| \le r$, and consequently
4. u has a unique minimum at $x = 0$ with

$$u'(x) < 0 \quad \text{for } -r \le x < 0, \quad u'(0) = 0, \quad \text{and} \quad u'(x) > 0 \quad \text{for } 0 < x \le r.$$

We give two approaches to the positivity. The first uses the well-known **intermediate value theorem**:

Theorem 16 *(intermediate value theorem) if $f \in C^0[a,b]$ and $v \in \mathbb{R}$ satisfies*

$$f(a) < v < f(b) \qquad \text{or} \qquad f(b) < v < f(a),$$

then there exists some $x_* \in (a,b)$ with $f(x_*) = v$.

Under the assumption $0 < \gamma < \pi/2$, we know $0 < \cos\gamma < 1$, and the boundary condition

$$\frac{u'(\pm r)}{\sqrt{1 + u'(\pm r)^2}} = \pm\cos\gamma$$

implies $u'(-r) < 0 < u'(r)$. Thus, applying the intermediate value theorem to u', we obtain some x_* with $|x_*| < r$ and

$$u'(x_*) = 0.$$

We next use some version of the **local existence and uniqueness theorem for initial value problems for ODEs**. This theorem is usually stated for systems of ODEs, so we will state it that way first. Also, our ODE is **autonomous**, so we will only state a result in that case.

Theorem 17 *(local existence and uniqueness for ODEs) Let* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *is a* C^1 *vector field, which means each of the coordinate functions* f_j *in* $\mathbf{F} = (f_1, f_2, ..., f_n)$ *for* $j = 1, 2, \ldots, n$ *satisfies* $f_j \in C^1(\mathbb{R}^n)$. *If* $\mathbf{x}_0 \in \mathbb{R}^n$ *and* $t_0 \in \mathbb{R}$, *then there exists some* $\delta > 0$ *such that the initial value problem*

$$\begin{cases} \mathbf{x}' = \mathbf{F}(\mathbf{x}), & |t - t_0| < \delta \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

has a unique solution $\mathbf{x} : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}^n$ *with each coordinate function* x_j *in* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *for* $j = 1, 2, \ldots, n$ *satisfying* $x_j \in C^1(t_0 - \delta, t_0 + \delta)$.

We will apply this result in the case $n = 2$ to our second order nonlinear ODE of prescribed signed curvature. Thus, a more directly applicable statement is the following:

Theorem 18 *(local existence and uniqueness for regular second order ODEs) Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *satisfy* $f \in C^1(\mathbb{R}^2)$. *If* $(t_0, u_0, u_0') \in \mathbb{R}^3$, *then there exists some* $\delta > 0$ *such that the initial value problem*

$$\begin{cases} u'' = f(u, u'), & |t - t_0| < \delta \\ u(t_0) = u_0, & u'(t_0) = u_0' \end{cases}$$

has a unique solution $u \in C^2(t_0 - \delta, t_0 + \delta)$.

Exercise 3.2 *Assume Theorem 17 and prove Theorem 18.*

For our application of Theorem 18 we will take

$$f(u, u') = \kappa u(1 + u'^2)^{3/2}$$

and write the 2-D capillary equation as

$$u'' = u(1 + u'^2)^{3/2}.$$

We show first that $u(x_*) > 0$. Our proof is by contradiction. We consider two distinct complementary cases:

1. $u(x_*) = 0$.

2. $u(x_*) < 0$.

It is especially attractive to consider these two cases because an immediate explicit solution is available in the first case.

If $u(x_*) = 0$, then since $u'(x_*) = 0$, we can consider the initial value problem

$$\begin{cases} u'' = cu(1+u'^2)^{3/2}, & |x| < r \\ u(x_*) = 0, & u'(x_*) = 0. \end{cases}$$

One solution of this problem, in this case is $u_0 \equiv 0$. It will be noted, in fact, that the constant function u_0 satisfies the global initial value problem

$$\begin{cases} u'' = cu(1+u'^2)^{3/2}, & x \in \mathbb{R} \\ u(x_*) = 0, & u'(x_*) = 0. \end{cases}$$

The existence and uniqueness theorem does not give us uniqueness immediately on the entire interval, but it does tell us that our original solution $u \in C^2(-r,r) \cap C^1[-r,r]$ satisfies

$$u(x) \equiv u_0 = 0 \qquad \text{for } |x - x_*| < \delta$$

where $\delta > 0$ is some positive number given by Theorem 18. We wish to show, of course, that $u(x) \equiv 0$ for $|x| \le r$. This will then imply $u'(\pm r) = 0$ contradicting our assumption/boundary condition

$$u'(\pm r) = \pm \cos\gamma \sqrt{1 + u'(\pm r)^2} \neq 0.$$

We have shown the two sets

$$A = \{a \in [-r, x_*) : u(x) \equiv 0 \quad \text{for } a < x \le x_*\}$$

and

$$B = \{b \in (x_*, r] : u(x) \equiv 0 \quad \text{for } x_* \le x < b\}$$

are both nonempty with

$$(x_* - \delta, x_*) \subset A \qquad \text{and} \qquad (x_*, x_* + \delta) \subset B.$$

Letting $a_0 = \inf A$ and $b_0 = \sup B$, we can attempt to apply the existence and uniqueness theorem again at the points a_0 and b_0. The requirements for such an application at $x = a_0$ are the following:

1. $u(a_0) = 0$, and

2. $u'(a_0) = 0$.

It is relatively easy to show $u(x) \equiv 0$ for $a_0 < x \le x_*$, that is, $a_0 \in A$. In fact, if this were not the case, then there would exist some x_1 with $a_0 < x_1 < x_*$ and $u(x_1) \ne 0$. This means every point a' with $r \le a' < x_1$ satisfies $a' \notin A$ and $a_0 = \inf A \ge x_1$. This contradicts the inequality $a_0 < x_1$. Finally, we obtain by continuity

$$u(a_0) = \lim_{x \searrow a_0} u(x) = 0.$$

Exercise 3.3 *Show* $u(b_0) = 0$.

It follows, furthermore, from the definition of the derivative that

$$u'(a_0) = \lim_{h \searrow 0} \frac{u(a_0 + h) - u(a_0)}{h} = 0.$$

Therefore, the existence and uniqueness theorem gives us some $\delta_0 > 0$ for which the initial value problem

$$\begin{cases} u'' = cu(1 + u'^2)^{3/2}, & |x - a_0| < \delta_0 \\ u(a_0) = 0, & u'(a_0) = 0 \end{cases}$$

has a unique solution. That solution is, of course, $u_0 \equiv 0$. If we assume, at this point, that $-r < a_0 < x_*$, then our application gives us a value

$$a_1 = \max\{-r, a_0 - \delta_0\}$$

for which $u(x) \equiv u_0$ for $a_1 < x \le x_*$. This, again contradicts the definition of a_0 because we find

$$a_0 \le a_1 = \max\{-r, a_0 - \delta_0\} < a_0.$$

We conclude $a_0 = -r$ and $u(x) \equiv 0$ for $-r \le x \le x_*$.

Exercise 3.4 *Carry out the details to show* $u(x) \equiv 0$ *for* $x_* \le x \le r$.

We have ruled out the case $u(x_*) = 0$. Next we consider the case $u(x_*) < 0$. In this case, we can apply the following sufficiency condition for a strict local maximum from calculus:

Theorem 19 *If $f \in C^2(a,b)$ and $x_* \in (a,b)$ satisfy*

$$u'(x_*) = 0 \qquad \text{and} \qquad u''(x_*) < 0,$$

then there exists some $\delta > 0$ for which $u(x) \le u(x_*)$ for $|x - x_*| < \delta$ with equality only for $x = x_*$.

Exercise 3.5 *Prove Theorem 19.*

It follows, in fact, that

$$u''(x) = \kappa u(x)(1 + u'(x)^2) \le \kappa u(x_*)(1 + u'(x)^2) < 0 \qquad \text{for } |x - x_*| < \delta$$

so that $-u$ is convex on the interval $x_* - \delta < x < x_* + \delta$ with

$$u'(x) = u'(x_*) + \int_{x_*}^{x} u''(\xi)\, d\xi > 0 \qquad \text{for } x_* - \delta < x < x_*$$

and

$$u'(x) = u'(x_*) + \int_{x_*}^{x} u''(\xi)\, d\xi < 0 \qquad \text{for } x_* < x < x_* + \delta.$$

The condition $u''(x) < 0$ and the monotonicity conditions on $u'(x)$ extend to the entire interval $[-r, r]$ in this case, so that $u(x_*)$ is a strict global maximum value. In order to see this, we will use the **mean value theorem**:

Theorem 20 *(mean value theorem) if $f \in C^1(a,b) \cap C^0[a,b]$, then there exists some $x \in (a,b)$ with*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Assume there is some $x \in (x_*, r]$ with $u(x) \ge u(x_*)$. Let

$$x_1 = \inf\{x \in (x_*, r] : u(x) \ge u(x_*)\}.$$

By continuity $u(x_1) \ge u(x_*)$, and according to the local behavior at x_* we know $x_1 \ge x_* + \delta > x_*$. Note also that $u(x) < u(x_*)$ for $x_* < x < x_1$. By the mean value theorem there exists $x_2 \in (x_*, x_1)$ with $u'(x_2) \ge 0$. By a second application of the mean value theorem there exists $x_3 \in (x_*, x_2)$ with $u''(x_3) \ge 0$. But then

$$u''(x_3) = \kappa u(x_3)(1 + u'(x_3)^2)^{3/2} \le \kappa u(x_*)(1 + u'(x_3)^2)^{3/2} < 0$$

which is a contradiction. This means there is no $x \in (x_*, r]$ with $u(x) \geq u(x_*)$.

Similarly, $u(x) < u(x_*) < 0$ for $-r \leq x < x_*$. We conclude

$$u''(x) = \kappa u(x)(1 + u'(x)^2) \leq \kappa u(x_*)(1 + u'(x)^2) < 0 \qquad \text{for } |x| < r.$$

Finally using the fundamental theorem of calculus to write

$$u'(x) = u'(x_*) + \int_{x_*}^{x} u''(\xi)\, d\xi$$

as above, we obtain $u'(x) > 0$ for $-r < x < x_*$ and $u'(x) < 0$ for $x_* < x < r$. It follows by continuity that $u'(-r) \geq 0$ and $u'(r) \leq 0$. Each of these inequalities contradicts the boundary condition

$$u'(\pm r) = \pm \cos\gamma \sqrt{1 + u'(\pm r)^2} \neq 0.$$

We have obtained the contradiction according to which we can assert $u(x_*) > 0$ whenever $0 < \gamma < \pi/2$ and $u'(x_*) = 0$ for some $x_* \in (-r, r)$. Technically, we have not shown, as claimed in the second (contradictory) case, that $u'(-r) > 0$, $u'(r) < 0$, and $u''(\pm r) < 0$. In fact, on the face of it, the condition $u''(\pm r)$ does not really make sense at least until we obtain an extension $\bar{u}$ of u to an open interval containing $\pm r$ where $\bar{u}''$ is well-defined.

Exercise 3.6 *Show that if $u'(x_*) = 0$ with $u(x_*) < 0$ and $u \in C^2(-r, r) \cap C^1[-r, r]$ satisfies the 2-D capillary equation*

$$\frac{u''}{(1 + u'^2)^{3/2}} = \kappa u,$$

on $(-r, r)$, then $u'(-r) > 0$ and $u'(r) < 0$. Show, moreover, that there exists some $\delta > 0$ and an extension $\bar{u} \in C^2(-r - \delta, r + \delta)$ satisfing

$$\bar{u}\Big|_{[-r,r]} \equiv u \qquad \text{and} \qquad \bar{u}''(\pm r) < 0.$$

In fact, show (you can take) $\bar{u} \in C^\infty(r - \delta, r + \delta)$.

Exercise 3.7 *It may be noted that we did not use the existence and uniqueness theorem for ODEs in the (contradictory) case $u(x_*) < 0$. Can you obtain a contradiction in the case $u(x_*) = 0$ of the proof above* **without** *using the existence and uniqueness theorem?*

Let us make a new start at this point, and give a second somewhat different approach to showing $u(x_*) > 0$. This approach is based on the **extreme value theorem** from calculus:

Theorem 21 *(extreme value theorem) If $f \in C^0[a,b]$, then there exists some $x_* \in [a,b]$ such that the absolute minimum of f is attained at x_*. That is,*

$$u(x_*) \le u(x) \qquad \text{for } x \in [a,b].$$

We claim first that the point of absolute minimum for u cannot occur at one of the boundary points $x = \pm r$. Take, for example, $x = -r$. We know from the boundary condtion that

$$u'(-r) = -\cos\gamma\sqrt{1 + u'(-r)^2} < 0.$$

Therefore, the **Taylor approximation formula** (or the mean value theorem along with continuity) tells us

$$u(x) = u(-r) + u'(\xi)(x+r) < u(-r) \qquad \text{when } |x+r| \text{ is small enough}$$

and where ξ is some point satisfying $-r < \xi < x$. This means there is not even a local minimum at $x = -r$. A similar observation applies to the other endpoint $x = r$.

Therefore, the global min point x_* from the extreme value theorem is an **interior point** satisfying $-r < \xi < r$. To such a min value we may apply the **necessary condition for a minimum** from calculus:

Theorem 22 *(first order necessary condition) If $f \in C^1(a,b)$ has a local interior minimum at $x \in (a,b)$, then $f'(x) = 0$.*

This tells us $u'(x_*) = 0$, and we can proceed somewhat as before: If $u(x_*) = 0$, then $u \equiv 0$ by the existence and uniqueness argument. But if $u(x_*) < 0$, we can obtain an immediate contradiction because

$$u''(x_*) = \kappa u(x_*)(1 + u'(x_*)^2) < 0.$$

(This gives us the hypotheses of the sufficient condition for a strict local maximum as described above—so of course, a strict local maximum cannot also be an interior local minimum.)

Alternatively, we could use the second order version of the necessary condition:

Theorem 23 *(second order necessary condition) If $f \in C^2(a,b)$ has a local interior minimum at $x \in (a,b)$, then $f'(x) = 0$ and $f'(x) \geq 0$.*

We have established the existence of an interior point $x_* \in (-r, r)$ satisfying

$$u'(x_*) = 0 \qquad \text{and} \qquad u(x_*) > 0.$$

We may assume $u(x_*)$ is the global minimum value of u. We can also prove $u(x_*) < u(x)$ for $x \in [-r, r] \backslash \{x_*\}$ directly using the reasoning above once we know $u(x_*) > 0$ (which we have established) and the convexity

$$u''(x) = \kappa u(x)(1 + u'(x)^2)^{3/2} \geq \kappa u(x_*)(1 + u'(x)^2)^{3/2} > 0.$$

Thus, we have our first lower bound for u:

$$u(x) > 0 \qquad \text{for } |x| < r.$$

It is not a stellar lower bound, but it is a useful one. Before we attempt to improve our lower bound, let us establish symmetry.

Theorem 24 *Any solution of (3.1) satisfies $u(-x) = u(x)$. In particular, if $0 < \gamma < \pi/2$, then the unique global min occurs at $x = 0$ with value $u(0) > 0$.*

Proof: As above, we continue to restrict attention to the case $0 < \gamma < \pi/2$. We have established the positivity $u > 0$ and the consequent convexity $u'' > 0$ with a unique global interior minimum point $x = x_*$. Let us assume by way of contradiction that $-r < x_* < 0$. Then the minimum is closer to the left wall $x = -r$. We construct another solution of the capillary equation as follows:

$$\tilde{u}(x) = \begin{cases} u(2x_* - x), & -r \leq x \leq x_* \\ u(x), & x_* \leq x \leq r. \end{cases}$$

This is the function obtained by reflecting the values of u to the right of the minimum across the line $x = x_*$.

Notice the value $\tilde{u}(x_*)$ is well-defined though the cases overlap. We may conclude from this that $\tilde{u} \in C^0[-r, r]$. Also, it is clear that $u \in C^2(-r, x_*) \cap C^2(x_*, r)$. We need to determine the regularity at the reflection point $x = x_*$.

The derivative of $\tilde{u}$ at $x = x_*$ from the right is well-defined and given by

$$\tilde{u}'(x_*^+) = u'(x_*) = 0.$$

The derivative of $\tilde{u}$ at $x = x_*$ from the left is also well-defined and

$$\tilde{u}'(x_*^-) = -u'(x_*) = 0.$$

This implies $\tilde{u} \in C^1[-r, r]$. Similarly,

$$\tilde{u}''(x_*^+) = u''(x_*)$$

and

$$\tilde{u}''(x_*^+) = (-1)(-1)u''(x_*) = u''(x_*).$$

Thus, $\tilde{u} \in C^2[-r, r]$ is a global solution of the 2-D capillary equation.

By the local uniqueness for ODEs, there is some $\delta > 0$ such that

$$\tilde{u}(x) \equiv u(x) \qquad \text{for } x_* - \delta < x < x_* + \delta.$$

We define sets

$$A = \{a \in [-r, x_*] : \tilde{u}(x) = u(x) \quad \text{for } a < x \leq x_*\}$$

and

$$B = \{b \in [x_*, r] : \tilde{u}(x) = u(x) \quad \text{for } x_* \leq x < b\}.$$

Letting $a_0 = \inf A$ and $b_0 = \sup B$, we see that $a_0 \in A$, $b_0 \in B$, and, according to the local existence and uniqueness theorem, there is no other possiblity than $a_0 = -r$ and $b_0 = r$. That is, $\tilde{u} \equiv u$.

In particular, u is symmetric with respect to the line $x = x_*$ satisfying

$$u(2x_* - x) = u(x) \qquad \text{for } -r \leq x \leq 2x_* + r.$$

This implies $u'(2x_* + r) = -u'(-r)$. In particular,

$$\frac{u'(2x_* + r)}{\sqrt{1 + u'(2x_* + r)^2}} = -\frac{u'(-r)}{\sqrt{1 + u'(-r)^2}} = \cos\gamma.$$

Finally, by the convexity $u''(x) > 0$ and using our assumption $x_* < 0$,

$$u'(r) > u'(2x_* + r).$$

Since the function

$$f(p) = \frac{p}{\sqrt{1 + p2}}$$

is strictly increasing and well-defined for all p, we conclude

$$\frac{u'(r)}{\sqrt{1+u'(r)^2}} > -\frac{u'(2x_*r)}{\sqrt{1+u'(2x_*+r)^2}} = \cos\gamma.$$

And this is a contradiction of the right boundary condition. We conclude $x_* \geq 0$.

Exercise 3.8 *Give your own argument (better than mine above) that $x_* \leq 0$.*

We conclude that $x_* = 0$. The symmetry following from reflection and the uniqueness theorem for ODEs now implies $u(x) = u(-x)$. □

3.8.1 Comparison to circular arcs

Now things get fun. Let $\sigma : [-1/(\kappa u(0)), -1/(\kappa u(0))] \to \mathbb{R}$ by

$$\sigma(x) = u(0) + \frac{1}{\kappa u(0)} - \sqrt{\frac{1}{\kappa^2 u(0)^2} - x^2}.$$

It will be noted that the minimum of the 2-D meniscus, which we have shown to be located at $(0, u(0))$ with $u(0) > 0$ determines a radius of curvature according to the equation $k = \kappa u(0)$. That radius is $1/(\kappa u(0))$, and σ determines a lower semi-circular graph tangent to the graph of u at the minimum of the 2-D meniscus with

$$\left(\frac{\sigma'}{\sqrt{1+\sigma'^2}}\right)' = \frac{\sigma''}{(1+\sigma'^2)^{3/2}} \equiv \kappa u(0).$$

We wish to show the following:

1. $\sigma(x)$ is defined on the entire interval $-r < x < r$ with $\sigma \in C^2[-r, r]$, that is,
$$\frac{1}{\kappa u(0)} > r.$$
2. $\sigma(x) \leq u(x)$ for $|x| \leq r$ with equality only for $x = 0$.

This is a much nicer lower bound than $u(x) > 0$. In fact, we already have, or can easily obtain, something a little better:

$$u(x) > u(0) > 0 \qquad \text{for } 0 < |x| \leq r.$$

We also know, and have essentially used, that $u''(x) > 0$ for $|x| < r$, or more precisely,

$$\frac{u''(x)}{(1+u'(x)^2)^{3/2}} \geq \frac{u''(0)}{(1+u'(0)^2)^{3/2}} = \kappa u(0) > 0.$$

We now sharpen these estimates as follows:

Theorem 25 *If $u \in C^2(-r,r) \cap C^1[-r,r]$ satisfies*

$$\begin{cases} \left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \kappa u & \textit{for } |x| < r \\ \frac{u'(\pm r)}{\sqrt{1+u'(\pm r)^2}} = \pm\cos\gamma, \end{cases}$$

then

$$r < \frac{1}{\kappa u(0)},$$

$$\sigma(x) < u(x) \qquad \textit{for } 0 < |x| \leq r$$

and

$$\sigma'(x) < u'(x) \qquad \textit{for } 0 < x \leq r.$$

Proof: We will again use the important fact, mentioned above, that $f : \mathbb{R} \to (-1,1)$ by

$$f(p) = \frac{p}{\sqrt{1+p^2}} \tag{3.3}$$

is strictly increasing with $f'(p) > 0$. For this argument, let us set

$$r_0 = \min\left\{r, \frac{1}{\kappa u(0)}\right\}.$$

If we assume, by way of contradiction that

$$\text{there is some } x_0 \text{ with } 0 < x_1 < r_0 \text{ and } u'(x_0) \leq \sigma'(x_0), \tag{3.4}$$

then by the monotonicity of f we get

$$f(u'(x_0)) = \frac{u'(x_0)}{\sqrt{1+u'(x_0)^2}} \leq \frac{\sigma'(x_0)}{\sqrt{1+\sigma'(x_0)^2}} = f(\sigma'(x_0)).$$

By the mean value theorem applied to

$$g(x) = \frac{u'(x)}{\sqrt{1+u'(x)^2}} - \frac{\sigma'(x)}{\sqrt{1+\sigma'(x)^2}}$$

which satisfies $g(0) = 0$ and $g(x_0) \le 0$, there is some x_1 with $0 < x_1 < x_0$ and $g'(x_1) \le 0$. That is,

$$\kappa u(x_1) = \frac{u''(x_1)}{(1 + u'(x_1)^2)^{3/2}} \le \frac{\sigma''(x_1)}{(1 + \sigma'(x_1)^2)^{3/2}} = \kappa u(0).$$

This contradicts our previous result that $u(x_1) > u(0)$. We have established, so far, that

$$\sigma'(x) \le u'(x) \qquad \text{for } 0 \le x < r_0 \text{ with equality only for } x = 0. \tag{3.5}$$

Integration now gives

$$\sigma(x) \le u(x) \qquad \text{for } 0 \le x \le r_0 \text{ with equality only for } x = 0.$$

If we assume

$$r_0 = \min\left\{r, \frac{1}{\kappa u(0)}\right\} = \frac{1}{\kappa u(0)},$$

then we find from (3.5)

$$u'\left(\frac{1}{\kappa u(0)}\right) = \lim_{x \nearrow r_0} u'(x) \ge \lim_{x \nearrow r_0} \sigma'(x) = +\infty$$

which is a contradiction since $u \in C^1[0, r_0]$. Therefore,

$$\frac{1}{\kappa u(0)} > r$$

and the inequality of (3.5) extends with strict inequality to $x = r$. $\square$

Nathan Soedjak's Proof

Here is another, rather more elegant, proof: We know $u'(x) > 0$ for $x > 0$, and this means $u(x) > u(0)$ for $x > 0$. We then have from the prescribed curvature equation

$$\kappa u(0) = \left(\frac{\sigma'(x)}{\sqrt{1 + \sigma'(x)^2}}\right)' \le \left(\frac{u'(x)}{\sqrt{1 + u'(x)^2}}\right)' = \kappa u(x)$$

for $x \geq 0$ with strict inequality for $x > 0$. If we denote by ψ_σ the inclination angle along the graph of σ and recall that

$$\sin\psi = \frac{u'}{\sqrt{1+u'^2}},$$

then integrating from $x = 0$ to $x > 0$, we have

$$\sin\psi_\sigma(x) < \sin\psi(x).$$

Since sine is monotone increasing on $[0, \pi/2)$, we get

$$0 < \psi_\sigma(x) < \psi(x) < \frac{\pi}{2} \qquad \text{for } 0 < x \leq r.$$

Similarly the tangent function is monotone increasing on $[0, \pi/2)$, so

$$\sigma'(x) = \tan\psi_\sigma(x) < \tan\psi(x) = u'(x) \leq u'(r) < \infty \qquad \text{for } 0 < x \leq r.$$

This proves all the assertions of Theorem 25 because we also have

$$\sigma(x) = \sigma(0) + \int_0^x \sigma'(t)\,dt < u(0) + \int_0^x u'(t)\,dt = u(x) \qquad \text{for } 0 < x \leq r. \qquad \square$$

We have obtained a lower bound on $u(r)$ in terms of a lower semi-circular arc given by

$$\sigma(x) = u(0) + \frac{1}{\kappa u(0)} - \sqrt{\left(\frac{1}{\kappa u(0)}\right)^2 - x^2}.$$

We have shown, in particular, that the radius of this arc is greater than the radius r of the tube which means

$$r < \frac{1}{\kappa u(0)} \qquad \text{or equivalently} \qquad u(0) < \frac{1}{\kappa r}. \tag{3.6}$$

Note that this estimate for $u(0)$ is a bound from above. We would really like to obtain a lower bound for $u(0)$ which tells us the limit of $u(0) = u(0; r)$ as r tends to zero. In view of the fact that a semi-circular arc lying everywhere below the graph of $u = u(r)$ gave us the upper bound (3.6) on $u(0)$, it may not be surprising that a semi-circular arc lying everywhere above the graph of u will yield a lower bound on $u(0)$. In fact, this will turn out to be the case.

Since we are going to consider comparisons of u with other circular arcs, we henceforth denote the function whose graph is the lower semi-circular arc above by σ_0, so

$$\sigma_0(x) = u(0) + \frac{1}{\kappa u(0)} - \sqrt{\left(\frac{1}{\kappa u(0)}\right)^2 - x^2}.$$

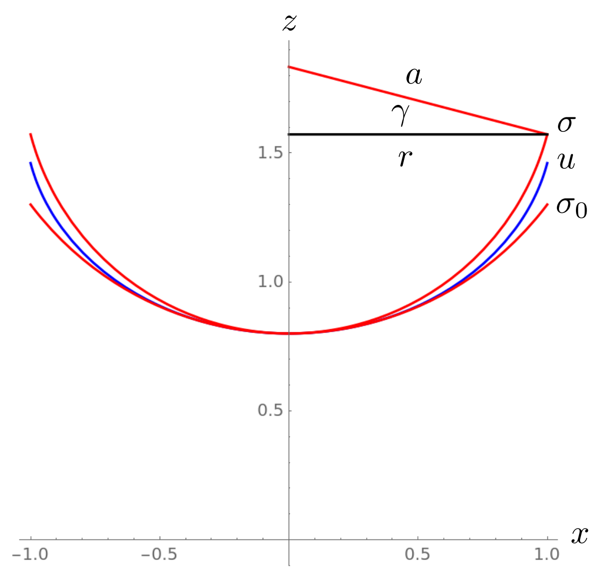

Figure 3.1: Upper and Lower Comparison Circles

Other circular arcs

We have obtained one comparison of the 2-D capillarity meniscus with a circular arc. It follows from this comparison that there holds, in particular,

$$\int_{-r}^{r} \sigma(x)\,dx < \int_{-r}^{r} u(x)\,dx.$$

In order to understand the consequences of this comparison of "raised volumes," we will calculate the integral on the left explicitly. Since we will make other comparisons of circular arcs where the same calculation will be used, it is convenient to consider the calculation somewhat more generally. Recall that we have denoted the particular circular comparison arc considered above by σ_0 instead of σ. We now consider the family of all lower circular arcs with center on the line $x = 0$ and write each such arc as the graph of a function $\sigma : [-a, a] \to \mathbb{R}$ by

$$\sigma(x) = z_0 + a - \sqrt{a^2 - x^2} \tag{3.7}$$

where $a > 0$ is the radius of the spherical arc and z_0 is the lowest (central) point. By taking $z_0 = u(0)$, we obtain a curve tangent to the capillary interface. Let us also denote various quantities associated with such a circular arc by using a subscript σ. The use of this notation should become clear below. We are primarily interested in the case $a \geq r$ in which σ is defined on $[-r, r]$. In this case,

$$\sigma'(r) = \frac{r}{\sqrt{a^2 - r^2}} \tag{3.8}$$

with

$$\sin \psi_\sigma(r) = \frac{\sigma'(r)}{\sqrt{1 + \sigma'(r)^2}} = \frac{r}{a} \tag{3.9}$$

where the quantity ψ_σ is the inclination angle as a function of x along the circular arc. The first expression $\sigma'(r)$ of the inclination is only defined for $a > r$, but the second expression $\sin \psi_\sigma$ is defined for $a \geq r$. In either case we see

$$\frac{d}{da}\sigma'(r) = -\frac{ar}{\sqrt{a^2 - r^2}} < 0 \qquad \text{and} \qquad \frac{d}{da}\sin \psi_\sigma(r) = -\frac{r}{a^2} < 0. \tag{3.10}$$

That is, the inclination of the circular arc at $x = r$ is decreasing with a with $\psi_\sigma(r) = \psi_\sigma(r; a)$ taking the values $\pi/2$ at $a = r$ and limiting to $\psi_\sigma = 0$ as $a \to +\infty$. The corresponding "contact angle" for the circular arcs with radius $a \geq r$ is given by

$$\gamma_\sigma = \frac{\pi}{2} - \psi_\sigma(r).$$

This is the angle at which the circular arc meets the vertical line $x = r$ measured from inside the enclosed region $\{(x, z) : 0 < z < \sigma(x),\ |x| < r\}$, and this angle $\gamma_\sigma = \gamma_\sigma(a)$ increases from 0 (when $a = r$) to $\pi/2$ (when a tends to $+\infty$).

Recall that the contact angle γ satisfies

$$\gamma = \frac{\pi}{2} - \psi(r)$$

where $\psi = \psi(x)$ is the inclination angle of the capillary graph as a function of x given by

$$(\cos\psi, \sin\psi) = \left(\frac{1}{\sqrt{1+u'(x)2}}, \frac{u'(x)}{\sqrt{1+u'(x)2}} \right).$$

The semi-circular enclosed region

We now calculate the area of

$$\{(x,z) : 0 < z < \sigma(x),\ |x| < r\}$$

under the assumption $a \geq r$. On the one hand, we can do this directly by integration:

$$\begin{aligned}
\int_{-r}^{r} \sigma(x)\,dx &= 2\int_0^r \sigma(x)\,dx \\
&= 2(z_0+a)r - 2\int_0^r \sqrt{a^2-x^2}\,dx \\
&= 2(z_0+a)r - 2a^2 \int_0^{\sin^{-1}(r/a)} \cos^2\theta\,d\theta \\
&= 2(z_0+a)r - a^2 \int_0^{\sin^{-1}(r/a)} [1+\cos(2\theta)]\,d\theta \\
&= 2(z_0+a)r - a^2\sin^{-1}(r/a) - a^2 \int_0^{\sin^{-1}(r/a)} \cos(2\theta)\,d\theta \\
&= 2(z_0+a)r - a^2\sin^{-1}(r/a) - \frac{a^2}{2}\sin(2\theta)\Big|_0^{\sin^{-1}(r/a)} \\
&= 2(z_0+a)r - a^2\sin^{-1}(r/a) - ar\cos(\sin^{-1}(r/a)) \\
&= 2(z_0+a)r - a^2\sin^{-1}\left(\frac{r}{a}\right) - r\sqrt{a^2-r^2}.
\end{aligned}$$

We have used the trigonometric substitution $x = a\sin\theta$ according to which $dx = a\cos\theta\,d\theta$ and $\sqrt{a^2-x^2} = a\cos\theta$. This same value can be obtained using elementary geometry, which also provides a check on our integration:

The enclosed area above is given by removing the area of the segment

$$\{(x,z) : \sigma(x) < z < \sigma(r),\ |x| < r\}$$

from the rectangle $[-r, r]\times[0, \sigma(r)]$. The area of the segment is the area A_{sector} of the corresponding sector in the circle determined by the angle $2\psi_\sigma(r) = 2\sin^{-1}(r/a)$ with the area of a triangular region of area $r\sqrt{a^2 - r^2}$ removed:

$$A_{\text{segment}} = A_{\text{sector}} - r\sqrt{a^2 - r^2} = a^2 \sin^{-1}\left(\frac{r}{a}\right) - r\sqrt{a^2 - r^2}$$

since

$$\frac{A_{\text{sector}}}{\pi a^2} = \frac{2\psi_\sigma(r)}{2\pi}.$$

According to this geometric calculation, the area of the enclosed region is

$$\begin{aligned}\int_{-r}^{r} \sigma(x)\,dx &= 2r\sigma(r) - A_{\text{segment}}\\ &= 2r(z_0 + a) - 2r\sqrt{a^2 - r^2} - a^2\sin^{-1}\left(\frac{r}{a}\right) + r\sqrt{a^2 - r^2}\\ &= 2r(z_0 + a) - r\sqrt{a^2 - r^2} - a^2\sin^{-1}\left(\frac{r}{a}\right)\end{aligned}$$

in agreement with our direct calculation by integrating.

We note finally the alternative expression for the area of the segment

$$A_{\text{segment}} = \frac{a^2}{2}\left(2\psi_\sigma(r) - \sin(2\psi_\sigma(r))\right]$$

and

$$\int_{-r}^{r} \sigma(x)\,dx = 2r(z_0 + a) - \frac{a^2}{2}\left(2\psi_\sigma(r) + \sin(2\psi_\sigma(r))\right].$$

Applying this discussion to σ_0, which gives a circular arc of curvature $\kappa u(0)$ matching the lowest point on the meniscus, we know

$$\begin{aligned}\frac{2\cos\gamma}{\kappa} &= \int_{-r}^{r} u(x)\,dx \qquad (3.11)\\ &> \int_{-r}^{r} \sigma_0(x)\,dx\\ &= 2r\left(u(0) + \frac{1}{\kappa u(0)}\right) - r\sqrt{\frac{1}{\kappa^2 u(0)^2} - r^2} - \frac{1}{\kappa^2 u(0)^2}\sin^{-1}\left(r\kappa u(0)\right).\end{aligned}$$

An upper semi-circular arc

We next turn our attention to the lower semi-circular arc determined by σ with $z_0 = u(0)$ and $a_1 = r/\cos\gamma > r$. Here we are starting with a radius $a = a_1 > r$, so the extent of domain of

$$\sigma_1(x) = u(0) + \frac{r}{\cos\gamma} - \sqrt{\frac{r^2}{\cos^2\gamma} - x^2} \tag{3.12}$$

is not in question. The relation between the curvature

$$\frac{1}{a_1} = \frac{\cos\gamma}{r}$$

of the circular arc determinded by σ_1 and $\kappa u(0)$, however, is not immediately clear.

Lemma 26 *The curvature of the semi-circular arc determined by σ_1 with radius $a_1 = r/\cos\gamma$ satisfies*

$$\frac{\cos\gamma}{r} > \kappa u(0) \qquad \textit{or equivalently} \qquad a_1 = \frac{r}{\cos\gamma} < \frac{1}{\kappa u(0)} \tag{3.13}$$

where we recall $1/(\kappa u(0))$ is the radius of the lower semi-circular arc determined by σ_0.

Proof: Note that a general semi-circular arc with radius $a > r$ determined by a function σ as given in (3.7) satisfies

$$\sin\psi_\sigma(r) = \frac{r}{a}$$

and computed in (3.9) above. Theorem 25 also gives

$$0 < \psi_0(r) < \frac{\pi}{2} - \gamma = \psi_1(r) = \psi(r) < \frac{\pi}{2}$$

where ψ_0 is the inclination of the semi-circular graph determined by σ_0 at $x = r$. Applying (3.9) to σ_0 and σ_1 we have

$$\cos\gamma = \frac{r}{a_1} = \sin\psi_1(r) > \sin\psi_0(r) = \kappa u(0)\, r.$$

It follows then that $a_1 < 1/(\kappa u(0))$ as desired. $\square$

Finn's proof of the corresponding result for axially symmetric capillary surfaces is slightly different. In this case Finn would say

$$\sigma_1'(r) = u'(r) > \sigma_0'(r).$$

The calculation of σ' given in (3.8) allows us to write this as

$$\frac{r}{\sqrt{a_1 - r^2}} > \frac{r}{\sqrt{\left(\frac{1}{\kappa u(0)}\right)^2 - r^2}}.$$

Rearranging this inequality gives

$$a_1 < \frac{1}{\kappa u(0)}$$

as desired.

A third alternative proof can be based on the general calculation (3.10). This says the inclination is a decreasing function of a for radii of semi-circular arcs $a > r$. Therefore, since $r < a_1$ and $r < a_0 = 1/(\kappa u(0))$ with the inclination $\sigma_1'(r) = u'(r) > \sigma_0'(r)$, or equivalently $\psi_1(r) = \pi/2 - \gamma > \psi_0(r)$ or, yet again equivalently,

$$\sin\psi_1(r) = \cos\gamma > \sin\psi_0(r),$$

it must be the case that $a_1 < a_0 = 1/(\kappa u(0))$.

Theorem 27 *If $\sigma_1 = \sigma_1(x)$ is given by (3.12) then*

$$\sigma_1(x) > u(x) \qquad \textit{for } 0 < x \le r, \tag{3.14}$$

and

$$\sigma_1'(x) > u'(x) \qquad \textit{for } 0 < x < r \tag{3.15}$$

where $u \in C^2(-r, r) \cap C^1[-r, r]$ is a solution of the 2-D capillary problem (3.1). Note that equality holds in (3.15) for $x = r$.

Will Nute's Proof: Due to the strict curvature inequality

$$\sigma_1''(0) = \frac{\sigma_1''(0)}{(1 + \sigma_1'(0)^2)^{3/2}} = \frac{\cos\gamma}{r} > \kappa u(0) = \frac{u''(0)}{(1 + u'(0)^2)^{3/2}} = u''(0)$$

and the continuity of u'', there is some $\epsilon > 0$ for which

$$\sigma_1''(x) > u''(x) \qquad \text{for } 0 \le x < \epsilon.$$

We have then, by integration,

$$\sigma_1'(x) = \sigma_1'(0) + \int_0^x \sigma_1''(t)\,dt > u'(0) + \int_0^x u''(t)\,dt = u'(x)$$

and consequently

$$\sigma_1(x) = \sigma_1(0) + \int_0^x \sigma_1'(t)\,dt > u(0) + \int_0^x u'(t)\,dt = u(x)$$

for $0 < x \le \epsilon$.

Let us assume

$$A = \{x \in (0,r] : \sigma_1(x) \le u(x)\}$$

is a non-empty set. The set A is then clearly bounded below by ϵ and has a positive infimum $x_1 = \inf A$. By the definition of the infimum we know

1. $\sigma_1(x) > u(x)$ for $0 < x < x_1$ with
2. $\sigma_1(0) = u(0)$ and $\sigma_1(x_1) = u(x_1)$.

It follows from the extreme value theorem that $\sigma_1 - u$ has a positive maximum value on $[0, \epsilon]$ at some $x_0 \in (0, \epsilon)$. Furthermore, we know

$$\sigma_1'(x_0) = u'(x_0) \qquad \text{and} \qquad \sigma_1''(x_0) \le u''(x_0).$$

This implies the curvature inequality

$$\frac{1}{a_1} = \frac{\sigma_1''(x_0)}{(1+\sigma_1'(x_0)^2)^{3/2}} \le \frac{u''(x_0)}{(1+u'(x_0)^2)^{3/2}} = \kappa u(x_0).$$

On the other hand, we know the curvature of the semi-circular arc remains constant on the interval $[x_0, r]$ while the curvature of the meniscus increases. That is,

$$\frac{d}{dx}(\sin\psi_1(x)) \equiv \frac{1}{a_1} \le \kappa u(x) = \frac{d}{dx}(\sin\psi(x)) \qquad \text{for } x_0 \le x \le r$$

with strict inequality except possibly at $x = x_0$. Noting also that

$$\sin\psi_1(x_0) = \frac{\sigma_1'(x_0)}{\sqrt{1+\sigma_1'(x_0)^2}} = \frac{u'(x_0)}{\sqrt{1+u'(x_0)^2}} = \sin\psi(x_0)$$

integration gives

$$\begin{aligned}\sin\psi_1(x) &= \sin\psi_1(x_0) + \int_{x_0}^{x} \frac{d}{dt}(\sin\psi_1(t))\,dt \\ &< \sin\psi(x_0) + \int_{x_0}^{x} \kappa u(t)\,dt \\ &= \sin\psi(x) \qquad \text{for } x_0 < x \le r.\end{aligned}$$

This gives, in particular,

$$\cos\gamma = \sin\psi_1(r) < \sin\psi(r) = \cos\gamma$$

which is a contradiction. This contradiction shows the set A is empty. Consequently, $u(x) < \sigma_1(x)$ for $0 < x \le r$. □

Exercise 3.9 *Will did not prove the inequality between the derivatives, and it is not entirely clear from Figure 3.1 that this inequality holds. Can you give a proof/complete the proof of Theorem 27?*

The inequality between u and σ_1 does imply the inequality for the enclosed regions/lifted volumes:

$$\begin{aligned}\frac{2\cos\gamma}{\kappa} &= \int_{-r}^{r} u(x)\,dx \\ &< \int_{-r}^{r} \sigma_1(x)\,dx \\ &= 2r\left[u(0) + \frac{r}{\cos\gamma}\right] - \frac{r^2}{\cos^2\gamma}\left(\frac{\pi}{2}-\gamma\right) - r\sqrt{\frac{r^2}{\cos^2\gamma} - r^2} \\ &= 2ru(0) + \frac{r^2}{\cos\gamma}\left[2 - \frac{1}{\cos\gamma}\left(\frac{\pi}{2}-\gamma\right) - \sin\gamma\right].\end{aligned}$$

It follows that

$$\frac{\cos\gamma}{\kappa r} - \frac{r}{2\cos\gamma}\left[2 - \frac{1}{\cos\gamma}\left(\frac{\pi}{2}-\gamma\right) - \sin\gamma\right] < u(0) < \frac{\cos\gamma}{\kappa r}. \tag{3.16}$$

We established the last inequality from the simple estimate $u(x) \geq u(0)$ with equality only for $x = 0$ since

$$2\kappa r u(0) < \kappa \int_{-r}^{r} u(t)\, dt = \int_{-r}^{r} \left(\frac{u'(t)}{\sqrt{1 + u'(t)^2}} \right)' dt = 2\cos\gamma.$$

Notice that as $r \searrow 0$, the estimate (3.16) from below and above on $u(0)$ implies that $u(0)$ tends to $+\infty$ like $\cos\gamma/(\kappa r)$. More precisely,

$$\frac{\cos\gamma}{\kappa r} - u(0) = \left| \frac{\cos\gamma}{\kappa r} - u(0) \right| < \frac{r}{\cos\gamma}. \tag{3.17}$$

There are quite a few more directions to consider with respect to the comparison of a 2-D capillary meniscus with circular arcs, and I will summarize some of these in the form of exercises below. Before that, however, let's consider an alternative proof of Theorem 27.

Finn's Proof of Theorem 27

Instead of considering the set A in Nute's proof, Finn considers

$$C = \{c \in (0, r] : u(x) < \sigma_1(x) \text{ for } 0 < x \leq c\}.$$

He then sets $x_1 = \sup C$.

Exercise 3.10 *Show* $\inf A = \sup C$*, so that the value* x_1 *is the same in Finn's proof and Nute's proof.*

Finn cannot show C is empty, but he shows $x_1 = r$. Thus, assuming $x_1 < r$, it is still true that

$$\sigma_1(x) > u(x) \qquad \text{for } 0 < x < x_1$$

with equality $\sigma_1(x_1) = u(x_1)$. He argues then that

$$\psi_1(x_1) \leq \psi(x_1). \tag{3.18}$$

One way to see this is via the difference quotient:

$$(\sigma_1 - u)' = \lim_{h \searrow 0} \frac{\sigma_1(x_1 - h) - u(x_1 - h)}{-h} \leq 0.$$

Thus, $\sigma_1'(x_1) \le u'(x_1)$ and (3.18) follows from the monotonicity of the function $f = f(p)$ by

$$f(p) = \frac{p}{\sqrt{1+p^2}}$$

as noted around (3.3) in the proof of Theorem 25. That is,

$$\sin\psi_1(x_1) \le \sin\psi(x_1).$$

This implies the value of the integral

$$\delta = \int_0^{x_1} \frac{d}{dt}\left(\sin\psi(t) - \sin\psi_1(t)\right)\, dt$$

is positive. This implies the integrand must also be somewhere strictly positive. That is, for some x_0 with $0 < x_0 < x_1$

$$\kappa u(x_0) = \frac{d}{dx}(\sin\psi(x))\Big|_{x=x_0} > \frac{d}{dx}(\sin\psi_1(x))\Big|_{x=x_0} = \frac{\cos\gamma}{r}.$$

As in Nute's proof, the monotonicity of the curvature of the meniscus is now used: We know that for $x > x_0$

$$\kappa u(x) = \frac{d}{dx}\sin\psi(x) > \kappa u(x_0) > \frac{d}{dx}(\sin\psi_1(x)) \equiv \frac{\cos\gamma}{r}.$$

Therefore,

$$0 < \int_{x_1}^{r} \frac{d}{dt}\left(\sin\psi(t) - \sin\psi_1(t)\right)\, dt = \sin\psi(r) - \sin\psi_1(r) - \delta = -\delta < 0.$$

Here the contradiction implies $x_1 = r$. $\square$

It should perhaps be said that this proof of Finn's was adapted from his proof of a nominally more difficult result (page 20), however, I believe Will Nute's approach (which seems to be somewhat more straightforward) will also work in that more complicated context. This needs to be checked.

Here are a couple exercises indicating further directions in the comparison of 2-D capillary graphs with circular arcs. Both are, at some level, aimed at improving the basic estimate (3.16).

Exercise 3.11 *We have shown* $\sigma_1(x) > u(x)$ *for* $0 < x \leq r$. *We also know the curvature of the circular arc determined by* σ_1 *is strictly greater than the curvature* $\kappa u(0)$ *of the meniscus interface at its lowest point. This means that if the radius of a circular arc given by*

$$\sigma(x) = u(0) + a - \sqrt{a^2 - x^2}$$

is increased to a value a slightly greater than $a_1 = r/\cos\gamma$, *then by continuity, it will still be true that* $\sigma(x) > u(x)$ *for* $0 < x \leq r$. *If we let* a_2 *be the supremum of radii for which this is the case, i.e., for which*

$$\sigma(x) = u(0) + a - \sqrt{a^2 - x^2} > u(x) \qquad \text{for } 0 < x \leq r,$$

then show

$$\sigma_2(x) = u(0) + a_2 - \sqrt{a_2^2 - x^2}$$

satisfies

$$\sigma_2(x) > u(x) \qquad \text{for } 0 < x < r$$

with $\sigma_2(r) = u(r)$. *What does this tell you about the comparison of* $\sigma_2'(x)$ *with* $u'(x)$ *for* $0 < x < r$*?*

Exercise 3.12 *Does the circular arc from Exercise 3.11 give any improvement in regard to the estimation of* $u(0)$*? Does it tell you something about* $u(r)$*?*

Exercise 3.13 *We have obtained above the inequality*

$$\frac{2\cos\gamma}{\kappa} > 2r\left(u(0) + \frac{1}{\kappa u(0)}\right) - r\sqrt{\frac{1}{\kappa^2 u(0)^2} - r^2} - \frac{1}{\kappa^2 u(0)^2}\sin^{-1}\left(r\kappa u(0)\right).$$

See (3.11). What does this inequality tell you about $u(0)$*? In particular, does it give an improvement of the upper bound in (3.16)?*

Asymptotics for $u(0)$

In regard to Exercise 3.13, we can attempt to follow Finn's approach to the 3-D axially symmetric central rise height estimate. This would be along the following lines:

Observe

$$\frac{\cos\gamma}{\kappa r} > u(0) + \frac{1}{\kappa u(0)} - \frac{1}{2}\left[\sqrt{\frac{1}{\kappa^2 u(0)^2} - r^2} + \frac{1}{r\kappa^2 u(0)^2}\sin^{-1}\left(r\kappa u(0)\right)\right]. \quad (3.19)$$

We can write this as

$$\frac{\cos\gamma}{\kappa r} > F\left(u(0), \frac{1}{\kappa u(0)}\right)$$

where

$$F(z,a) = z + a - \frac{1}{2}\left[\sqrt{a^2 - r^2} + \frac{a^2}{r}\sin^{-1}\left(\frac{r}{a}\right)\right].$$

We also consider $\Phi : [u(0), 1/(\kappa r)] \to \mathbb{R}$ by

$$\Phi(z) = F\left(z, \frac{1}{\kappa z}\right) = z + \frac{1}{\kappa z} - \frac{1}{2}\left[\sqrt{\frac{1}{\kappa^2 z^2} - r^2} + \frac{1}{r\kappa^2 z^2}\sin^{-1}\left(r\kappa z\right)\right].$$

We claim first that

$$\Phi(u(0)) < \frac{\cos\gamma}{\kappa r} < \Phi\left(\frac{1}{\kappa r}\right).$$

The first inequality is just (3.19) which follows from the comparison of enclosed areas involving σ_0 and u. For the inequality on the right, we observe

$$\Phi\left(\frac{1}{\kappa r}\right) = \frac{1}{\kappa r} + r - \frac{r}{2}\left[\sin^{-1}(1)\right]$$
$$= \frac{1}{\kappa r} + r\left[1 - \frac{\pi}{4}\right].$$

Since $1/(\kappa r) > \cos\gamma/(\kappa r)$ and $1 - \pi/4 > 0$, the inequality on the right holds.

Next we claim $\Phi'(z) > 1$ for $u(0) \le z < 1/(\kappa r)$ with

$$\lim_{z \nearrow 1/(\kappa r)} \Phi'(z) = 1 + \kappa r[\sin^{-1}(r) - r].$$

To see this, note that

$$\Phi'(z) = \frac{\partial F}{\partial z} - \frac{1}{\kappa z^2}\frac{\partial F}{\partial a} = 1 - \frac{1}{\kappa z^2}\frac{\partial F}{\partial a}$$

where

$$\frac{\partial F}{\partial a} = \frac{\partial F}{\partial a}\left(z, \frac{1}{\kappa z}\right) = p\left(\frac{1}{\kappa z}\right)$$

and $p : (r, 1/(\kappa u(0))] \to \mathbb{R}$ by

$$\begin{aligned} p(a) &= 1 - \frac{1}{2}\left[\frac{a}{\sqrt{a^2 - r^2}} + \frac{2a}{r}\sin^{-1}\left(\frac{r}{a}\right) - \frac{a}{\sqrt{a^2 - r^2}}\right] \\ &= 1 - \frac{a}{r}\sin^{-1}\left(\frac{r}{a}\right) \\ &= 1 - \sin^{-1}(t)/t. \end{aligned} \tag{3.20}$$

Since

$$\lim_{t\searrow 0}\frac{\sin^{-1}(t)}{t} = \lim_{t\searrow 0}\frac{1}{\sqrt{1-t^2}} = 1 \qquad \text{and} \qquad \sin^{-1}(t) > t \quad \text{for } 0 < t < 1,$$

we see $p < 0$ and $\Phi' > 1$. The limiting value follows simply from substitution since the expression in (3.20) is nonsingular at $a = r$.

We have shown there exists a unique value z_* for which

$$\Phi(z_*) = z_* + \frac{1}{\kappa z_*} - \frac{1}{2}\left[\sqrt{\frac{1}{\kappa^2 z_*^2} - r^2} + \frac{1}{r\kappa^2 z_*^2}\sin^{-1}(r\kappa z_*)\right] = \frac{\cos\gamma}{\kappa r}. \tag{3.21}$$

We know also from the monotonicity that

$$u(0) < z_*.$$

We claim also that

$$\Phi\left(\frac{\cos\gamma}{\kappa r}\right) > \frac{\cos\gamma}{\kappa r}. \tag{3.22}$$

And this means

$$u(0) < z_* < \frac{\cos\gamma}{\kappa r}.$$

To see (3.22) we can compute

$$\begin{aligned}\Phi\left(\frac{\cos\gamma}{\kappa r}\right) &= \frac{\cos\gamma}{\kappa r} + \frac{r}{\cos\gamma} - \frac{r}{2\cos\gamma}\left[\sin\gamma + \frac{1}{\cos\gamma}\sin^{-1}(\cos\gamma)\right]\\ &= \frac{\cos\gamma}{\kappa r} + \frac{r}{2\cos\gamma}\left[2 - \sin\gamma - \frac{1}{\cos\gamma}\left(\frac{\pi}{2} - \gamma\right)\right]\\ &= \frac{\cos\gamma}{\kappa r} + \frac{r}{4\cos^2\gamma}\left[4\cos\gamma - \sin(2\gamma) - \pi + 2\gamma\right].\end{aligned}$$

Introducing the function $p : [0, \pi/2] \to \mathbb{R}$ by

$$p(\gamma) = 4\cos\gamma + 2\gamma - \sin(2\gamma) - \pi,$$

we see

$$p(0) = 4 - \pi > 0; \qquad p\left(\frac{\pi}{2}\right) = 0,$$

and

$$p'(\gamma) = -4\sin\gamma + 2 - 2\cos(2\gamma),$$

so that

$$p'(0) = 0 \qquad \text{and} \qquad p'\left(\frac{\pi}{2}\right) = 0.$$

Now if we assume $p'(\gamma) = 0$ for any $\gamma \in (0, \pi/2)$, then we know there are two values γ_1 and γ_2 with $0 < \gamma_1 < \gamma_2 < \pi/2$ and $p''(\gamma_1) = 0 = p''(\gamma_2)$. But

$$p''(\gamma) = 4(\sin(2\gamma) - \cos\gamma) = -4\cos\gamma(1 - 2\sin\gamma),$$

and $\cos\gamma \neq 0$ for $0 < \gamma < \pi/2$ and $1 - 2\sin\gamma = 0$ only for $\gamma = \pi/6$, so there are not two distinct arguments γ where p'' vanishes, and $p'(\gamma) < 0$ for $0 < \gamma < \pi/2$. This implies $p(\gamma) > 0$ for $0 \le \gamma < \pi/2$ and, in particular,

$$\Phi\left(\frac{\cos\gamma}{\kappa r}\right) = \frac{\cos\gamma}{\kappa r} + \frac{r}{4\cos^2\gamma}\,p(\gamma) > \frac{\cos\gamma}{\kappa r}.$$

We have established the following result:

Theorem 28 *If $\kappa > 0$ and $\gamma \in (0, \pi/2)$ are given and fixed and $u \in C^2(-r, r) \cap C^1[-r, r]$ is a solution of (3.1), then*

$$\frac{\cos\gamma}{\kappa r} - \frac{r}{2\cos\gamma}\left[2 - \frac{1}{\cos\gamma}\left(\frac{\pi}{2} - \gamma\right) - \sin\gamma\right] < u(0) < z_* < \frac{\cos\gamma}{\kappa r}$$

where z_* is determined uniquely by the equation

$$z_* + \frac{1}{\kappa z_*} - \frac{1}{2}\left[\sqrt{\frac{1}{\kappa^2 z_*^2} - r^2} + \frac{1}{r\kappa^2 z_*^2}\sin^{-1}(r\kappa z_*)\right] = \frac{\cos\gamma}{\kappa r}.$$

The estimate from below in Theorem 28 tells us that both $u(0)$ and z_* tend to infinity as r tends to zero. We have furthermore an estimate on how much these values can differ from each other and from the growth term

$$\frac{\cos\gamma}{\kappa r}.$$

In particular, we know

$$0 < \frac{\cos\gamma}{\kappa r} - z_* < \frac{\cos\gamma}{\kappa r} - u(0) < \frac{r}{2\cos\gamma}\left[2 - \frac{1}{\cos\gamma}\left(\frac{\pi}{2} - \gamma\right) - \sin\gamma\right].$$

We seek now to improve this estimate.

The following approach is called **formal asymptotic expansion**. We assume $u(0)$ and z_* can be written in the following forms

$$u(0) = \frac{\cos\gamma}{\kappa r} + \sum_{j=0}^{\infty} a_j r^j \qquad \text{and} \qquad z_* = \frac{\cos\gamma}{\kappa r} + \sum_{j=0}^{\infty} b_j r^j.$$

Under this assumption, our estimate above already tells us that we must have $a_0 = b_0 = 0$. Thus, we start with

$$u(0) = \frac{\cos\gamma}{\kappa r} + \sum_{j=1}^{\infty} a_j r^j \qquad \text{and} \qquad z_* = \frac{\cos\gamma}{\kappa r} + \sum_{j=1}^{\infty} b_j r^j. \tag{3.23}$$

Let us consider only z_*. In fact, it is not difficult to implement a numerical rootfind algorithm to obtain values of $z_* = z_*(r)$; see Figure 3.2.

Under the formal assumption (3.23), the defining equation for z_* can be written as

$$\sum_{j=1}^{\infty} b_j r^j + \frac{1}{\kappa} T_1 - \frac{1}{2} T_2 - \frac{1}{2} T_3 = 0$$

where

$$T_1 = z_*^{-1} = \left(\frac{\cos\gamma}{\kappa r} + \sum_{j=1}^{\infty} b_j r^j\right)^{-1},$$

$$T_2 = \sqrt{\frac{1}{\kappa^2} T_1^2 - r^2},$$

$$T_3 = \frac{1}{r\kappa^2} T_1^2 \; P,$$

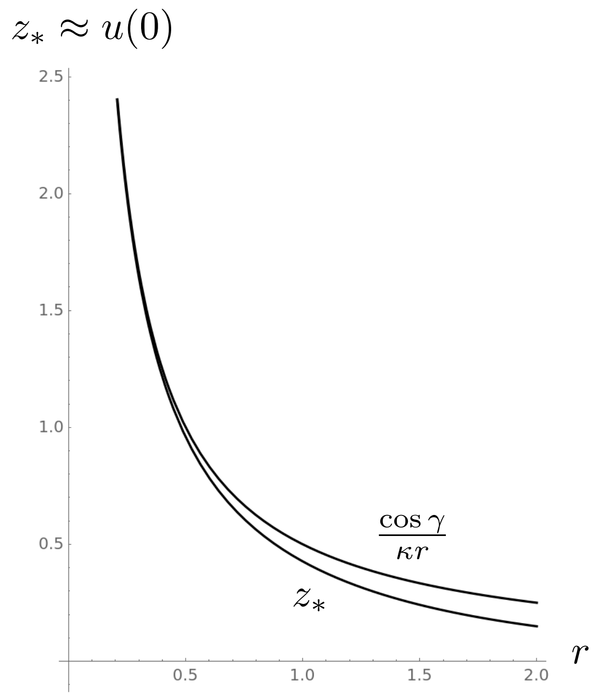

Figure 3.2: Asymptotic approximation of $u(0)$. (The computation for this figure was with $\kappa = 1$ and $\gamma = \pi/4$.)

$$T_1^2 = \left(\frac{\cos\gamma}{\kappa r} + \sum_{j=1}^{\infty} b_j r^j \right)^{-2},$$

and

$$P = \sin^{-1}\left(\cos\gamma + r\kappa \sum_{j=1}^{\infty} b_j r^j \right).$$

In analyzing/expanding the terms T_1, T_2, and T_3 a couple preliminary comments may be useful. First, we will use the "big O" notation as follows:

$$\sum_{j=1}^{\infty} b_j r^j = b_1 r + \mathbf{O}(r^2) = b_1 r + b_2 r^2 + \mathbf{O}(r^3) = \cdots = \sum_{j=1}^{k} b_j r^j + \mathbf{O}(r^{j+1})$$

where $\mathbf{O}(r^k)$ may be read as "terms of order k and higher." Technically,

$\mathbf{O}(r^k)$ represents a function of r having the property that

$$\frac{\mathbf{O}(r^k)}{r^k} \qquad \text{is bounded.}$$

That is, there exists a constant C independent of r for which

$$\left| \frac{\mathbf{O}(r^k)}{r^k} \right| \leq C.$$

Thus,

$$\lim_{r \to 0} \frac{\mathbf{O}(r^k)}{r^{k-1}} = 0.$$

This kind of condition can also be expressed with the "small O" notation, so that $\mathbf{o}(r^k)$ represents some function of r for which

$$\lim_{r \to 0} \frac{\mathbf{o}(r^k)}{r^k} = 0.$$

We will also use power series expansions and a generalized binomial expansion in particular. Thus, for a real analytic function of x expanded at $x = x_0$, we can write

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x)}{j!}(x - x_0)^j$$

and

$$(a+b)^p = a^p + \sum_{j=1}^{\infty} \frac{p(p-1)(p-2)\cdots[p-(j-1)]}{j!} a^{p-j} b^j.$$

Taking $p = -1$ in this last formula, we get

$$\begin{aligned} T_1 &= \left(\frac{\cos\gamma}{\kappa r} + \sum_{j=1}^{\infty} b_j r^j \right)^{-1} \\ &= \frac{\kappa r}{\cos\gamma} - \frac{\kappa^2 r^2}{\cos^2\gamma} \sum_{j=1}^{\infty} b_j r^j + \frac{(-1)(-2)}{2!} \frac{\kappa^3 r^3}{\cos^3\gamma} \left(\sum_{j=1}^{\infty} b_j r^j \right)^2 + \mathbf{O}(r^7) \\ &= \frac{\kappa r}{\cos\gamma} - b_1 \frac{\kappa^2 r^3}{\cos^2\gamma} - b_2 \frac{\kappa^2 r^4}{\cos^2\gamma} + \left(b_1^2 \frac{\kappa^3}{\cos^3\gamma} - b_3 \frac{\kappa^2}{\cos^2\gamma} \right) r^5 \\ &\qquad + \left(2b_1 b_2 \frac{\kappa^3}{\cos^3\gamma} - b_4 \frac{\kappa^2}{\cos^2\gamma} \right) r^6 + \mathbf{O}(r^7). \end{aligned}$$

Similarly, taking the power $p = -2$ in the generalized binomial expansion, we get

$$\begin{aligned}
T_1^2 &= \left(\frac{\cos\gamma}{\kappa r} + \sum_{j=1}^{\infty} b_j r^j\right)^{-2} \\
&= \frac{\kappa^2 r^2}{\cos^2\gamma} - 2\frac{\kappa^3 r^3}{\cos^3\gamma}\sum_{j=1}^{\infty} b_j r^j + \frac{(-2)(-3)}{2!}\frac{\kappa^4 r^4}{\cos^4\gamma}\left(\sum_{j=1}^{\infty} b_j r^j\right)^2 + \mathbf{O}(r^8) \\
&= \frac{\kappa^2 r^2}{\cos^2\gamma} - 2b_1\frac{\kappa^3 r^4}{\cos^3\gamma} - 2b_3\frac{\kappa^3 r^5}{\cos^3\gamma} + \left(3b_1^2\frac{\kappa^4}{\cos^4\gamma} - 2b_3\frac{\kappa^3}{\cos^3\gamma}\right) r^6 + \mathbf{O}(r^7).
\end{aligned}$$

From this expansion and using the power $p = 1/2$ we obtain

$$\begin{aligned}
T_2 &= \sqrt{\frac{1}{\kappa^2}\,T_1^2 - r^2} \\
&= \frac{1}{\kappa}\left(\kappa^2\tan^2\gamma\; r^2 - 2b_1\frac{\kappa^3 r^4}{\cos^3\gamma} - 2b_3\frac{\kappa^3 r^5}{\cos^3\gamma} + \mathbf{O}(r^6)\right)^{1/2} \\
&= \frac{1}{\kappa}\left(\kappa\tan\gamma\; r - \frac{\cot\gamma}{\kappa r}\left(b_1\frac{\kappa^3 r^4}{\cos^3\gamma} + b_3\frac{\kappa^3 r^5}{\cos^3\gamma}\right)\right. \\
&\qquad\qquad\qquad \left. + \left(\frac{(1/2)(-1/2)}{2}\;\frac{\cot^3\gamma}{\kappa^3 r^3}\;\mathbf{O}(r^8)\right)\right) + \mathbf{O}(r^5) \\
&= \tan\gamma\; r - \frac{\cot\gamma}{\kappa^2 r}\left(b_1\frac{\kappa r^4}{\cos^3\gamma} + b_3\frac{\kappa^3 r^5}{\cos^3\gamma}\right) + \mathbf{O}(r^5) \\
&= \tan\gamma\; r - b_1\frac{\kappa^2 r^3}{\cos^2\gamma\sin\gamma} - b_3\frac{\kappa r^4}{\cos^2\gamma\sin\gamma} + \mathbf{O}(r^5).
\end{aligned}$$

Next, we use the Taylor expansion for $\sin^{-1}(x)$ with center at $x = \cos\gamma$ to obtain the expansion for P. Recall that $\sin^{-1}(\cos\gamma) = \pi/2 - \gamma$ and note that

$$(\sin^{-1})'(x) = \frac{1}{\sqrt{1-x^2}}, \qquad (\sin^{-1})''(x) = \frac{x}{(1-x^2)^{3/2}},$$

$$(\sin^{-1})'''(x) = \frac{2x^2+1}{(1-x^2)^{5/2}}, \qquad \text{and} \qquad (\sin^{-1})''''(x) = \frac{3x(2x^2+3)}{(1-x^2)^{7/2}},$$

so that

$$
\begin{aligned}
&(\sin^{-1})'(\cos\gamma) = \csc\gamma,\\
&(\sin^{-1})''(\cos\gamma) = \cot\gamma\csc^2\gamma,\\
&(\sin^{-1})'''(\cos\gamma) = (2\cos^2\gamma+1)\csc^5\gamma = (3\cot^2\gamma+1)\csc^3\gamma, \text{ and}\\
&(\sin^{-1})''''(\cos\gamma) = 3\cos\gamma(5\cos^2\gamma+3\sin^2\gamma)\csc^7\gamma = 3\cot\gamma(5\cot^2\gamma+3)\csc^4\gamma,
\end{aligned}
$$

and

$$
\begin{aligned}
\sin^{-1}(x) = \frac{\pi}{2} - \gamma \ + \ \csc\gamma(x-\cos\gamma) \ + \ &\frac{\cos\gamma\csc^2\gamma}{2}(x-\cos\gamma)^2\\
&+\frac{1}{3!}(\sin^{-1})'''(\cos\gamma)(x-\cos\gamma)^3 + \frac{1}{4!}(\sin^{-1})''''(\cos\gamma)(x-\cos\gamma)^4\\
&+\mathbf{O}(x-\cos\gamma)^5.
\end{aligned}
$$

This gives

$$
\begin{aligned}
P &= \sin^{-1}(\kappa r z_*)\\
&= \sin^{-1}\left(\cos\gamma + r\kappa\sum_{j=1}^{\infty} b_j r^j\right)\\
&= \frac{\pi}{2} - \gamma + \cos\gamma\ r\kappa\sum_{j=1}^{\infty} b_j r^j + \frac{\cos\gamma\csc^2\gamma}{2}\left(r\kappa\sum_{j=1}^{\infty} b_j r^j\right)^2\\
&\qquad + \frac{1}{3!}(\sin^{-1})'''(\cos\gamma)\left(r\kappa\sum_{j=1}^{\infty} b_j r^j\right)^3\\
&\qquad\qquad + \frac{1}{4!}(\sin^{-1})''''(\cos\gamma)\left(r\kappa\sum_{j=1}^{\infty} b_j r^j\right)^4 + \mathbf{O}(r^4)\\
&= \frac{\pi}{2} - \gamma + \kappa\cos\gamma\ b_1 r^2 + \kappa\cos\gamma\ b_2 r^3\\
&\qquad + \left(\kappa\cos\gamma\ b_3 + \kappa^2\frac{\cos\gamma\csc^2\gamma}{2}\ b_1\right) r^4 + \mathbf{O}(r^4)\\
&= \frac{\pi}{2} - \gamma + \kappa\cos\gamma\ b_1 r^2 + \kappa\cos\gamma\ b_2 r^3 + \mathbf{O}(r^4).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
T_3 &= \frac{1}{r\kappa^2} T_1^2 \; P \\
&= \frac{1}{r\kappa^2}\left(\frac{\pi}{2} - \gamma + \kappa\cos\gamma \; b_1 r^2 + \kappa\cos\gamma \; b_2 r^3 + \mathbf{O}(r^4)\right) \\
&\qquad \left(\frac{\kappa^2 r^2}{\cos^2\gamma} - 2b_1\frac{\kappa^3 r^4}{\cos^3\gamma} - 2b_3\frac{\kappa^3 r^5}{\cos^3\gamma} + \left(3b_1^2\frac{\kappa^4}{\cos^4\gamma} - 2b_3\frac{\kappa^3}{\cos^3\gamma}\right) r^6 + \mathbf{O}(r^7)\right) \\
&= \left(\frac{\pi}{2} - \gamma + \kappa\cos\gamma \; b_1 r^2 + \kappa\cos\gamma \; b_2 r^3 + \mathbf{O}(r^4)\right) \\
&\qquad \left(\frac{r}{\cos^2\gamma} - 2b_1\frac{\kappa r^3}{\cos^3\gamma} - 2b_3\frac{\kappa r^4}{\cos^3\gamma} + \left(3b_1^2\frac{\kappa^2}{\cos^4\gamma} - 2b_3\frac{\kappa}{\cos^3\gamma}\right) r^5 + \mathbf{O}(r^6)\right) \\
&= \frac{1}{\cos^2\gamma}\left(\frac{\pi}{2} - \gamma\right) r + b_1\left(\frac{\kappa}{\cos\gamma} - 2\frac{\kappa}{\cos^3\gamma}\left(\frac{\pi}{2} - \gamma\right)\right) r^3 \\
&\qquad + \left(b_2\frac{\kappa}{\cos\gamma} - 2b_3\frac{\kappa}{\cos^3\gamma}\left(\frac{\pi}{2} - \gamma\right)\right) r^4 + \mathbf{O}(r^5).
\end{aligned}
$$

Returning to the defining equation

$$\sum_{j=1}^{\infty} b_j r^j + \frac{1}{\kappa} T_1 - \frac{1}{2} T_2 - \frac{1}{2} T_3 = 0$$

we have

$$
\begin{aligned}
&\left(b_1 + \frac{1}{\cos\gamma} - \frac{\tan\gamma}{2} - \frac{1}{2\cos^2\gamma}\left(\frac{\pi}{2} - \gamma\right)\right) r + b_2 r^2 \\
&\quad + \left(b_3 + b_1\frac{\kappa}{\cos^2\gamma} + b_1\frac{\kappa^2}{2\cos^2\gamma\sin\gamma} - b_1\frac{\kappa}{2\cos\gamma}\left(1 - \frac{2}{\cos^2\gamma}\left(\frac{\pi}{2} - \gamma\right)\right)\right) r^3 \\
&\qquad + \mathbf{O}(r^4) = 0.
\end{aligned}
$$

Equating coefficients, we find

$$b_1 = \frac{1}{2\cos\gamma}\left(\sin\gamma + \frac{1}{\cos\gamma}\left(\frac{\pi}{2} - \gamma\right) - 2\right),$$

$b_2 = 0$, and

$$
\begin{aligned}
b_3 &= \frac{\kappa}{2\cos^2\gamma}\left(\cos\gamma - \frac{2}{\cos\gamma}\left(\frac{\pi}{2} - \gamma\right) - \frac{\kappa}{\sin\gamma} - 2\right) b_1 \\
&= \frac{\kappa}{4\cos^3\gamma}\left(2 + \frac{2}{\cos\gamma}\left(\frac{\pi}{2} - \gamma\right) + \frac{\kappa}{\sin\gamma} - \cos\gamma\right)\left(2 - \frac{1}{\cos\gamma}\left(\frac{\pi}{2} - \gamma\right) - \sin\gamma\right) \\
&> 0.
\end{aligned}
$$

It will be noted, first of all, that the formal coefficient b_1 matches the coefficient of r appearing in the lower bound for $u(0)$ in Theorem 28. In particular, we have confirmed that the formal expansion for $u(0)$ should also have this coefficient for the first order term. The same comment applies to the (zero) coefficent of r^2. In particular, substituting the formal expansion for z_* into the estimate of Theorem 28 we obtain the formal improvement

$$\mathfrak{L}_1(r) < u(0) < \mathfrak{L}_1(r) + \mathbf{O}(r^3)$$

where

$$\mathfrak{L}_1(r) = \frac{\cos\gamma}{\kappa r} - \frac{r}{2\cos\gamma}\left[2 - \frac{1}{\cos\gamma}\left(\frac{\pi}{2} - \gamma\right) - \sin\gamma\right].$$

We have not shown, in fact, either that the expansion of z_* is valid (even to first order) nor that this estimate holds. I believe it is a correct estimate, however, and can be shown. This belief is based on the assumption that there is a valid and complete asymptotic expansion[1] and that if there is a valid asymptotic expansion, it must agree with the formal asymptotic expansion obtained above.

Exercise 3.14 *Show carefully with estimates that we do have a valid zero order asymptotic expansion so that*

$$0 < \frac{\cos\gamma}{\kappa r} - u(0) = \mathbf{O}(r)$$

and

$$0 < \frac{\cos\gamma}{\kappa r} - z_* = \mathbf{O}(r).$$

In particular, $u(0) \sim \cos\gamma/(\kappa r) + a_0$ *and* $b_* \sim \cos\gamma/(\kappa r) + b_0$ *with* $a_0 = b_0 = 0$.

3.15 Parametric Solutions

An important fact from the theory of elementary ODEs is that any (regular) ODE is equivalent to a first order system of ODEs. There are various ways

[1] This is almost certainly true as it is true for the 3-D axially symmetric capillary problem. The full asymptotic expansion in the 3-D axially symmetric case was obtained by Erich Miersemann. He also obtained his result for every value of $\sqrt{x^2+y^2}$, depending on $\sqrt{x^2+y^2}$. This expansion is not in Finn's book. It is/should be in the papers of Miersemann, though I have not read them in detail. I also have lecture notes from Miersemann.

one can write down a system equivalent to the 2-D capillary interface equation

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \kappa u.$$

One way was discussed in Exercise 1.13(b) where the first equation is $u' = v$, and the second equation is

$$v' = \kappa u(1+v^2)^{3/2}.$$

We consider next two crucial observations:

1. If one writes the 2-D capillary interface equation as a system of equations in a way that allows parametric curves for solutions (instead of just graphs), then one finds there are such solutions. These solutions contain the graphs we have considered above, and they offer a perspective on those graphs which can be extremely useful in various ways.

2. The condition on a curve that its signed curvature is a linear function of height makes perfectly good sense for parametric curves.

Just to be clear, when we talk about a **parametric curve** we mean a curve which may not project simply as a graph. We have considered semicircular graphs given as

$$\sigma(x) = z_0 + a - \sqrt{a^2 - x^2}$$

above. But as curves of constant curvature, it is much more natural to consider circles as parametric curves with a **parameterization** along the lines of

$$\alpha(\theta) = a(\cos\theta, \sin\theta). \tag{3.24}$$

Any parameterization $\alpha : (a, b) \to \mathbb{R}^2$ of a planar curve is said to be nonsingular if the "velocity" vector α' does not vanish. And any such curve may be reparameterized by **arclength**. If a nonsingular curve α has parameter $t \in (a, b)$, then the arclength can be written as

$$s = \int_{t_0}^{t} |\alpha'(\tau)|\, d\tau. \tag{3.25}$$

Interpreting this integral in terms of speed (with time parameter t) and a Riemann sum, it is a manifestation of the familiar formula

$$(\text{rate}) \;\times\; (\text{time}) \;=\; (\text{distance})$$

where s is the "distance" (traveled by a point moving along the curve). The identity (3.25) moreover may be interpreted in two important ways. First, we can think of (3.25) as defining s as a function of the parameter t. From this point of view, the fundamental theorem of calculus tells us

$$\frac{ds}{dt} = |\alpha'(t)| > 0.$$

Having made this observation, we see $s = s(t)$ is an invertible function. (Any real valued function which has a positive derivative on an interval is increasing and has an increasing inverse.) Let us denote the co-domain of $s = s(t)$ by $(-\ell, m)$ where one or both of ℓ and m may be $+\infty$. Then we have an inverse function $t : (-\ell, m) \to (a, b)$ giving the time required to travel along the curve a "distance" $s \in (-\ell, m)$ (where distance, of course, can be positive or negative to indicate direction starting from t_0). This is the second way to interpret (3.25): We can think of

$$s = \int_{t_0}^{t} |\alpha'(\tau)|\, d\tau$$

as defining $t = t(s)$. Then using the fundamental theorem of calculus and the chain rule, we find

$$1 = |\alpha'(t)|\, \frac{dt}{ds},$$

so

$$\frac{dt}{ds} = \frac{1}{\alpha'(t)}. \tag{3.26}$$

It is from this second interpretation that we can obtain a **parameterization by arclength** of a nonsingular curve. This works as follows: We define a new parameterization $\gamma : (-\ell, m) \to \mathbb{R}^2$ given by

$$\gamma(s) = \alpha(t(s)).$$

Now by the chain rule and (3.26) we find

$$\dot{\gamma}(s) = \frac{\alpha'}{|\alpha'|} \tag{3.27}$$

which is a unit vector corresponding to the fact that α is a unit speed parameterization (as expected).

There are couple technicalities to note here. Technically, we should use different symbols to denote "s the arclength function" and "s the arclength parameter," but we're using the same symbol for both of them (and hoping context will tell us what we mean, or at least that there is not too much confusion as a result. In particular, it should be noted that when we write

$$\frac{ds}{dt} = |\alpha'|$$

we have $\alpha' = \alpha'(t)$, so the independent parameter t is appearing on the right. When we write

$$\frac{dt}{ds} = \frac{1}{|\alpha'|}$$

then $\alpha' = \alpha'(t) = \alpha'(t(s))$, and we are looking at a fundamentally implicit formula. In particular, it may not always be possible to write down an explicit formula expressing the function $t = t(s)$ appearing in the limit of integration in (3.25).

Exercise 3.16 *Reparameterize the circle defined in (3.24) in terms of arclength.*

The use of a "dot" to denote derivatives with respect to arclength, as opposed to a "prime" for derivatives with respect to some other parameter, as in (3.27) is intended to help mitigate the confusion between these parameters.

Applying this discussion to a 2-D capillary graph given by $u = u(x)$, let us attempt to write such a graph as a parameterized curve $\gamma(s) = (x(s), z(s))$ in terms of an arclength parameter. In fact, the parameterization $\alpha(x) = (x, u(x))$ is nonsingular since $\alpha' = (1, u')$ does not vanish. In view of the relation

$$z(s) = u(x(s))$$

we have $\dot{z} = u'(x(s))\,\dot{x}$, so $\dot{\gamma} = \dot{x}(1, u'(x))$, and we obtain the relation

$$\dot{x} = \frac{1}{\sqrt{1 + u'(x)^2}}$$

as long as $\dot{x} > 0$. This means

$$\dot{\gamma} = (\dot{x}, \dot{z}) = \left(\frac{1}{\sqrt{1 + u'(x)^2}}, \frac{u'(x)}{\sqrt{1 + u'(x)^2}} \right).$$

It will be observed that everything we have obtained so far applies to any graph. At this point, we may introduce the inclination angle ψ as a third dependent variable and eliminate u using the 2-D capillary equation to obtain

$$\begin{cases} \dot{x} = \cos\psi \\ \dot{z} = \sin\psi \\ \dot{\psi} = \kappa z. \end{cases} \tag{3.28}$$

The last equation involves a function $\psi = \psi(s)$ having the same name as the non-parametric inclination $\psi = \psi(x)$ considered above, but considered as a function of the arclength parameter. To obtain the third equation completing this parametric system, it may be useful to temporarily use different symbols for these two functions. Let us denote the new inclination angle (as a function of arclength appearing in (3.28) by $\Psi = \Psi(s)$. Then we have the relation $\Psi(s) = \psi(x(s))$. Thus,

$$\dot{\Psi} = \psi'(x(s))\,\dot{x} = \psi'(x(s))\,\cos\psi(x(s)) = \psi'(x(s))\,\cos\Psi(s).$$

In order to determine the value of $\psi'(x(s))$, we can simply recall that

$$\frac{d}{dx}(\sin\psi(x)) = \kappa u(x)$$

and observe

$$\frac{d}{dx}(\sin\psi(x)) = \cos\psi(x)\frac{d}{dx}\psi(x) = \cos\psi(x)\psi'(x).$$

In particular,

$$\cos\psi(x(s))\psi'(x(s)) = \cos\Psi(s)\psi'(x(s)) = \kappa u(x(s)) = \kappa z(s).$$

Having made this calculation and obtained (3.28), it turns out this system essentially supersedes the non-parametric version of the capillary equation, so it is convenient to adopt the symbol ψ for the parametric inclination $\psi = \psi(s)$ and sometimes (if not henceforth) use a different symbol for the non-parametric inclination angle if a distinction needs to be made.

Since we're on this topic of symbols, it may be pointed out that we are intending to use the symbol x for the horizontal component $x = x(s)$ along a 2-D capillary interface/interfacial curve. We will continue, for the most part, to use the same symbol for the independent horizontal variable x, again, hoping the context will make clear the role played by this particular abused symbol.

The 2-D capillary graphs considered above (in the 2-D capillary tube problem) are distinguished among the solutions of (3.28) by the appended initial values

$$\begin{cases} \dot{x} = \cos\psi, & x(0) = 0 \\ \dot{z} = \sin\psi, & z(0) = z_0 \\ \dot{\psi} = \kappa z, & \psi(0) = 0. \end{cases} \tag{3.29}$$

It may be observed that beginning with the arclength $s = 0$ at the central meniscus point $(0, u(0)) = (0, z_0)$ is natural but also somewhat arbitrary.

Exercise 3.17 *Write down the initial value problem for (3.28) with $s = 0$ corresponding to the left point $(-r, u(-r))$ or the right point $(r, u(r))$.*

It should be immediately remarked that we have not discussed existence and uniqueness for the 2-D capillary boundary value problem. We have mostly assumed the existence of solutions and attempted to obtain consequences of that existence. We do have (at least) local existence and uniqueness for the system (3.28) and for the system (3.29) in particular. If local solutions of (3.29) around $s = 0$ extend to global solutions giving graphs that solve (3.1) remains to be seen.

3.18 Solving the ODE

The following is generally not the most useful direction for solving the 2-D capillary equation, but it is suggestive that an explicit solution of some form should be available. Also, this approach may have some interesting consequences/uses, and I have not seen it elsewhere, and I think it provides a kind of nice introduction. If we start with

$$\frac{u''}{(1 + u'^2)^{3/2}} = \kappa u$$

we can multiply both sides by u' and write

$$\frac{u'}{(1 + u'^2)^{3/2}}\, u'' = \kappa u u' = \frac{\kappa}{2}(u^2)'.$$

Integrating, say from $x = 0$ where $u = u(0)$ and $u'(0) = 0$, we get

$$\int_0^{u'} \frac{t}{(1+t^2)^{3/2}}\,dt = \frac{\kappa}{2}[u^2 - u(0)^2]$$

or using the change of variables $v = 1 + t^2$,

$$\int_1^{1+u'^2} v^{-3/2}\,dv = -2\left[\frac{1}{\sqrt{1+u'^2}} - 1\right] = \kappa[u^2 - u(0)^2].$$

From this we obtain

$$\frac{1}{\sqrt{1+u'^2}} = 1 - \frac{\kappa}{2}[u^2 - u(0)^2] = p(u)$$

where $p(u) = -\kappa u^2/2 + 1 + \kappa u(0)^2/2$ is an even quadratic polynomial in u. Rearranging this equation we find

$$\frac{p}{\sqrt{1-p^2}}\,u' = \pm 1$$

or

$$\int_{u(0)}^{u} \frac{p(t)}{\sqrt{1-p(t)^2}}\,dt = \pm x.$$

Thus, we see $u = u(x)$ is given implicitly in terms of the function

$$g(u) = \int_{u(0)}^{u} \frac{p(t)}{\sqrt{q(t)}}\,dt$$

where q is the even quartic polynomial $q(t) = 1 - p(t)^2$. Functions having the form of g are called **elliptic integrals**. Here is some background:

Elliptic Integrals

You may remember from calculus that there are certain integrals expressible in terms of "elementary functions." Elaborating on this observation a little, every polynomial $p = p(x)$ has an antiderivative

$$P(x) = \int_a^x p(t)\,dt$$

which is also a polynomial. In particular, of we let $\mathcal{P} = \mathbb{R}[x]$ denote the ring of polynomials with real coefficients, then $\mathcal{P}$ is a vector space over $\mathbb{R}$, and differentiation

$$\frac{d}{dx} : \mathcal{P} \to \mathcal{P}$$

is a linear function with null space the collection C of polynomials of order zero (the constants). We can consider the quotient space $\mathcal{P}/C$ and differentiation induces a vector space isomorphism

$$\frac{d}{dx} : \mathcal{P}/C \to \mathcal{P}$$

with inverse given essentially by antidifferentiation (indefinite integration). The relation between differentiation and integration as operations on sets of functions is not quite so tidy if we move beyond the class of polynomials. In particular, we know that integration of the rational function $f(x) = 1/x$ involves something essentially new:

$$\int_1^x \frac{1}{t}\, dt = \ln x. \tag{3.30}$$

With the addition of this new antiderivative, we can integrate many, but not all, rational functions

$$f(x) = \frac{p(x)}{q(x)}$$

where $p, q \in \mathcal{P}$. In particular, we can integrate any rational function with denominator $q = q(x)$ which factors as a product of affine polynomials:

$$q(x) = a_n(x - x_1)(x - x_2)\cdots(x - x_n).$$

In fact, if we allow complex roots, then the fundamental theorem of algebra tells us every such polynomial factors this way. But that is moving on to $\mathbb{C}[x]$ the ring of polynomials with complex coefficients. In that ring, in fact, all rational functions can be integrated using $\ln : \mathcal{L} \to \mathbb{C}$, but this logarithm is a somewhat different animal from the one defined in (3.30). In particular, the domain $\mathcal{L}$ of the complex logarithm is a Riemann surface, and understanding this, in itself, requires the use of a new real (and transcendental) function.

Returning to $\mathcal{P} = \mathbb{R}[x]$, the trouble is irreducible quadratic factors:

$$\frac{1}{x^2 - 1} = \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right] = \frac{d}{dx}\left(\frac{1}{2}\ln(x-1)\right) - \frac{d}{dx}\left(\frac{1}{2}\ln(x+1)\right) \tag{3.31}$$

at least as long as $x > 1$, but

$$\int_0^x \frac{1}{t^2+1}\,dt = \tan^{-1}(x),$$

and this is something new.

Exercise 3.19 *What is the antiderivative of $1/(x^2-1)$ if $|x| < 1$. What about when $x < -1$? How would the plot of*

$$\left(\frac{d}{dx}\right)^{-1}\left(\frac{1}{x^2-1}\right)$$

look in $\mathcal{R}/C$ where $\mathcal{R}$ is the ring of rational functions $f(x) = p(x)/q(x)$ with $p, q \in \mathcal{P}$ and C is still the subspace of constant valued functions? Notice these rational functions have singularities; that is to be expected.

In some ways the irreducible quadratic case is the nicer of the two. There is no singularity in the arctangent function.

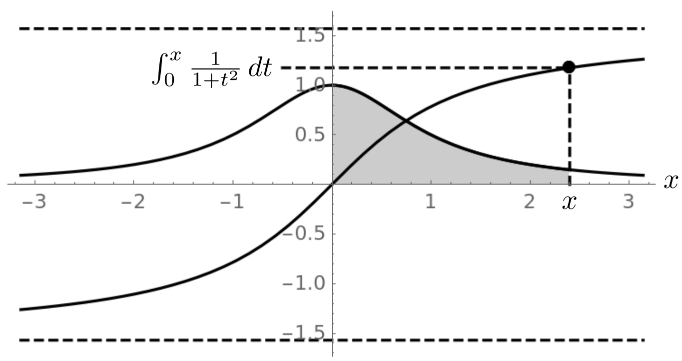

Figure 3.3: The graphs of $\tan^{-1}(x)$ and its derivative.

With these two new functions, we can make two new vector spaces:

$$\mathcal{V}_{\ln} = \left\{\sum_{j=1}^{k} a_j \ln|f(x)| : f \in \mathcal{R},\ a_1, a_2, \ldots, a_k \in \mathbb{R}\right\}$$

and

$$\mathcal{V}_{\arctan} = \left\{\sum_{j=1}^{k} a_j \tan^{-1} f(x) : f \in \mathcal{R},\ a_1, a_2, \ldots, a_k \in \mathbb{R}\right\}.$$

In each case, we are taking all finite linear combinations in the ring of functions with isolated singularities containing the rational functions $\mathcal{R}$. Note that $\mathcal{V}_{\ln}$ and $\mathcal{V}_{\arctan}$ are not rings themselves because they are not closed under multiplication, but they are subspaces in the function ring and they intersect with each other and with $\mathcal{R}$ in the subring C. Finally, taking the subspace of all finite linear combinations of functions from $\mathcal{R}$, $\mathcal{V}_{\ln}$, and $\mathcal{V}_{\arctan}$, we get a subspace

$$\mathcal{T} = \{\sum_{j=1}^{k} a_j f_j(x) : f_1, f_2, \ldots, f_k \in \mathcal{R} \cup \mathcal{V}_{\ln} \cup \mathcal{V}_{\arctan},\ a_1, a_2, \ldots, a_k \in \mathbb{R}\},$$

and

$$\frac{d}{dt} : \mathcal{T}/C \to \mathcal{R}$$

is a vector space isomorphism with inverse given by antidifferentiation. In short, you can integrate all rational functions in terms of rational functions and the compositions of the logarithm and arctangent with rational functions.

At some point one realizes this approach can be extended even further. For example,

$$\int_0^x \frac{1}{\sqrt{1+t^2}}\, dt = \ln\left(x + \sqrt{x^2+1}\right),$$

and in fact (though it's not so obvious) a rather nice "closed system of integration" is obtained by considering

$$\frac{d}{dt} : \mathcal{T}^{\text{root}}/C \to \mathcal{R}^{\text{root}}$$

where

$$\mathcal{R}^{\text{root}} = \{f(x, \sqrt{q(x)}) : f \in \mathcal{R}_2,\ q \in \mathcal{P},\ \deg(q) \le 2\},$$
$$\mathcal{R}_2 = \left\{\frac{P(x,y)}{Q(x,y)} : P, Q \in \mathcal{P}_2 = \mathbb{R}[x,y]\right\},$$

and

$$\mathcal{T}^{\text{root}} = \{\sum_{j=1}^{k} a_j f_j(x) : f_1, f_2, \ldots, f_k \in \mathcal{R}^{\text{root}} \cup \mathcal{V}_{\ln}^{\text{root}} \cup \mathcal{V}_{\arctan}^{\text{root}},\ a_1, a_2, \ldots, a_k \in \mathbb{R}\}.$$

In words, $\mathcal{R}_2$ is the ring of rational functions in two variables, $\mathcal{R}^{\text{root}}$ is obtained by substituting the square root of a quadratic (or lower degree) polynomial in for the second variable, and $\mathcal{T}^{\text{root}}$ is the span of rational functions in x and $\sqrt{q(x)}$ along with the composition spaces

$$\mathcal{V}_{\ln}^{\text{root}} = \left\{ \sum_{j=1}^{k} a_j \ln |f(x)| : f \in \mathcal{R}^{\text{root}},\ a_1, a_2, \ldots, a_k \in \mathbb{R} \right\}$$

and

$$\mathcal{V}_{\text{arctan}}^{\text{root}} = \left\{ \sum_{j=1}^{k} a_j \tan^{-1} f(x) : f \in \mathcal{R}^{\text{root}},\ a_1, a_2, \ldots, a_k \in \mathbb{R} \right\}.$$

The integration required at this point is covered in a standard elementary calculus course. In practice, there may be some functions in $\mathcal{R}^{\text{root}}$ which calculus students will have trouble integrating, but in principle, all techniques required to find an antiderivative of such a function in $\mathcal{T}^{\text{root}}$ are in the elementary texts. The structure described above is somewhat obscured in elementary calculus courses due to the natural introduction of other transcendental functions for convenience. For example, it is very natural to define

$$\sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt$$

with derivative included in our class $\mathcal{V}^{\text{root}}$. It may also be noted that this construction takes us out of the ring of functions with isolated singularities as

$$f(x) = \frac{1}{\sqrt{x-a}}$$

has natural domain $x > a$. But aside from being careful about taking square roots of negative numbers, antiderivatives can be found using the three kinds of functions: rational functions of x and $\sqrt{q(x)}$ where $\deg q(x) \le 2$ and compositions of the logarithm and arctangent on these functions.

What is usually not considered in elementary calculus is the possibility of generally allowing the square root of a polynomial with degree greater than two. It seems to be Legendre who first realized that if we invite the indefinite integrals

$$\int^t f(t, \sqrt{q(t)})\, dt$$

with $f \in \mathcal{R}_2 = \mathbb{R}[x, y]$ and $q \in \mathcal{P}$ with $\deg q(x) \le 4$ to the party, then we can again achieve a closed system of integration with the introduction of a relatively small collection of additional "standard" indefinite integrals. By way of review: We start with rational functions $\mathcal{R}_2$ in two variables as a base, then we consider functions of the form

$$f(x, \sqrt{q(x)}) \tag{3.32}$$

with $f \in \mathcal{R}_2$ and $q = q(x)$ a quartic (or lower degree) polynomial. In order to find an antiderivative for such a function we will need the rational functions $\mathcal{R}$ in one variable and, more generally, functions with the form of the integrand (3.32), that is, functions in

$$\mathcal{R}^{\text{root}} = \{f(x, \sqrt{q(x)}) : f \in \mathcal{R}_2,\ q \in \mathcal{P},\ \deg(q) \leq 4\}.$$

We will also need the standard indefinite integrals

$$\ln x = \int_1^x \frac{1}{t}\, dt \qquad \text{and} \qquad \tan^{-1}(x) = \int_0^x \frac{1}{t^2+1}\, dt$$

and compositions of these with functions in $\mathcal{R}^{\text{root}}$, our new class of integrands. Finally, we need the following: For $0 < m < 1$,

$$F(x\big|m) = \int_0^x \frac{1}{\sqrt{(1-t^2)(1-mt^2)}}\, dt.$$

This is the **elliptic integral of the first kind**. You can think of it as a generalization of the arcsin which is precisely what you get when $m = 0$. The quarter period $F(1\big|m)$ of the inverse $\text{sn} = \text{sn}(t\big|m)$ is different from $\pi/2$ when $m \neq 0$.

Exercise 3.20 *Show* $\pi/2 < F(1\big|m) < \infty$ *for* $0 < m < 1$. *Show the inverse of* F *extends to a well-defined smooth (real analytic) function* $\text{sn} : \mathbb{R} \to [-1, 1]$. *Hint: When the inverse is considered, the parameter m is fixed, but indeed* $\text{sn} = \text{sn}(y\big|m)$. *Once you have obtained an extension* $\text{sn} \in C^1(\mathbb{R})$, *it may be helpful to note that there exists a function* $\text{cn} : [0, F(1\big|m)] \to [-1, 1]$ *defined by*

$$\text{cn}y = \sqrt{1 - \text{sn}^2 y}.$$

Furthermore, $\text{cn} = \text{cn}(y\big|m)$ *has an extension* $\text{cn} \in C^1(\mathbb{R})$ *such that the pair of functions* (sn, cn) *satisfies the IVP*

$$\begin{cases} \text{sn}' = \text{cn}\sqrt{1 - m\ \text{sn}^2}, & \text{sn}(0) = 0 \\ \text{cn}' = -\text{sn}\sqrt{1 - m\ \text{sn}^2}, & \text{cn}(0) = 1. \end{cases}$$

Incidentally, the function dn : $\mathbb{R} \to [\sqrt{1-m}, 1]$ *defined by*

$$\text{dn} y = \sqrt{1 - m \ \text{sn}^2 y}$$

also satisfies dn $\in C^1(\mathbb{R})$ *and the IVP above can be extended to include this third elliptic function.*

Also for $0 < m < 1$,

$$E(x|m) = \int_0^x \frac{\sqrt{1-mt^2}}{\sqrt{1-t^2}}\, dt.$$

This is the **elliptic integral of the second kind**. The notation is a little weird with the " $|$ " separating the arguments instead of a comma or semicolon, but it is standard. The elliptic integral of the second kind has properties somewhat similar to the elliptic integral of the first kind but, as you can see from the formula, it is a different function, and you need both kinds of get a closed system of integration.

Exercise 3.21 *Show* $0 < E(1|m) < \pi/2$ *for* $0 < m < 1$.

We also need one more function (or one more kind of function). These are, you might guess, the **elliptic integrals of the third kind**. For $0 < n, k < 1$,

$$\Pi(n; x|m) = \int_0^x \frac{1}{1-nt^2} \frac{1}{\sqrt{(1-mt^2)(1-t^2)}}\, dt.$$

These three kinds of functions are "standard" in the sense that their values are tabulated and available in standard mathematical software; many of thier properties are known like the properties of arcsin and arccosine are known.

At this point, however, let us return to our "implicit" solution of the capillary equation

$$\int_{u(0)}^u \frac{p(t)}{\sqrt{1-p(t)^2}}\, dt = \pm x.$$

Restricting to $x > 0$, we can consider the function $g : [z_0, \sqrt{z_0^2 + 4/\kappa}] \to \mathbb{R}$ by

$$g(z) = \int_{z_0}^z \frac{p(t)}{\sqrt{1-p(t)^2}}\, dt = \int_{z_0}^z \frac{1 + \kappa z_0^2/2 - \kappa t^2/2}{\sqrt{1 - (1 + \kappa z_0^2/2 - \kappa t^2/2)^2}}\, dt$$

directly as we would one of the standard elliptic integrals or one of the inverse trigonometric functions $\sin^{-1}$ or $\tan^{-1}$. In this case, we see the integrand is singular, but integrable, at both endpoints.

Note that $1-p^2 = (1-p)(1+p)$ with $1-p(t) = \kappa(t-z_0)(t+z_0)/2$ and

$$1+p(t) = -\kappa(t-c)(t+c)/2 \quad \text{with} \quad c = \sqrt{z_0^2 + \frac{4}{\kappa}} > z_0 > 0.$$

In particular, $1-p(t)^2 = -\kappa^2(t-z_0)(t-c)(t+z_0)(t+c)/4$ has four simple roots $-c < -z_0 < z_0 < c$; see Figure 3.4. Basic estimates for the integral

$$\int_{z_0}^{z} \frac{p(t)}{\sqrt{1-p(t)^2}}\,dt = \frac{2}{\kappa}\int_{z_0}^{z} \frac{p(t)}{\sqrt{-(t-z_0)(t-c)(t+z_0)(t+c)}}\,dt$$

are the following:

$$-\frac{1}{b\sqrt{b-z_0}}\int_b^c \frac{1}{\sqrt{c-t}}\,dt < \int_{z_0+\epsilon}^{c-\epsilon} \frac{p(t)}{\sqrt{1-p(t)^2}}\,dt < \frac{1}{z_0\sqrt{c-b}}\int_{z_0}^{b} \frac{1}{\sqrt{t-z_0}}$$

where

$$b = \sqrt{z_0^2 + \frac{2}{\kappa}} \qquad \text{with} \qquad z_0 < b < c.$$

Since

$$\int_b^c \frac{1}{\sqrt{c-t}}\,dt = 2\sqrt{c-b} \qquad \text{and} \qquad \int_{z_0}^{b} \frac{1}{\sqrt{t-z_0}} = 2\sqrt{b-z_0}$$

are both well-defined real numbers, we see the function $g : [z_0, c] \to [0, g_{\max}]$ where

$$g_{\max} = g(b) = \int_{z_0}^{b} \frac{p(t)}{\sqrt{1-p(t)^2}}\,dt$$

is well-defined with $g(z_0) = 0$, $g'(z) > 0$ for $z_0 < z < b$ and $g'(z) < 0$ for $b < z < c$ as indicated in Figure 3.5. In particular, the function g restricted to $z_0 \le z \le b$ has a well-defined increasing inverse $g^{-1} : [0, g_{\max}] \to [z_0, b]$ with $g'(0) = 0$ and

$$\lim_{x \nearrow g_{\max}} \frac{dg^{-1}}{dx}(x) = +\infty.$$

Notice we can write $u(x) = g^{-1}(x)$ and u gives the unique maximal capillary (semi) graph ($x \ge 0$) with $u(0) = x_0$ and $u'(0) = 0$. The maximal extent of

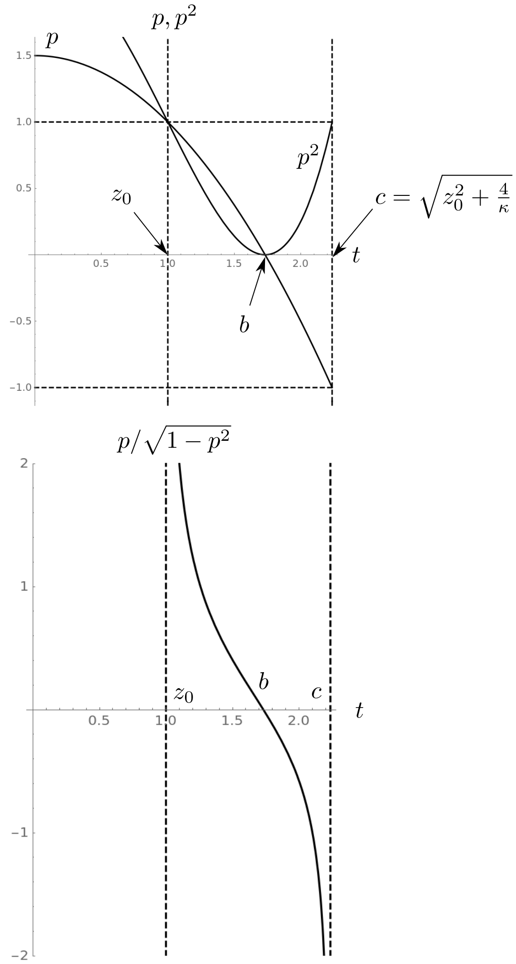

Figure 3.4: The integrand of an elliptic integral: The quadratic polynomial $p = p(t)$ and the quartic polynomial p^2 (top); The integrand $p/\sqrt{q}$ where $q = q(t)$ is the quartic polynomial $q = 1 - p^2$ (bottom).

this graph is $R = g_{\max}$, and there is a unique tube radius r with $0 < r < R$ for which

$$u'(r) = \tan\gamma = \frac{dg^{-1}}{dx}(r). \tag{3.33}$$

Now, of course, we've been thinking of the tube radius r as prescribed and looking for the solution of (3.1). From this point of view, it is natural to take

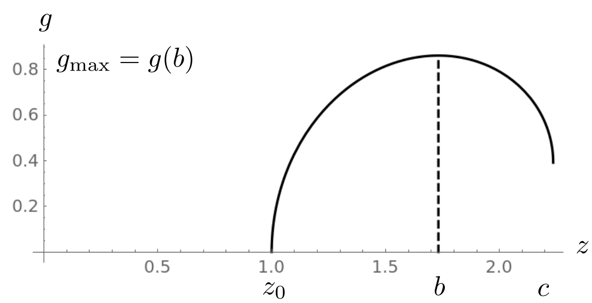

Figure 3.5: Values of an elliptic integral.

a modified version of (3.33)

$$u'(\rho) = \tan\gamma = \frac{dg^{-1}}{dx}(\rho). \tag{3.34}$$

as the definition of a certain positive radius $\rho = \rho(\gamma) = \rho(\gamma; z_0)$, and attempt to solve the equation $\rho(\gamma) = r$ for any positive $r > 0$. For this one needs a monotonicity result:

Lemma 29 *The function $\rho = \rho(\gamma; z_0)$ defined by (3.34) satisfies $\rho \in C^1((0, \pi/2)\times(0,\infty))$ with*

$$\frac{\partial\rho}{\partial\gamma} < 0 \qquad \text{and} \qquad \frac{\partial\rho}{\partial z_0} > 0.$$

In addition,

$$\lim_{z_0 \searrow 0} \rho(\gamma; z_0) = +\infty \qquad \text{and} \qquad \lim_{z_0 \nearrow 0} \rho(\gamma; z_0) = 0.$$

Exercise 3.22 *Use the lemma above to prove the existence and uniqueness of solutions for (3.1).*

Exercise 3.23 *Prove the lemma.*

The Extension of g

Looking at the graph of the elliptic integral g, there are a couple obvious questions/observations which are worth mentioning. First of all, notice that in Figure 3.5 $u(c) > 0$. This condition is equivalent to the inequality

$$\int_{z_0}^{b} \frac{p(t)}{\sqrt{1 - p(t)^2}}\, dt > -\int_{b}^{c} \frac{p(t)}{\sqrt{1 - p(t)^2}}\, dt. \tag{3.35}$$

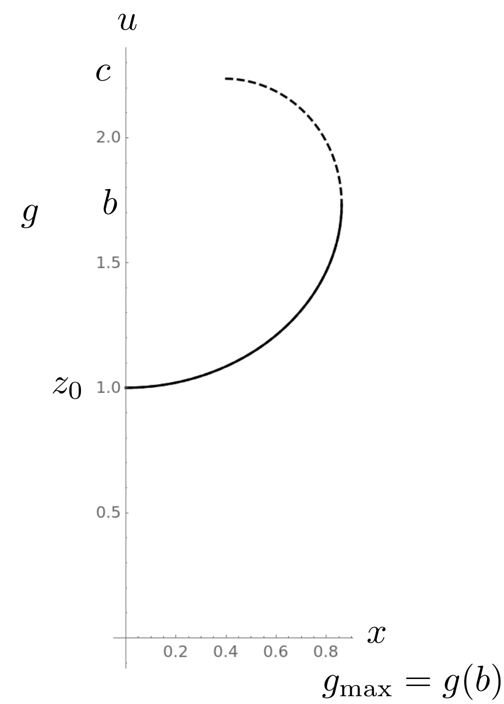

Figure 3.6: The inverse $u = g^{-1}$ of g is a 2-D capillary graph.

Exercise 3.24 *Prove (3.35) holds in general. (This exercise is perhaps not so easy.)*

We know also, the parametric profile curves $s \mapsto (x(s), z(s))$ associated with the parametric initial value problem (3.29) have the following properties:

1. There is a unique first positive arclength $s = s_1$ for which $\psi(s) = \pi/2$ corresponding to a vertical point on the profile curve. We know, in fact, the parametric profile is identical to the graph of $u = g^{-1}$ determined by the elliptic integral g.

2. There is a unique first positive arclength $s = \bar{s}$ for which $\psi(s) = \pi$. This corresponds to the maximum value of z given, according to the conserved quantity, by

$$\frac{\kappa}{2} z_0^2 + 1 = \frac{\kappa}{2} z(\bar{s})^2 - 1 \qquad \text{or} \qquad z(\bar{s}) = \sqrt{z_0^2 + \frac{4}{\kappa}}.$$

Notice this value coincides with the root c of the quartic polynomial $q(t) = 1 - p(t)^2$.

Exercise 3.25 *Show the "reflection" of the parametric profile given by the image of* $\alpha(s) = (z(s), x(s))$ *for* $0 \leq s \leq \bar{s}$ *coincides precisely with the graph of the elliptic integral* $g : [z_0, c] \to [0, g_{\max}]$.

Note finally, that in view of Exercise 3.25, the inequality (3.35) of Exercise 3.24 is precisely the inequality required to verify that parametric solutions of the 2-D capillary equation (starting with initial conditions $x(0) = 0$, $z(0) = z_0 > 0$, and $\psi(0) = 0$) "loop to the right."

Chapter 4

Floating Objects (2-D)

Chapter 5

3-D Capillairy Surfaces

5.1 The Axially Symmetric Tube

Perhaps a reasonable way to start the discussion of capillary surfaces in three-dimensional space, where we really encounter models of the bounded surfaces we observe—rather than idealized infinitely long surfaces considered in cross-section in the 2-D case—is with a discussion of the axially symmetric tube considered in the introductory chapter. We will again proceed for some time without an existence or uniqueness theorem, though these are available in some generality as may be discussed later. Thus, we assume the existence of and consider a function $u \in C^2(B_r(\mathbf{0})) \cap C^1(\overline{B_r(\mathbf{0})})$ satisfying

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \kappa u & \text{on } B_r(\mathbf{0}), \\ \dfrac{Du}{\sqrt{1+|Du|^2}} \cdot \mathbf{n} = \cos\gamma & \text{on } \partial B_r(\mathbf{0}). \end{cases}$$

In this boundary value problem $r > 0$ and

$$B_r(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < r\}$$

as usual; $\mathbf{n} = \mathbf{x}/r$ is the outward unit normal to $\partial B_r(\mathbf{0})$. Let us also assume, for the moment, $0 < \gamma < \pi/2$.

It can be shown, in fact, that this problem always has a unique solution, the solution is positive, and the solution is axially symmetric in the sense that

$$u(x, y) = \phi\left(\sqrt{x^2 + y^2}\right)$$

for some function $\phi \in C^\infty[0, r]$ which extends to a positive even function $\phi \in C^\infty[-r, r]$. The graph of u may be considered as the surface of rotation generated by the graph of $\phi = \phi(x)$ which is called the **meridian** of the axially symmetric capillary graph. Letting Π_m denote the x, z-plane, we can compute the mean curvature of the graph of u along the meridian $\Pi_m \cap \text{graph}(u)$ to obtain an ODE for ϕ. The plane Π_m contains the upward normal to the graph of u, and one normal curvature with respect to this normal can be taken as the curvature of the meridian:

$$k_m = \frac{\phi''}{(1 + \phi'^2)^{3/2}} = \left(\frac{\phi'}{\sqrt{1 + \phi'^2}}\right)' = (\sin\psi)'$$

where ψ is the inclination of ϕ and the derivatives are with respect to x at the point $(x, 0, \phi(x)) \in \text{graph}(u)$.

The plane Π_ℓ containing the upward unit normal N to graph(u) which is orthogonal to Π_m intersects graph(u) locally near $(x, 0, \phi(x))$ in a curve which is somewhat difficult to treat directly. There is a simple curve, however, passing through $(x, 0, \phi(x))$ tangent to graph$(u) \cap \Pi_\ell$, namely the (longitudinal) circle graph$(u) \cap \{(x, y, \phi(x)) : (x, y) \in \mathbb{R}^2\}$. Assuming $x > 0$, this horizontal circle has curvature vector

$$\frac{1}{x}(-1, 0, 0).$$

The theorem of Meusnier asserts that the curvature of graph$(u) \cap \Pi_\ell$ with respect to N is the projection of the curvature vector of any curve lying on the surface and tangent to graph$(u) \cap \Pi_\ell$ onto N. An equivalent statement is that the projections onto the normal of the curvature vectors of all curves on a surface which are tangent to one another at a point on the surface are the same. In any case, the normal N is given by

$$N = \frac{(-\phi', 0, 1)}{\sqrt{1 + \phi'^2}}$$

and consequentely the orthogonal normal curvature of $\Pi_\ell \cap \text{graph}(u)$ is

$$k_\ell = \frac{1}{x}(-1, 0, 0) \cdot N = \frac{1}{x}\frac{\phi'}{\sqrt{1 + \phi'^2}}.$$

Thus, the (doubled) mean curvature of graph(u), and the ODE for ϕ is

$$k_\ell + k_m = \frac{1}{x}\frac{\phi'}{\sqrt{1+\phi'^2}} + \left(\frac{\phi'}{\sqrt{1+\phi'^2}}\right)' = \kappa\phi.$$

This ODE can also be written as

$$\left(\frac{x\phi'}{\sqrt{1+\phi'^2}}\right)' = \kappa x\phi$$

or

$$(\sin\psi)' = \kappa\phi - \frac{1}{x}\sin\psi.$$

Note that the last form prescribes the curvature of the meridian

$$\frac{d\psi}{ds} = \frac{d}{dx}(\sin\psi).$$

The parametric version (for $s \mapsto (x(x), z(s))$ in terms of arclength and inclination angle) is

$$\begin{cases} \dot{x} = \cos\psi \\ \dot{z} = \sin\psi \\ \dot{\psi} = \kappa z - \frac{1}{x}\sin\psi \end{cases}$$

with associated initial value problem

$$\begin{cases} \dot{x} = \cos\psi, & x(0) = 0 \\ \dot{z} = \sin\psi, & z(0) = z_0 = u(\mathbf{0}) \\ \dot{\psi} = \kappa z - \frac{1}{x}\text{sn}\psi, & \psi(0) = 0. \end{cases}$$

In all forms, however, the ODEs we have written are singular at $x = 0$. The graph of u, however, and the solution functions $u = u(x, y)$, $\phi = \phi(x)$, $x = x(s)$, $z = z(s)$, and $\psi = \psi(s)$ have no singularity. In particular, the limit

$$\lim_{s\to 0}\frac{\sin\psi}{x}$$

may be calculated using L'Hopital's rule as follows: First note that

$$\lim_{s\to 0}\frac{\sin\psi}{x} = \lim_{s\to 0}\frac{\cos\psi\,\dot{\psi}}{\dot{x}} = \kappa u(\mathbf{0}) - \lim_{s\to 0}\frac{\sin\psi}{x}.$$

Therefore,

$$\lim_{s\to 0} \frac{\sin\psi}{x} = \frac{\kappa u(\mathbf{0})}{2}.$$

Consequently, the curvature of the meridian curve at the minimum point $(0, \phi(0))$ corresponding to $(0, 0, u(\mathbf{0}))$ is also $\kappa u(\mathbf{0})/2$, and the sphere tangent to the interface and having the same mean curvature as graph(u) has radius

$$a_0 = \frac{2}{\kappa u(\mathbf{0})}.$$

Finn gives an argument to show the lower hemisphere determined by

$$\sigma_0(\mathbf{x}) = u(\mathbf{0}) + a_0 - \sqrt{a_0^2 - |\mathbf{x}|^2}$$

satisfies $\sigma_0(\mathbf{x}) < u(\mathbf{0})$ for $0 < |\mathbf{x}| \le r$, or equivalently, $\sigma_0(x) < \phi(x)$ for $0 < x \le r$ where σ_0 denotes also the real valued function giving the lower semi-circular graph with lowest point at $u(\mathbf{0})$ and radius a_0. This result is similar to (though a bit harder than) Theorem 25.

Let us assume this result for the moment and consider the resulting volume comparison. First of all, the raised volume can be computed using the divergence theorem as follows:

$$\kappa \int_{B_r(\mathbf{0})} u = \int_{B_r(\mathbf{0})} \text{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \int_{\partial B_r(\mathbf{0})} \frac{Du}{\sqrt{1+|Du|^1}} \cdot \mathbf{n} = 2\pi r\cos\gamma.$$

Thus, the raised volume is

$$\int_{B_r(\mathbf{0})} u = \frac{2\pi r\cos\gamma}{\kappa}.$$

This means, in particular, that

$$\frac{2\pi r\cos\gamma}{\kappa} > \int_{B_r(\mathbf{0})} \sigma_0.$$

As before, let us compute the volume enclosed below a general lower hemispherical cap of radius $a \ge r$ within a cylinder of radius r determined by

$$\sigma(\mathbf{x}) = z_0 + a - \sqrt{a^2 - |\mathbf{x}|^2}.$$

This volume is

$$2\pi \int_0^r \sigma_0(x)\ x\,dx = \pi \left\{ (z_0+a)r^2 + \int_{a^2}^{a^2-r^2} \sqrt{v}\,dv \right\}$$
$$= \pi \left\{ (z_0+a)r^2 - \frac{2}{3}\left[a^3 - (a^2-r^2)^{3/2}\right] \right\}.$$

Finn concludes

$$\frac{2\pi r\cos\gamma}{\kappa} > \pi F\left(u(\mathbf{0}), \frac{2}{\kappa u(\mathbf{0})}\right)$$

where

$$F(z,a) = (z+a)r^2 - \frac{2}{3}\left[a^3 - (a^2-r^2)^{3/2}\right].$$

At this point Finn introduces his displayed equation (2.21) on page 21 with a technically correct but, as far as I can tell,[1] more or less irrelevant, and at least inadequate, observation.[2] I think what he really wants to say is something like the following: If we set

$$\Phi(z) = F\left(z, \frac{2}{\kappa z}\right),$$

then the existence of a solution $u \in C^1(\overline{B_r(\mathbf{x})})$ implies

$$a_0 = \frac{2}{\kappa u(\mathbf{0})} > r \qquad \text{or} \qquad u(\mathbf{0}) < \frac{2}{\kappa r}.$$

Thus, we are interested in in the continuous function $\Phi = \Phi(z)$ for $u(\mathbf{0}) \le z \le 2/(\kappa r)$. Note that

$$\Phi(u(\mathbf{0})) < \frac{2r\cos\gamma}{\kappa} \qquad \text{and} \qquad \Phi\left(\frac{2}{\kappa r}\right) = \frac{2r}{\kappa} + \frac{r^3}{3} > \frac{2r\cos\gamma}{\kappa}.$$

[1] Of course, maybe I'm missing something here.

[2] For direct comparison it should be pointed out, furthermore, that Finn is essentially treating the special case $r = 1$ with $\kappa = B$. On the other hand, there is certainly no loss of generality in his argument which is based on a homogeneous scaling of $\mathbb{R}^3$ by a factor $1/r$. The resulting capillary constant B in this case is called the **Bond number** after Wilfrid Noel Bond. I've left this scaling out based on the observation that the argument, once you've got it figured out, is not that much more difficult with the inclusion of the general radius r.

Thus, the equation

$$\Phi(z) = F\left(z, \frac{2}{\kappa z}\right) = \frac{2r\cos\gamma}{\kappa} \tag{5.1}$$

has at least one solution z_*. We wish to show there is exactly one solution.

Lemma 30 $\Phi'(z) > 0$ *for* $u(\mathbf{0}) \le z \le 2/(\kappa r)$.

Proof: First of all

$$\Phi'(z) = \frac{\partial F}{\partial z}\left(z, \frac{2}{\kappa z}\right) - \frac{2}{\kappa z^2}\frac{\partial F}{\partial a}\left(z, \frac{2}{\kappa z}\right) = r^2 - \frac{2}{\kappa z^2}\frac{\partial F}{\partial a}\left(z, \frac{2}{\kappa z}\right).$$

We claim in fact

$$\frac{\partial F}{\partial a}\left(z, \frac{2}{\kappa z}\right) < 0 \qquad \text{for} \qquad u(\mathbf{0}) \le z \le \frac{2}{\kappa r}, \tag{5.2}$$

and therefore $\Phi'(z) \ge r^2 > 0$. Notice that

$$g(a) = \frac{\partial F}{\partial a} = r^2 - 2a^2 + 2a\sqrt{a^2 - r^2}.$$

Thus, it is enough[3] to show

$$g(a) = r^2 - 2a^2 + 2a\sqrt{a^2 - r^2} < 0 \qquad \text{for} \qquad r \le a = \frac{2}{\kappa z} \le \frac{2}{\kappa u(\mathbf{0})}. \tag{5.3}$$

Observe that $g(r) = -r^2 < 0$. Furthermore, if for some a we have $g(a) = 0$, then

$$\sqrt{a^2 - r^2} = a - \frac{r^2}{2a} \qquad \text{and} \qquad a^2 - r^2 = a^2 - r^2 + \frac{r^4}{4a^2},$$

so that

$$\frac{r^4}{4a^2} = 0.$$

But we know $r^4/(4a^2) > 0$, so we can (attempt to) reverse the algebra for the inequality. Specifically, we know

$$a^2 - r^2 < a^2 - r^2 + \frac{r^4}{4a^2} = \left(a - \frac{r^2}{2a}\right)^2. \tag{5.4}$$

[3] At this point a much more streamlined proof of the desired fact, $g(a) < 0$, can be given, but I have included a number of, arguably unnecessary, details which I hope provide motivation.

Since $h(a) = a - r^2/(2a)$ satisfies

$$h(r) = \frac{r}{2} > 0 \qquad \text{and} \qquad h'(a) = 1 + \frac{r^2}{2a^2} > 0,$$

we obtain from (5.4)

$$\sqrt{a^2 - r^2} < \left| a - \frac{r^2}{2a} \right| = a - \frac{r^2}{2a}.$$

Hence, $r^2 - 2a^2 + 2a\sqrt{a^2 - r^2} < 0$. That is, (5.3) is established, and this implies (5.2). $\square$

Letting z_* be the unique solution of the equation (5.1), that is, the unique value z_* with

$$u(\mathbf{0}) < z_* < \frac{2}{\kappa r}$$

for which

$$\Phi(z_*) = \left(z_* + \frac{2}{\kappa z_*} \right) r^2 - \frac{2}{3} \left[\frac{8}{\kappa^3 z_*^3} - \left(\frac{4}{\kappa^2 z_*^2} - r^2 \right)^{3/2} \right] = \frac{2r\cos\gamma}{\kappa},$$

we obtain

$$u(\mathbf{0}) < z_*. \tag{5.5}$$

This, it is claimed, is an improvement over the estimate obtained directly from the volume comparison

$$\pi r^2 u(\mathbf{0}) < \int_{B_r(\mathbf{0})} u = \frac{2\pi r \cos\gamma}{\kappa}.$$

To see this is the case, we compute

$$\begin{aligned}
\Phi\left(\frac{2\cos\gamma}{\kappa r}\right) &= F\left(\frac{2\cos\gamma}{\kappa r}, \frac{r}{\cos\gamma}\right) \\
&= \frac{2r\cos\gamma}{\kappa} + \frac{r^3}{\cos\gamma} - \frac{2}{3}\left[\frac{r^3}{\cos^3\gamma} - \left(\frac{r^2}{\cos^2\gamma} - r^2\right)^{3/2}\right] \\
&= \frac{2r\cos\gamma}{\kappa} + \frac{r^3}{\cos\gamma} - \frac{2r^3}{3}\left[\frac{1}{\cos^3\gamma} - \left(\frac{1}{\cos^2\gamma} - 1\right)^{3/2}\right] \\
&= \frac{2r\cos\gamma}{\kappa} + \frac{r^3}{\cos\gamma} - \frac{2r^3}{3\cos^3\gamma}\left[1 - \sin^3\gamma\right] \\
&= \frac{2r\cos\gamma}{\kappa} + \frac{r^3}{3\cos^3\gamma}\left[3\cos^2\gamma - 2 + 2\sin^3\gamma\right] \\
&= \frac{2r\cos\gamma}{\kappa} + \frac{r^3}{3\cos^3\gamma}\left[1 - 3\sin^2\gamma + 2\sin^3\gamma\right].
\end{aligned}$$

The cubic polynomial $p(t) = 2t^3 - 3t^2 + 1$ satisfies

$$p(0) = 1 > 0 = p(1) \qquad \text{and} \qquad p'(t) = 6t(t-1) < 0 \quad \text{for } 0 < t < 1.$$

Therefore, the quantity $1 - 3\sin^2\gamma + 2\sin^3\gamma > 0$ and

$$\Phi\left(\frac{2\cos\gamma}{\kappa r}\right) > \frac{2r\cos\gamma}{\kappa}.$$

This means that, indeed, takinig account of the monotonicity of Φ we have

$$z_* < \frac{2\cos\gamma}{\kappa r}.$$

On the next page (page 22) I think Finn must mean expansion in (2.21) which defines z_* rather than expansion in (2.20). For us, that would be expansion in

$$\left(z_* + \frac{2}{\kappa z_*}\right) r^2 - \frac{2}{3}\left[\frac{8}{\kappa^3 z_*^3} - \left(\frac{4}{\kappa^2 z_*^2} - r^2\right)^{3/2}\right] = \frac{2r\cos\gamma}{\kappa}.$$

Of course, Finn can expand in the Bond number $B = \kappa r^2$, and this will require a bit of extra manipulation in our case.

In retrospect, though the argument above is correct I think, it may be more natural to divide both sides of the original estimate by πr^2 to obtain

$$\frac{2\cos\gamma}{\kappa r} > u(\mathbf{0}) + \frac{2}{\kappa u(\mathbf{0})} - \frac{2}{3r^2}\left[\frac{8}{\kappa^3 u(\mathbf{0})^3} - \left(\frac{4}{\kappa^2 u(\mathbf{0})^2} - r^2\right)^{3/2}\right].$$

Then we can call the right side of this inequality

$$F\left(u(\mathbf{0}), \frac{2}{\kappa u(\mathbf{0})}\right),$$

and the terms after $u(\mathbf{0})$ may be naturally considered a correction.

Chapter 6

Existence and Uniqueness

In this section I give a global existence and uniqueness theorem based on an in-class presentation by Nathan Soedjak. He pieced together a proof of the following theorem for autonomous equations, by his own account, from Wikipedia.[1]

Theorem 31 *If* $\mathbf{f} \in C^1(\mathbb{R}^n \to \mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n \to \mathbb{R}^n)$*, then the initial value problem*

$$\begin{cases} \mathbf{y}' = \mathbf{f}(\mathbf{y}) & \textit{for } t \in \mathbb{R} \\ \mathbf{y}(t_0) = \mathbf{y}_0 \end{cases}$$

has a unique solution $\mathbf{y} \in C^1(\mathbb{R} \to \mathbb{R}^n)$ *for any initial* $t_0 \in \mathbb{R}$ *and any initial value* $\mathbf{y}_0 \in \mathbb{R}^n$*.*

Note: I do not see in the proof where the differentiability assumption on $\mathbf{f}$ is used. I think it is enough to assume $\mathbf{f} \in \mathrm{Lip}(\mathbb{R}^n \to \mathbb{R}^n)$. See also the generalization for nonautonomous systems below.

Soedjak left as exercises certain aspects of the proof and also the generalization to the nonautonomous case. I will attempt to fill in these details/do this exercise and obtain the above result as a special case of the nonautonomous version in particular.

Two of our main applications are to the 2-D parametric system

$$\begin{cases} \dot{x} = \cos\psi, \\ \dot{z} = \sin\psi, \\ \dot{\psi} = \kappa z, \end{cases}$$

[1] Soedjak referenced the articles on **The Picard-Lindelöf Theorem**, **Picard Iteration**, and **The Banach Fixed Point Theorem** in particular.

and to the 3-D axially symmetric capillary equation which, in parametric form, is

$$\begin{cases} \dot{x} = \cos\psi, \\ \dot{z} = \sin\psi, \\ \dot{\psi} = \kappa z - \sin\psi/x \end{cases}$$

These are both autonomous systems with respect to arclength along the parameterized interface, though it will be observed that the 3-D system has a nominal (and obvious) singularity at $x = 0$ in the third equation. This will require separate consideration. The result above applies to the 2-D system giving the existence of solutions for all arclengths $s \in \mathbb{R}$. In this case, $\mathbf{f}(x, z, \psi) = (\cos\psi, \sin\psi, \kappa z)$ with

$$D\mathbf{f} = \begin{pmatrix} 0 & 0 & -\sin\psi \\ 0 & 0 & \cos\psi \\ 0 & \kappa & 0 \end{pmatrix}.$$

Since it is clear that all components in $D\mathbf{f}$ are continuous, we have[2] $\mathbf{f} \in C^1(\mathbb{R}^3 \to \mathbb{R}^3)$. The Lipschitz requirement is that there is a constant $\lambda \geq 0$ for which

$$\|\mathbf{f}(x_2, z_2, \psi_2) - \mathbf{f}(x_1, z_1, \psi_2)\| \leq \lambda \|(x_2, z_2, \psi_2) - (x_1, z_1, \psi_2)\|. \tag{6.1}$$

In the notes above, I have normally denoted the Euclidian distance

$$\|(x_2, z_2, \psi_2) - (x_1, z_1, \psi_2)\| = \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2 + (\psi_2 - \psi_1)^2} \tag{6.2}$$

by $|(x_2, z_2, \psi_2) - (x_1, z_1, \psi_2)|$. I am going to incorporate the change of notation suggested in (6.2) and (6.1) to draw attention to the presence of vector valued functions in a context which has not been used much above and in contrast to the usual absolute value on $\mathbb{R}$. More precisely, we will be integrating vector valued functions below. Briefly, a vector valued function $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ of one variable defined and continuous on an interval $[a, b]$ has integral (defined by)

$$\int_a^b \mathbf{g}(x)\, dx = \left(\int_a^b g_1(x)\, dx, \int_a^b g_2(x)\, dx, \ldots, \int_a^b g_3(x)\, dx \right).$$

[2] Again, I don't think we need this, but it is true for the 2-D capillary system, so I went ahead and recorded the verification.

This is also a convenient place to point out that we will be using, in particular, the inequality

$$\left\| \int_a^b \mathbf{g}(x)\,dx \right\| \le \int_a^b \|\mathbf{g}(\mathbf{x})\|\,dx.$$

Returning to (6.1),

$$\begin{aligned} &\sqrt{(\cos\psi_2 - \cos\psi_1)^2 + (\sin\psi_2 - \sin\psi_1)^2 + \kappa(z_2 - z_1)^2} \\ &\qquad \le \sqrt{(\psi_2 - \psi_1)^2 + (\psi_2 - \psi_1)^2 + \kappa(z_2 - z_1)^2} \\ &\qquad \le \max\{2, \kappa\} \|(x_2, z_2, \psi_2) - (x_1, z_1, \psi_1)\|. \end{aligned}$$

Thus, the existence theorem above applies to the 2-D capillary problem.

My generalization to non-autonomous systems is the following:

Theorem 32 *If* $\mathbf{f} \in C^0(\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n)$ *and there is a constant* $\Lambda \ge 0$ *such that for each fixed* $t \in \mathbb{R}$ *the function* $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ *by*

$$\mathbf{g}(\mathbf{y}) = \mathbf{f}(t, \mathbf{y})$$

satisfies[3]

$$\|\mathbf{g}(\mathbf{y}_2) - \mathbf{g}(\mathbf{y}_1)\| \le \Lambda \|\mathbf{y}_2 - \mathbf{y}_1\|,$$

then the initial value problem

$$\begin{cases} \mathbf{y}' = \mathbf{f}(t, \mathbf{y}) & \text{for } t \in \mathbb{R} \\ \mathbf{y}(t_0) = \mathbf{y}_0 \end{cases} \tag{6.3}$$

has a unique solution $\mathbf{y} \in C^1(\mathbb{R} \to \mathbb{R}^n)$ *for any initial* $t_0 \in \mathbb{R}$ *and any initial value* $\mathbf{y}_0 \in \mathbb{R}^n$.

The proof will be based on a corollary of the **Banach fixed point theorem** which I will state here and discuss later. The statement of the Banach fixed point theorem (also called[4] the contraction mapping theorem) we will use is as follows:

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[3]In particular $\mathbf{g} \in \mathrm{Lip}(\mathbb{R}^n \to \mathbb{R}^n)$, but we are also requiring the Lipschitz constant to be uniform with respect to the choice of $t \in \mathbb{R}$ in the first argument of $\mathbf{f}$. There may be weaker hypotheses, but this one gives the result with only minor modifications of Soedjak's proof.

[4]This result, incidentally, is included in Rudin's *Principles of Mathematical Analysis* (i.e., baby Rudin). Rudin uses it to prove the inverse function theorem.

Theorem 33 *If X is a complete metric space and $\Gamma : X \to X$ is a* **contraction**, *that is, there is some c with $0 \le c < 1$ such that*

$$d(\Gamma(x_2), \Gamma(x_1)) \le c\, d(x_2, x_1) \qquad \text{for all } x_1, x_2 \in X,$$

then there exists a unique fixed point $x_* \in X$, that is a point with $\Gamma(x_*) = \Gamma(x_*)$. In fact,

$$x_* = \lim_{n\to\infty} \Gamma^n(x)$$

where x is any point in X and Γ^n is the composition of Γ with itself n times.

The corollary is the following:

Corollary 34 *If X is a complete metric space and $\Gamma : X \to X$ has the property that for some $k \in \mathbb{N}$, the function $\Gamma^k : X \to X$ is a* **contraction**, *then there exists a unique fixed point $x_* \in X$, that is a point with $\Gamma(x_*) = \Gamma(x_*)$. In fact,*

$$x_* = \lim_{n\to\infty} \Gamma^n(x)$$

where x is any point in X and Γ^n is the composition of Γ with itself n times.

I will prove both of these results below as well as give a brief discussion of what it means to be a **complete metric space** and some related topics.

Owing to the fact that $C^0(\mathbb{R} \to \mathbb{R}^n)$ is not a complete metric space, we will prove a restricted version of the global existence and uniqueness theorem, Theorem 32, which may be viewed as a technical lemma:

Lemma 35 *Let $R > 0$ and $t_0 \in \mathbb{R}$. If $\mathbf{f} \in C^0([t_0 - 3R, t_0 + 3R] \times \mathbb{R}^n \to \mathbb{R}^n)$ and there is a constant $\Lambda \ge 0$ such that for each fixed $t \in [t_0 - 3R, t_0 + 3R]$ the function $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ by*

$$\mathbf{g}(\mathbf{y}) = \mathbf{f}(t, \mathbf{y})$$

satisfies

$$\|\mathbf{g}(\mathbf{y}_2) - \mathbf{g}(\mathbf{y}_1)\| \le \Lambda \|\mathbf{y}_2 - \mathbf{y}_1\|,$$

then the initial value problem

$$\begin{cases} \mathbf{y}' = \mathbf{f}(t, \mathbf{y}) & \text{for } t \in [t_0 - R, t_0 + R] \\ \mathbf{y}(t_0) = \mathbf{y}_0 \end{cases} \tag{6.4}$$

has a unique solution $\mathbf{y} \in C^1([t_0 - R, t_0 + R] \to \mathbb{R}^n)$ for any initial value $\mathbf{y}_0 \in \mathbb{R}^n$.

Proof: We begin by noting $X = C^0([t_0 - 2R, t_0 + 2R] \to \mathbb{R}^n)$ is a Banach space (i.e., a normed vector space which is a complete metric space with respect to the metric induced by the norm) under the uniform norm[5]

$$\|\mathbf{y}_2\|_\infty = \max_{|t-t_0|\le 2R} \|\mathbf{y}(t)\|.$$

Note the use of two different norms. The space $X = C^0([t_0 - 2R, t_0 + 2R] \to \mathbb{R}^n)$ is complete owing to the following two facts from analysis:

1. A Cauchy sequence of functions $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \dots$ in $C^0([t_0 - 2R, t_0 + 2R] \to \mathbb{R}^n)$ with respect to the uniform norm converges pointwise to some function
$$\mathbf{y}(t) = \lim_{j\to\infty} \mathbf{y}_j(t).$$
This is just the Cauchy completeness theorem for $\mathbb{R}^n$ since for fixed t the sequence $\{\mathbf{y}_j(t)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^n$.

2. The uniform limit of a sequence of continuous functions on a closed interval is continuous. (One also needs to show the pointwise limit $\mathbf{y}$ is also a uniform limit.)

It is to a mapping on the space $X = C^0([t_0 - 2R, t_0 + 2R] \to \mathbb{R}^n)$ to which we will apply the Banach fixed point theorem.

Consider $\Gamma : C^0([t_0 - 2R, t_0 + 2R] \to \mathbb{R}^n) \to C^0([t_0 - 2R, t_0 + 2R] \to \mathbb{R}^n)$ by

$$\Gamma[\mathbf{y}](t) = \mathbf{y}_0 + \int_{t_0}^t \mathbf{f}(\tau, \mathbf{y}(\tau))\, d\tau. \tag{6.5}$$

We need to verify this mapping satisfies the requirements of the corollary of the Banach fixed point theorem. That the mapping is well-defined is clear. In fact, by the fundamental theorem of calculus

$$\frac{d}{dt}\Gamma[\mathbf{y}](t) = \mathbf{f}(t, \mathbf{y}(t)). \tag{6.6}$$

Thus, Γ maps into the even smaller subspace $C^1([t_0 - 2R, t_0 + 2R] \to \mathbb{R}^n)$. Note the definition of $\Gamma[y]$ given in (6.5) applies for $|t - t_0| \le 3R$, so the

[5]This norm is also called the **supremum norm** or the **infinity norm** especially in the context of the space $L^\infty[t_0 - 2R, t_0 + 2R]$ of essentially bounded functions. This is where we get the notation.

derivative given in (6.6) is also well-defined and continuous up to the endpoints in $[t_0 - 2R, t_0 + 2R]$. We need to show some iteration of compositions Γ^k is a contraction. We obtain first, by induction, a pointwise estimate[6]

$$\|\Gamma^k[\mathbf{y}_2](t) - \Gamma^k[\mathbf{y}_1](t)\| \le \frac{\Lambda^k}{k!}\|\mathbf{y}_2 - \mathbf{y}_1\|_\infty |t - t_0|^k \tag{6.7}$$

holding for all $k \in \mathbb{N}$, $\mathbf{y}_2, \mathbf{y}_1 \in X$ and t with $|t - t_0| < 2R$. The base case for $k = 1$ is as follows:

$$\begin{aligned}\|\Gamma[\mathbf{y}_2](t) - \Gamma[\mathbf{y}_1](t)\| &= \left|\int_{t_0}^{t} [\mathbf{f}(\tau, \mathbf{y}_2(\tau)) - \mathbf{f}(\tau, \mathbf{y}_1(\tau))]\, d\tau\right| \\ &\le \left|\int_{t_0}^{t} \|\mathbf{f}(\tau, \mathbf{y}_2(\tau)) - \mathbf{f}(\tau, \mathbf{y}_1(\tau))\|\, d\tau\right| \\ &\le \Lambda \left|\int_{t_0}^{t} \|\mathbf{y}_2(\tau) - \mathbf{y}_1(\tau)\|\, d\tau\right| \\ &\le \Lambda \|\mathbf{y}_2 - \mathbf{y}_1\|_\infty \left|\int_{t_0}^{t} 1\, d\tau\right| \\ &\le \Lambda \|\mathbf{y}_2 - \mathbf{y}_1\|_\infty |t - t_0|.\end{aligned}$$

Now we take (6.7) as inductive hypothesis. Then we have

$$\begin{aligned}\|\Gamma^{k+1}[\mathbf{y}_2](t) - \Gamma^{k+1}[\mathbf{y}_1](t)\| &= \left|\int_{t_0}^{t} [\mathbf{f}(\tau, \Gamma^k\mathbf{y}_2(\tau)) - \mathbf{f}(\tau, \Gamma^k\mathbf{y}_1(\tau))]\, d\tau\right| \\ &\le \left|\int_{t_0}^{t} \|\mathbf{f}(\tau, \Gamma^k\mathbf{y}_2(\tau)) - \mathbf{f}(\tau, \Gamma^k\mathbf{y}_1(\tau))\|\, d\tau\right| \\ &\le \Lambda \left|\int_{t_0}^{t} \|\Gamma^k\mathbf{y}_2(\tau) - \Gamma^k\mathbf{y}_1(\tau)\|\, d\tau\right| \\ &\le \frac{\Lambda^{k+1}}{k!} \|\mathbf{y}_2 - \mathbf{y}_1\|_\infty \left|\int_{t_0}^{t} |\tau - t_0|^k\, d\tau\right| \\ &= \frac{\Lambda^{k+1}}{k!} \|\mathbf{y}_2 - \mathbf{y}_1\|_\infty \left|\int_{t_0}^{t} (\tau - t_0)^k\, d\tau\right| \\ &= \frac{\Lambda^{k+1}}{(k+1)!} \|\mathbf{y}_2 - \mathbf{y}_1\|_\infty |t - t_0|^{k+1}.\end{aligned}$$

[6] Again, note carefully the distinction between norms. I am not using, and cannot express the content of this estimate using, an infinity norm on the left.

Having established what may be called the **pointwise polynomial estamate** (6.7) for all $k = 1, , 2, 3, \ldots$, we proceed to an estimate in the Banach space $X = C^0([t_0 - 2R, t_0 + 2R] \to \mathbb{R}^n)$:

$$\|\Gamma^k[\mathbf{y}_2] - \Gamma^k[\mathbf{y}_1]\|_\infty \leq \frac{(2R\Lambda)^k}{k!} \|\mathbf{y}_2 - \mathbf{y}_1\|_\infty. \tag{6.8}$$

This estimate follows immediately from the pointwise polynomial estimate by taking the maximum value[7] of both sides for $|t - t_0| \leq 2R$.

Finally, then, $\Gamma^k : X \to X$ is seen to be a contraction when

$$\frac{(2R\Lambda)^k}{k!} < 1.$$

The fact that for any fixed positive number M we have

$$\lim_{k \nearrow \infty} \frac{M^k}{k!} = 0$$

is standard from elementary calculus, but in the spirit of review (and just for fun) here is a direct proof based on the simpler facts

$$\lim_{k \nearrow \infty} \frac{M}{k} = 0 \qquad \text{and} \qquad \lim_{k \nearrow \infty} \frac{M}{2^k} = 0.$$

Let $n \in \mathbb{N}$ be fixed so that

$$\frac{M}{n} < \frac{1}{2}.$$

Then

$$\frac{M^{n+1}}{(n+1)!} = \frac{M}{n+1} \frac{M^n}{n!} < \frac{1}{2} \frac{M^n}{n!}$$

and we see by an easy induction that

$$\frac{M^{n+k}}{(n+k)!} < \frac{1}{2^k} \frac{M^n}{n!}.$$

Since $M^n/n!$ is fixed, we see that given any $\epsilon > 0$ we can make k large enough so that

$$\frac{M^{n+k}}{(n+k)!} < \epsilon.$$

[7] Note, however, that this estimate on the infinity norm is not equivalent to (6.7); you cannot get back the pointwise estimate.

In any case, the corollary of the Banach fixed point theorem applies, and we conclude there is a unique $\mathbf{y}_* \in C^0([t_0 - 2R, t_0 + 2R] \to \mathbb{R}^n)$ with $\Gamma[\mathbf{y}_*] = \mathbf{y}_*$. That is,

$$\mathbf{y}_* = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{y}_*(\tau))\, d\tau.$$

We see from the fundamental theorem of calculus and simple evaluation that $\mathbf{y}_*$ is a solution of the IVP

$$\begin{cases} \mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t)) & \text{for } |t - t_0| \le R \\ \mathbf{y}(t_0) = \mathbf{y}_0. \end{cases}$$

It also follows from the fundamental theorem of calculus that

$$\mathbf{y}_* \in C^1([t_0 - r, t_0 + r] \to \mathbb{R}^n) \qquad \text{for each } r \text{ with } 0 < r < 2R.$$

In particular, $\mathbf{y}_* \in C^1([t_0 - R, t_0 + R] \to \mathbb{R}^n)$. Thus, we have existence of the desired solution of Lemma 35.

If $\mathbf{y} \in C^1([t_0 - R, t_0 + R] \to \mathbb{R}^n)$ were another such solution, then we obtain $\mathbf{v} = \mathbf{y} - \mathbf{y}_*$ satisfies the particular initial value problem

$$\begin{cases} \mathbf{v}'(t) = \mathbf{0} & \text{for } |t - t_0| \le R \\ \mathbf{v}(t_0) = \mathbf{0}. \end{cases} \tag{6.9}$$

This particular IVP is susceptible to direct integration:

$$\mathbf{v}(t) = \mathbf{0} + \int_{t_0}^{t} \mathbf{0}\, d\tau \equiv \mathbf{0}$$

so that uniqueness follows. Note: It may be possible to conclude the uniqueness of the general (Lipschitz) IVP by using the uniqueness of the Banach fixed point theorem (or its corrollary), but our set-up for the proof of Lemma 35 does not make this so convenient. The reason is that it is not immediately obvious, given an alternative solution $\mathbf{y} \in C^1([t_0 - R, t_0 + R] \to \mathbb{R}^n)$ how to extend it to a solution in $X = C^1([t_0 - 2R, t_0 + 2R] \to \mathbb{R}^n)$. At some level this brings up the question: Is the consideration of the three distinct intervals $[t_0 - R, t_0 + R]$, $[t_0 - 2R, t_0 + 2R]$, and $[t_0 - 3R, t_0 + 3R]$ necessary? Of course, one could get away with $[t_0 - R, t_0 + R]$, $[t_0 - (R + \epsilon), t_0 + (R + \epsilon)]$, and $[t_0 - (R + 2\epsilon), t_0 + (R + 2\epsilon)]$ for any $\epsilon > 0$, but can one get away with only two such intervals, or is it overkill to consider any extended interval at all? The answer, in turn, seems to lie in consideration of the following question: If I have $\mathbf{y} \in C^0([t_0 - R, t_0 + R] \to \mathbb{R}^n)$, then can I conclude $\Gamma[\mathbf{y}]$ given by

$$\Gamma[\mathbf{y}](t) = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{y}(\tau))\, d\tau \tag{6.10}$$

satisfies $\Gamma[\mathbf{y}] \in C^1([t_0 - R, t_0 + R] \to \mathbb{R}^n)$? This assertion requires differentiability at the endpoints $t = t_0 \pm R$ which, at least in higher dimensions, is usually defined to mean there is an extension of $\mathbf{y}$ to a larger open set like $(t_0 - (R + \epsilon), t_0 + (R + \epsilon))$ to a continuously differentiable function. The construction I've given gives this. On the other hand, differentiability at the boundary of a closed interval in $\mathbb{R}^1$ is somewhat simpler, and one can consider one-sided derivatives. This consideration may make the use of nested intervals unnecessary in this case, but still, it may mean you need to consider one-sided derivatives at the endpoints of functions given by expressions like (6.10).

In any case, I think we've finished the proof of Lemma 35. □

Proof of Theorem 32: Given $t_1 \in \mathbb{R}$, the lemma gives us, for example, solutions $\mathbf{y}_R \in C^1([t_0 - R, t_0 + R] \to \mathbb{R}^n)$ of (6.4) for each $R > |t_1 - t_0|$. Since $\mathbf{y}_R \in C^1([t_0 - r, t_0 + r] \to \mathbb{R}^n)$ for every r with $|t_1 - t_0| < r < R$ and we have uniqueness of $\mathbf{y}_r$ as a solution, we must have $\mathbf{y}_R(t_1)$ taking a common value for all $R > |t_1 - t_0|$. Thus, a function $\mathbf{y}_* : \mathbb{R} \to \mathbb{R}^n$ given by this common value

$$\mathbf{y}_*(t) = \mathbf{y}_R(t) \qquad \text{for any } R > |t - t_0| \tag{6.11}$$

is well-defined.

In addition, since continuous differentiability and satisfying the ODE are local properties, the relation (6.11) gives us that $\mathbf{y}_* \in C^1(\mathbb{R} \to \mathbb{R}^n)$ is a solution of the global initial value problem (6.3). Uniqueness follows as in the proof of Lemma 35 from the uniqueness of the solution of (6.9). □

The Banach Fixed Point Theorem

We now turn to proofs of Theorem 33 and Corollary 34 as well as a brief discussion of complete metric spaces and metric spaces given by norms.

A **metric space** is a set X with a (distance) function $d : X \times X \to [0, \infty)$ satisfying

(i) $d(x, y) = d(y, x)$ for all $x, y \in X$. (symmetric)

(ii) $d(x, y) = 0$ if and only if $x = y$. (positive definite)

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. (triangle inequality)

This definition does not require any additional structure on the set X. In particular **any** subset $A \subset X$ of a metric space X is also a metric space obtained by simply restricting the distance function to the cross-product of A with itself.

In contrast, a **normed space** must be, apriori, a **vector space** which is a relatively complicated algebraic structure requiring, in particular, an operation of addition of vectors, a scalar field in the background, and an operation of scaling of vectors by elements of the scalar field. I will assume this structure is more or less familiar and/or can be looked up. Nevertheless, it should be noted that to really understand and appreciate the structure of vector spaces one already needs to understand and appreciate, at least to some extent, the general algebraic structures of groups, rings, and fields from abstract/modern algebra.

For our purposes, it is perhaps enough to note that the presence of a **norm** requires addition $v+w$ and scaling αv. More precisely, V is a **normed space** if V is a vector space over a field F with a function $\| \cdot \| : V \to [0, \infty)$ (called a norm) having the following properties:

(i) $\|\alpha v\| = |\alpha| \|v\|$ for all $\alpha \in F$ and $v \in V$. (nonnegative homogeneous)

(ii) $\|v\| = 0$ if and only if $v = \mathbf{0}$. (positive definite)

(iii) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$. (triangle inequality)

Every normed space is a metric space with distance given by

$$d(v, w) = \|v - w\|.$$

This is called the **norm induced distance** or norm induced metric.

Exercise 6.1 *Prove it.*

Completeness is a property of some metric spaces. Either a metric space is complete or it is not.

Definition 17 *A metric space X is* **complete** *if every Cauchy sequence converges to an element of X.*

To elaborate on this definition:

1. **Convergence** of sequences makes sense in any metric space: A sequence $\{x_j\}_{j=1}^{\infty} \subset X$ **converges to a limit** $x \in X$ if for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that
$$d(x_j, x) < \epsilon \qquad \text{whenever} \qquad j > N.$$

2. The **Cauchy property for sequences** makes sense in any metric space: A sequence $\{x_j\}_{j=1}^{\infty} \subset X$ is **Cauchy** if for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that
$$d(x_j, x_k) < \epsilon \qquad \text{whenever} \qquad j, k > N.$$
Notice that no limiting point appears in the definition of the Cauchy property.

3. Every sequence in a metric space which converges is Cauchy.

   Exercise 6.2 *Prove it.*

   Exercise 6.3 *Let $X = [-1, 1)$ be the metric subspace of $\mathbb{R}$ with the usual distance $d(x, y) = |x - y|$ given by the absolute value (norm) on $\mathbb{R}^1$.*

   (a) *Show $x_j = -1 + 1/j$ defines a sequence in X convergent to $-1 \in X$.*

   (b) *Show $x_j = (-1)^j/j$ defines a sequence in X convergent to $0 \in X$.*

   (c) *Show $x_j = 1 - 1/j$ defines a sequence in X which is Cauchy but* **does not converge** *to any element $x \in X$.*

From what we have said above, it should be clear that it makes sense to ask when a normed space is complete. By this we mean the following:

Take the normed space V and consider it as a metric space under the norm induced distance. A normed space which is complete (as a metric space) has a special name; it is called a **Banach space**.

Our theorem, the Banach fixed point theorem, is not properly about Banach spaces. It is about complete metric spaces, but of course it applies to Banach spaces, and our application of it was to a Banach space $X = C^0(I \to \mathbb{R}^n)$ where I is a closed interval and the norm is the infinity norm.

Proof of the Banach fixed point theorem: Take any point $x \in X$. Let us first prove the sequence $\{\Gamma^j(x)\}_{j=1}^{\infty}$ is a **bounded sequence** in X. Remember that in the statement of Theorem 33 we are givne a contraction $\Gamma : X \to X$ which is a function for which there is a constant c with $0 < c < 1$ such that

$$d(\Gamma(x_2), \Gamma(x_1)) \le c\ d(x_2, x_1) \qquad \text{for all } x_1, x_2 \in X.$$

By the triangle inequality

$$d(\Gamma^2(x), x) \le d(\Gamma^2(x), \Gamma(x)) + d(\Gamma(x), x) \le (c+1)\ d(\Gamma(x), x).$$

It follows by an easy induction that

$$d(\Gamma^k(x), x) \le \left(\sum_{j=0}^{k-1} c^j \right) d(\Gamma(x), x) \le \frac{1}{1-c}\, d(\Gamma(x), x).$$

Another thing that makes sense in any metric space is the notion of an open ball and, hence, the notion of an **open set**. In this case, we have shown

$$\Gamma^k(x) \in B_r(x) = \{\xi \in X : d(\xi, x) < r\} \qquad \text{for all } k \in \mathbb{N}$$

where, for example, $r = 1 + d(\Gamma(x), x)/(1-c) > 0$. This is essentially what it means to be bounded in a metric space.

We proceed to show our sequence of iterates is Cauchy:

$$d(\Gamma^j(x) - \Gamma^k(x)) \le c^{\min\{j,k\}}\ d(\Gamma^{\max\{j,k\}-\min\{j,k\}}, x) < c^{\min\{j,k\}}\ r.$$

Since $c < 1$ and $r > 0$, given any $\epsilon > 0$, there is some $N \in \mathbb{N}$ so that $c^N < \epsilon/r$ and then

$$d(\Gamma^j(x) - \Gamma^k(x)) < \epsilon \qquad \text{for } j, k > N.$$

Thus, $\{\Gamma^j(x)\}_{j=1}^{\infty}$ is a Cauchy sequence in a complete metric space. By the definition of completeness there is some $x_* \in X$ for which

$$\lim_{j\nearrow\infty} \Gamma^j(x) = x_*.$$

Now, we want to show x_* is a fixed point of Γ. One way to do this is to know **the distance function is continuous**:

$$d(\Gamma(x_*), x_*) = d\left(\lim_{j\nearrow\infty} \Gamma^{j+1}(x), \lim_{j\nearrow\infty} \Gamma^j(x)\right) = d(x_*, x_*) = 0.$$

Thus, $\Gamma(x_*) = x_*$ because the metric is positive definite. Alternatively, we can use the contraction property and the triangle inequality for a more direct proof:

$$d(\Gamma(x_*), x_*) \le d(\Gamma(x_*), \Gamma^k(x)) + d(\Gamma^k(x), x_*) \le (c+1)\ d(\Gamma^k(x), x_*).$$

The fact that

$$\lim_{k\nearrow\infty} d(\Gamma^k(x), x_*) = 0$$

is just the definition of convergence. Either way, we have established that x_* is a fixed point of Γ.

If x_{**} is another fixed point, then

$$d(x_{**}, x_*) = d(\Gamma(x_{**}), \Gamma(x_*)) \le c\ d(x_{**}, x_*).$$

Thus, $0 \le (1-c)\, d(x_{**}, x_*) \le 0$ from which it follows $d(x_{**}, x_*) = 0$ and $x_* = x_{**}$ is unique. $\square$

Proof of Corollary 34: Here we have a fixed point of the contraction $\Gamma^k : X \to X$. That is, there is a unique $x_* \in X$ for which $\Gamma^k(x_*) = x_*$. We first claim that x_* is a fixed point for $\Gamma : X \to X$. In fact,

$$d(\Gamma(x_*), x_*) = d(\Gamma^{k+1}(x_*), \Gamma^k(x_*)) \le c\ d(\Gamma(x_*), x_*), \tag{6.12}$$

so we obtain as in the argument above

$$0 \le (1-c) d(\Gamma(x_*), x_*) \le 0.$$

Of course, a fixed point of Γ is a fixed point of Γ^k. Since we know the fixed point of Γ^k is unique, clearly the fixed point x_* is unique as a fixed point of Γ as well. It remains to show

$$\lim_{j \nearrow \infty} \Gamma(x) = x_* \qquad \text{for any } x \in X. \tag{6.13}$$

This proposition seems a little delicate. However, it is not difficult to establish by induction that

$$d(\Gamma^{\ell k+m}(x), x_*) \le c^\ell M \qquad \text{for } \ell, m \in \mathbb{N}, \tag{6.14}$$

where $M = \max\{d(\Gamma^j(x), x_*) : 0 \le j < k\}$ is fixed and finite and c with $0 < c < 1$ is the contraction constant for Γ^k. This condition may be written equivalently as

$$d(\Gamma^j(x), x_*) \le c^\ell M \qquad \text{for } j \in \mathbb{N} \text{ with } j > \ell k.$$

It follows that given any $\epsilon > 0$ there is some $L \in \mathbb{N}$ for which $c^L < \epsilon/M$. Thus, taking $N = Lk \in \mathbb{N}$ we have for $j > N = Lk$ that

$$d(\Gamma^j(x), x_*) \le c^L M < \epsilon.$$

This establishes (6.13) and completes the proof. □

I was pretty happy with (6.12) and the proof that x_* is the unique fixed point for Γ. The argument giving the limit (6.13) using (6.14) seems a bit clumsy, and maybe you can give a better one. On the other hand, maybe an "eventual contraction" $\Gamma : X \to X$ can do something like expand for the interations Γ^j with $0 < j < k$, and maybe this makes something like (6.14) more or less necessary.

Chapter 7

Comparison Principles

Consider Figure 7.1, Figure 7.2, and Figure 7.3. The comparison principle captures (especially in higher dimensions) certain aspects of what is illustrated in these figures. One concept to keep in mind is that if one function "dominates" another, i.e., $f \le g$, then certain conditions hold at any points of equality (touching). This happens, in particular, at a local maximum when one function is dominated (locally) by a constant function. Closely related to the necessary conditions at a touching point is the fact that two functions satisfying some inequality between the second derivatives, $f'' > g''$, and having also an inequality over a boundary, $f(a) \le g(a)$ and $f(b) \le g(b)$ must satisfy a global pointwise inequality $f(x) < g(x)$ for $a < x < b$. Several figures illustrating this implication using quadratic functions (with constant second derivative) are shown in Figure 7.2.

Finally, these considerations carry over to certain other "second order inequalities." Notice that an inequality on curvatures,

$$\frac{f''}{(1+f'^2)^{3/2}} > \frac{g''}{(1+g'^2)^{3/2}}, \tag{7.1}$$

is not equivalent to the inequality $f'' > g''$.

Exercise 7.1 *Give examples of functions $f, g \in C^2[a,b]$ for which (7.1) holds on (a,b) but $f''(x_0) < g''(x_0)$ at some point $x_0 \in (a,b)$.*

Exercise 7.2 *Give examples of functions $f, g \in C^2[a,b]$ for which $f''(x_0) > g''(x_0)$ holds on (a,b) but*

$$\frac{f''(x_0)}{(1+f'(x_0)^2)^{3/2}} \le \frac{g''(x_0)}{(1+g'(x_0)^2)^{3/2}} \qquad \text{at some point } x_0 \in (a,b).$$

Nevertheless, the same global pointwise inequality $f(x) < g(x)$ for $a < x < b$ if $f, g \in C^2[a, b]$ are functions for which (7.1) holds with $f(a) \leq g(a)$ and $f(b) \leq g(b)$.

Consider "pushing up" on the graph of f, as indicated by the arrow in Figure 7.3, so that a single interior touching point between the graphs occurs but in such a way that the inequality

$$\frac{f''}{(1 + f'^2)^{3/2}} \geq \frac{g''}{(1 + g'^2)^{3/2}}$$

is maintained. If you "feel" that the only way to accomplish this task is by making f **identically equal** to g, then you are starting to see what Finn called the magic of the comparison principle. These are very simple ideas, but they are still rather magical. Being able to justify them rigorously is also not so difficult, and we will try to discuss what is involved with that now.

As a preliminary consideration, let us consider a real valued function of one variable with a (local) maximum at a point $x_0 \in \mathbb{R}$. In this situation we have

$$f'(x_0) = 0 \qquad \text{and} \qquad f''(x_0) \leq 0.$$

These necessary conditions are, in a certain sense, precursors to the maximum principle.

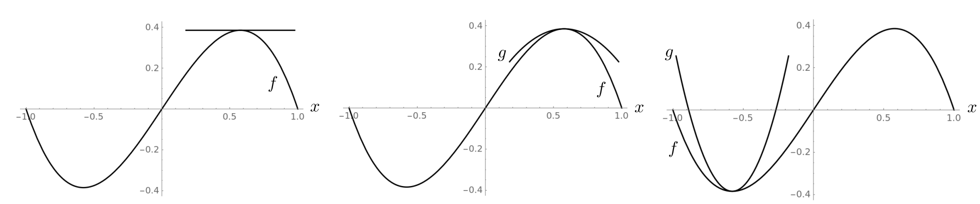

Figure 7.1: Touching Points

Exercise 7.3 *If $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ with $f, g \in C^2(a, b)$ and $f \leq g$, and $f(x_0) = g(x_0)$ for some $x_0 \in (a, b)$, then what can you say about the relations between*

(a) *$f'(x_0)$ and $g'(x_0)$, and*

(b) *$f''(x_0)$ and $g''(x_0)$?*

Next, let us consider two functions $f, g \in C^2[a,b]$ with $f(a) = g(a)$ and $f(b) = g(b)$. If we know

$$f''(x) \geq g''(x) \qquad \text{for } x \in (a,b), \tag{7.2}$$

then what can be said about the relation between f and g?

Exercise 7.4 *Given $f \in C^2[a,b]$ with $f(a) = f(b) = 0$ and $f''(x) \geq 0$ for $x \in (a,b)$, show*

(a) *$f(x) \leq$ for $x \in (a,b)$, and*

(b) *If $f(x_0) = 0$ for any one point $x_0 \in (a,b)$, then $f \equiv 0$.*

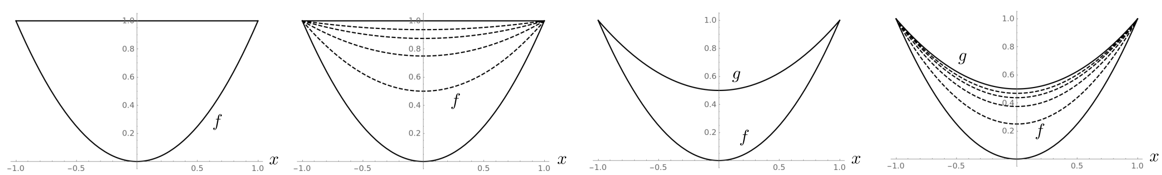

Figure 7.2: Functions with the same boundary values and satisfying a second order inequality

As a generalization of the previous exercise we may assume there is a point x for which $f(x) > g(x)$. This means we can apply the extreme value theorem to obtain a point x_0 for which

$$f(x_0) - g(x_0) = \max(f - g) > 0.$$

According to the necessary condition we find

$$f'(x_0) = g'(x_0) \qquad \text{and} \qquad f''(x_0) \leq g''(x_0).$$

Notice this is not enough to get a contradiction, but it is in the "right direction." That is to say, if we had $f''(x_0) < g''(x_0)$, then we would have a contradiction of (7.2). To obtain such a point, we can argue as follows: Consider a quadratic function h_0 having the form $h_0(x) = M - \epsilon(x - x_0)^2$ where $M = \max(f-g) > 0$ and ϵ is a small number to be chosen. Remember $(f - g)(a) = (f - g)(b) = 0$. We could introduce appropriate inequalities at the boundary instead of these equalities. At any rate, for $\epsilon > 0$ small enough we will have $h_0(a) = M - \epsilon(a - x_0)^2$ and $h_0(b) = M - \epsilon(b - x_0)^2$ both positive. Thus, the maximum value

$$c = \max\{f(x) - g(x) - h_0(x) : x \in [a, b]\} \geq 0$$

is achieved at some point $x_1 \in (a, b)$. Thus, we may set $h = h_0 + c$, and we have a function h satisfying

(i) $h \geq f - g$ on $[a, b]$,

(ii) $h(x_1) = (f - g)(x_1)$, and

(iii) $h'' \equiv -2\epsilon < 0$.

The necessary condition then implies $f''(x_1) - g''(x_1) \leq h''(x_1) = -2\epsilon < 0$. That is, $f''(x_1) < g''(x_1)$ and we do get a contradiction. This argument shows the following:

Theorem 36 *If $f, g \in C^2[a, b]$ with $f'' \geq g''$, $f(a) \leq g(a)$ and $f(b) \leq g(b)$, then $f \leq g$ on $[a, b]$.*

Exercise 7.5 *Under the hypotheses of Theorem 36 show that if equality $f(x_0) = g(x_0)$ holds at one point $x_0 \in (a, b)$, then $f \equiv g$ or equivalently, $f(x) < g(x)$ for $x \in (a, b)$ unless $f \equiv g$.*

7.6 The Maximum Principle

Let us first consider the **maximum principle** which applies to solutions or subsolutions of certain second order linear partial differential equations. The second order linear partial differential operator $L : C^2(\mathcal{U}) \to C^0(\mathcal{U})$ given by

$$Lu = \sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{j=1}^{n} b_j D_j u + cu \tag{7.3}$$

is said to be **elliptic** if the coefficient matrix $A = (a_{ij})$ is symmetric and positive definite. In defining the form of this operator the coefficients a_{ij}, b_j and c may be assumed to be continuous functions of a variable $\mathbf{x}$ defined on an open subset $\mathcal{U}$ of $\mathbb{R}^n$. We have denoted partial derivatives as follows:

$$D_j u = \frac{\partial u}{\partial x_j}; \quad D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

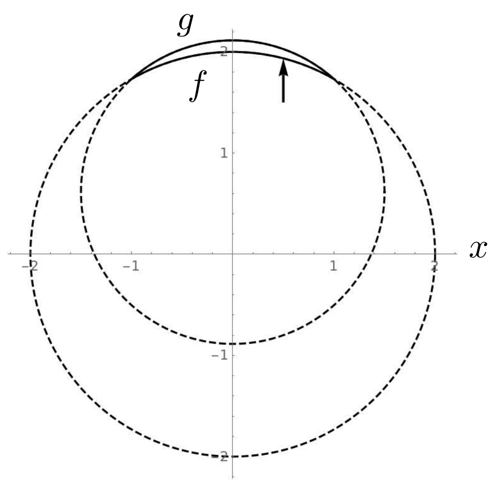

Figure 7.3: Circles with different curvatures and functions determined by them having the same boundary values at $a = -1$ and $b = 1$

Note that in principle one may allow the leading order coefficient matrix $A = (a_{ij})$ to be non-symmetric. In this case, the same operator may be expressed in terms of a symmetric leading order coefficient matrix $\tilde{A} = (\tilde{a}_{ij})$ with

$$\tilde{a}_{ij} = \frac{a_{ij} + a_{ji}}{2}.$$

Exercise 7.7 *What condition on a non-symmetric matrix $A = (a_{ij})$ implies the symmetrized matrix $\tilde{A} = (\tilde{a}_{ij})$ with*

$$\tilde{a}_{ij} = \frac{a_{ij} + a_{ji}}{2}.$$

is positive definite? Remember that an $n \times n$ real matrix M is positive definite if

$$M\mathbf{v} \cdot \mathbf{v} \geq 0 \qquad \textit{with equality only if } \mathbf{v} = \mathbf{0} \in \mathbb{R}^n.$$

The simplest example of a second order linear elliptic operator is the Laplace operator given by

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

In this case, the coefficient matrix is the $n \times n$ identity matrix and the coefficients of the lower order terms b_j and c are all zero.

A function $u \in C^2(\mathcal{U})$ is said to be a (classical) **subsolution** if $Lu \geq 0$ on $\mathcal{U}$. Similarly, u is a **supersolution** if $Lu \leq 0$ on $\mathcal{U}$ and a **classical solution** if $Lu = 0$. The **weak maximum principle** asserts, under various conditions, that the value of a subsolution $u \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ cannot exceed the value of u on the boundary of a bounded domain $\mathcal{U}$. There are many generalizations and variants, some of which we will consider below. Let us first consider cases where some or all of the lower order terms are not present to illustrate some fundamental mechanisms driving weak maximum principles.

Theorem 37 *If $\mathcal{U}$ is a open subset of $\mathbb{R}^n$ and L is an operator of the form (7.3) with leading coefficient matrix $A = (a_{ij})$ non-negative semi-definite and satisfying $c = 0$ on $\mathcal{U}$, then a* **strict subsolution** *$u \in C^2(\mathcal{U})$ satisfying*

$$Lu > 0 \qquad \text{on } \mathcal{U}$$

cannot have an interior local maximum.

Proof: The key here is that

$$Lu = \sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_j b_j D_j u = \text{tr}(AD^2 u) + \sum_j b_j D_j u$$

where $D^2 u = (D_{ij} u)$ is the Hessian matrix of u. Thus, at an interior local maximum $\mathbf{p}$, one has $Du(\mathbf{p}) = 0$ and $D^2 u(\mathbf{p}) \leq 0$ by which we mean $D^2 u(\mathbf{p})$ is non-positive semi-definite, or

$$D^2 u(\mathbf{p}) \mathbf{v} \cdot \mathbf{v} \geq 0 \qquad \text{for all } \mathbf{v} \in \mathbb{R}^n.$$

Exercise 7.8 *Review the proof of the necessary conditions $Du(\mathbf{p}) = 0$ and $D^2 u(\mathbf{p}) \leq 0$ at a local maximum $\mathbf{p} \in \mathcal{U}$ of a function $u \in C^2(\mathcal{U})$.*

Any $n \times n$ real symmetric matrix, for example $D^2 u(\mathbf{p})$ has real eigenvalues corresponding to a basis for $\mathbb{R}^n$ consisting of orthonormal vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Taking these vectors as the columns of a matrix Q^{-1}, we obtain a change of basis matrix Q satisfying $Q^{-1} = Q^T$ and having $M = QD^2 u(\mathbf{p}) Q^{-1}$ diagonal. The diagonal entries μ_{jj} of M are non-positive since

$$\mu_{jj} = QD^2 u(\mathbf{p}) Q^{-1} \mathbf{e}_j \cdot \mathbf{e}_j = D^2 u(\mathbf{p}) Q^{-1} \mathbf{e}_j \cdot Q^{-1} \mathbf{e}_j.$$

On the other hand, the trace is invariant under conjugation,[1] so we can write

$$Lu(\mathbf{p}) = \operatorname{tr}(A(\mathbf{p})D^2u(\mathbf{p})) = \operatorname{tr}(A(\mathbf{p})Q^{-1}MQ) = \operatorname{tr}(QA(\mathbf{p})Q^{-1}M) = \operatorname{tr}(\tilde{A}M)$$

where $\tilde{A} = QA(\mathbf{p})Q^{-1}$. The conjugation $\tilde{A}$ of A is also symmetric since $(QAQ^{-1})^T = (QAQ^T)^T = (Q^T)^TA^TQ^T$ and non-negative semi-definite since

$$QAQ^{-1}\mathbf{v}\cdot\mathbf{v} = AQ^{-1}\mathbf{v}\cdot Q^{-1}\mathbf{v}.$$

Furthermore, the product $\tilde{A}M = (\tilde{a}_{ij})M$ is easily seen to be the matrix obtained by multiplying each column of $\tilde{A}$ by the corresponding eigenvalue of M:

$$\tilde{A}M = (\mu_{jj}a_{ij}).$$

This means

$$\operatorname{tr}(\tilde{A}M) = \sum_{j=1}^{n} \tilde{a}_{jj}\mu_{jj}.$$

Finally, since each diagonal entry in $\tilde{A}$ satisfies

$$\tilde{a}_{jj} = \tilde{A}\mathbf{e}_j\cdot\mathbf{e}_j \geq 0,$$

and each diagonal element/eigenvalue of M satisfies

$$\mu_{jj} = D^2u(\mathbf{p})\mathbf{u}_j\cdot\mathbf{u}_j \leq 0,$$

we know

$$Lu(\mathbf{p}) = \operatorname{tr}(\tilde{A}M) \leq 0.$$

This contradicts our assumption $Lu(\mathbf{p}) > 0$. □

Note that one cannot just assume the product $A(\mathbf{p})D^2u(\mathbf{p})$ is non-positive semi-definite. This will be the case if the product happens to be a symmetric matrix, but not all products of real symmetric matrices are symmetric. The real symmetric matrices do not form a ring.

If the above result is applied to a bounded domain $\mathcal{U} \subset \mathbb{R}^n$, we obtain what is called a **strong maximum principle**. Note that when $\mathcal{U}$ is bounded,

[1]The fact that trace is invariant under conjugation is a consequence of the fact that for real square matrices A and B one has $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ which can be seen immediately from the formula $\operatorname{tr}(AB) = \sum_{i,j=1}^{n} a_{ij}b_{ji}$.

then the closure $\overline{\mathcal{U}}$ and the boundary $\partial\mathcal{U}$ are both compact sets, and if $u \in C^0(\overline{\mathcal{U}})$, then

$$M = \max\{u(\mathbf{x}) : \mathbf{x} \in \overline{\mathcal{U}}\} \qquad \text{and} \qquad m = \max\{u(\mathbf{x}) : \mathbf{x} \in \partial\mathcal{U}\}$$

are both well-defined finite numbers attained at points $\mathbf{p} \in \overline{\mathcal{U}}$ and $\mathbf{q} \in \partial\mathcal{U}$ with

$$u(\mathbf{p}) = M \qquad \text{and} \qquad u(\mathbf{q}) = m.$$

The strong maximum principle asserts $\mathbf{p} \in \partial\mathcal{U}\backslash\mathcal{U}$:

Theorem 38 *(preliminary strong maximum principle) If $\mathcal{U}$ is an open and bounded subset of $\mathbb{R}^n$ and L is an operator of the form (7.3) with leading coefficient matrix $A = (a_{ij})$ non-negative semi-definite and satisfying $c = 0$ on $\mathcal{U}$, then a* **strict subsolution** *$u \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ satisfying*

$$Lu > 0 \qquad \text{on } \mathcal{U}$$

satisfies

$$u(\mathbf{x}) < m = \max_{\partial\mathcal{U}} u \qquad \text{for } \mathbf{x} \in \mathcal{U}. \tag{7.4}$$

We next generalize the necessary condition for a local maximum in a superficially trival way:

Corollary 39 *If $u, v \in C^2(\mathcal{U})$ and $v - u$ has an interior local minimum at a point $\mathbf{p} \in \mathcal{U}$, then*

$$Du(\mathbf{p}) = Dv(\mathbf{p}) \qquad \text{and} \qquad D^2u(\mathbf{p}) \leq D^2v(\mathbf{p})$$

by which we mean $D^2u(\mathbf{p}) - D^2v(\mathbf{p})$ is non-positive semi-definite.

This allows us to illustrate the next fundamental mechanism which may be viewed as driving maximum principles.

Theorem 40 *(preliminary weak maximum principle) If $Lu \geq 0$ on a bounded domain $\mathcal{U} \subset \mathbb{R}^n$ with $u \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ and L has the form (7.3) with leading coefficient matrix $A = (a_{ij})$ positive definite, i.e., L is elliptic, and satisfying $b_j = c = 0$ on $\mathcal{U}$ for $j = 1, 2, \ldots, n$, then*

$$u(\mathbf{x}) \leq m = \max_{\partial\mathcal{U}} u \qquad \text{for all } \mathbf{x} \in \mathcal{U}. \tag{7.5}$$

Proof: Assume, by way of contradiction, there is some $\mathbf{p} \in \mathcal{U}$ with

$$u(\mathbf{p}) > m = \max_{\partial\mathcal{U}} u.$$

Then

$$\delta = \max_{\mathbf{x} \in \partial\mathcal{U}} |\mathbf{x} - \mathbf{p}| < \infty$$

and

$$v_0(\mathbf{x}) = u(\mathbf{p}) - \epsilon|\mathbf{x} - \mathbf{p}|^2$$

satisfies

$$v_0(\mathbf{x}) - u(\mathbf{x}) = u(\mathbf{p}) - u(\mathbf{x}) - \epsilon|\mathbf{x} - \mathbf{p}|^2 \geq u(\mathbf{p}) - m - \epsilon\delta^2 > 0 \qquad \text{for } \mathbf{x} \in \partial\mathcal{U}$$

if $\epsilon < (u(\mathbf{p}) - m)/\delta^2$. Thus,

$$\max_{\partial\mathcal{U}}(v_0 - u) \geq u(\mathbf{p}) - m - \epsilon\delta^2 > 0.$$

Also,

$$c = \max_{\overline{\mathcal{U}}}(u - v_0) = u(\mathbf{q}) - v(\mathbf{q}) \geq 0 \qquad \text{for some } \mathbf{q} \in \mathcal{U}$$

and $Lv_0 = -2\epsilon\,\mathrm{tr}(A) < 0$. Therefore, $v(\mathbf{x}) = v_0(\mathbf{x}) + c$ satisfies

$$\begin{cases} v \geq u \quad \text{on } \overline{\mathcal{U}}, \\ v(\mathbf{q}) = u(\mathbf{q}) \\ Lv = Lv_0 = -2\epsilon\,\mathrm{tr}(A) < 0. \end{cases}$$

Using the corollary

$$\begin{aligned} Lu(\mathbf{q}) &= \sum_{i,j=1}^{n} a_{ij}(\mathbf{q}) D_{ij} u(\mathbf{q}) \\ &= \mathrm{tr}(A(\mathbf{q}) D^2 u(\mathbf{q})) \\ &= \mathrm{tr}(A(\mathbf{q}) D^2 v(\mathbf{q})) + \mathrm{tr}(A(\mathbf{q})(D^2 u(\mathbf{q}) - D^2 v(\mathbf{q}))) \\ &\leq Lv(\mathbf{q}) \\ &< 0. \end{aligned}$$

This contradicts the hypothesis $Lu \geq 0$ on $\mathcal{U}$. □

Theorem 40 is a weak maximum principle by virtue of the weak inequality in (7.5) in contrast to the strong inequality of (7.4) in Theorem 38. We next generalize the weak maximum principle to allow for first order terms in the operator, but a new idea/observation is needed: Consider the function $g(\mathbf{x}) = \epsilon e^{\gamma x_1}$ for which

$$Lg = \epsilon(\gamma^2 a_{11} + \gamma b_1 + c)e^{\gamma x_1}.$$

Notice that for an elliptic operator we know $a_{11} > 0$. Therefore, as long as the lower order coordinate functions b_1 and c are bounded and a_{11} is bounded away from zero, it is possible to ensure $Lg > 0$ for every $\epsilon > 0$ by taking $\gamma > 0$ large enough.

Theorem 41 *(classical weak maximum principle) If $Lu \geq 0$ on a bounded domain $\mathcal{U} \subset \mathbb{R}^n$ with $u \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ and L has the form (7.3) with leading coefficient matrix $A = (a_{ij})$ positive definite (i.e., L is elliptic) and satisfying the following:*

(i) *There is some $\lambda > 0$ such that*

$$a_{11} = a_{11}(\mathbf{x}) \geq \lambda > 0 \qquad \text{for all } \mathbf{x} \in \mathcal{U},$$

(ii) *There is some $M_b > 0$ such that*

$$|b_j| = |b_j(\mathbf{x})| \leq M_b < \infty \qquad \text{for all } \mathbf{x} \in \mathcal{U} \text{ and } j = 1, 2, \ldots, n, \text{ and}$$

(iii) *$c = c(\mathbf{x}) \equiv 0$ on $\mathcal{U}$,*

then

$$u(\mathbf{x}) \leq m = \max_{\partial\mathcal{U}} u \qquad \text{for all } \mathbf{x} \in \mathcal{U}. \tag{7.6}$$

Proof: For any $\epsilon > 0$, if we take $\gamma > M_b/\lambda$, then

$$L(u + g) = Lu + \epsilon(\gamma^2 a_{11} + \gamma b_1)e^{\gamma x_1} > 0$$

where $g = \epsilon e^{\gamma x_1}$ as above.

Applying the preliminary strong maximum principle, Theorem 38, we conclude

$$u(\mathbf{x}) + \epsilon e^{\gamma x_1} < \max_{\partial\mathcal{U}} u \qquad \text{for every } \mathbf{x} \in \mathcal{U}.$$

Letting ϵ tend to zero wtih γ fixed gives

$$u(\mathbf{x}) \leq \max_{\partial\mathcal{U}} u \qquad \text{for every } \mathbf{x} \in \mathcal{U}$$

as desired. $\square$

Exercise 7.9 *Show the hypotheses (i) and (ii) of the classical maximum principle can be replaced with the condition that there is some constant* $\Lambda_b > 0$ *for which*

$$\frac{|b_j(\mathbf{x})|}{a_{11}(\mathbf{x})} \leq \Lambda_b < \infty \qquad \text{for every } \mathbf{x} \in \mathcal{U}.$$

Exercise 7.10 *(uniqueness of solutions of the Dirichlet problem) Show that if* L *is elliptic in a bounded domain* $\mathcal{U}$ *with* $c \leq 0$ *on* $\mathcal{U}$ *and*

(i) $u, v \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ *satisfying*

(ii) $Lu = Lv$ *on* $\mathcal{U}$*, and*

(iii) $u = v$ *on* $\partial\mathcal{U}$*,*

then $u \equiv v$ *on* $\mathcal{U}$*.*

Exercise 7.11 *(comparison principle) Show that if* L *is elliptic in a bounded domain* $\mathcal{U}$ *with* $c \leq 0$ *on* $\mathcal{U}$ *and*

(i) $u, v \in C^2(\mathcal{U}) \cap C^0(\overline{\mathcal{U}})$ *satisfying*

(ii) $Lu \geq Lv$ *on* $\mathcal{U}$*, and*

(iii) $u \leq v$ *on* $\partial\mathcal{U}$*,*

then $u \leq v$ *on* $\mathcal{U}$*.*

7.12 E. Hopf Strong Maximum Principle

The next result strengthens the preliminary strong maximum principle, Theorem 38, above:

Theorem 42 *(E. Hopf strong maximum principle) If* L *is* **uniformly elliptic** *on* $\mathcal{U}$*, i.e., there is some constant* $\lambda > 0$ *for which*

$$A\mathbf{v} \cdot \mathbf{v} = \sum_{i,j=1}^{n} a_{i,j} v_i v_j \geq \lambda |\mathbf{v}|^2 \qquad \text{for all } \mathbf{x} \in \mathcal{U} \text{ and all } \mathbf{v} \in \mathbb{R}^n,$$

and $c = 0$ *on* $\mathcal{U}$*, then any subsolution* $u \in C^2(\mathcal{U})$ *satisfies the following:*

If there is some $\mathbf{p} \in \mathcal{U}$ *such that*

$$u(\mathbf{p}) \geq u(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathcal{U},$$

then $u \equiv u(\mathbf{p})$ *is constant on the connected component of* $\mathcal{U}$ *containing* $\mathbf{p}$.

While there is a new condition on **connectedness**, Hopf's result only requires u to be a subsolution, rather than a strict subsolution as in Theorem 38, and there is no requirement that the domain $\mathcal{U}$ be bounded. Something fundamentally different is required to get a result of this generality.

Theorem 43 *(E. Hopf boundary point lemma) If* L *is uniformly elliptic in* $\mathcal{U}$ *with* $c \equiv 0$ *and* $u \in C^2(\mathcal{U}) \cap C^1(\overline{\mathcal{U}})$ *is a subsolution,* $Lu \geq 0$*, with maximum at a point* $\mathbf{p} \in \partial\mathcal{U}$ *for which there is a point* $\mathbf{q} \in \mathcal{U}$ *with*

$$B_r(\mathbf{q}) \subset \mathcal{U}$$

where $r = |\mathbf{p} - \mathbf{q}|$*, then* **either**

$$D_{\mathbf{n}} u(\mathbf{p}) > 0$$

where $\mathbf{n} = (\mathbf{p} - \mathbf{q})/|\mathbf{q} - \mathbf{p}|$ *is the outward normal to* $\partial\mathcal{U}$ *at* $\mathbf{p}$ **or** $u \equiv u(\mathbf{p})$ *is constant on the connected component of* $\mathcal{U}$ *containing* $\mathbf{q}$.

Proof: Let us begin with an additional assumption:

$$u(\mathbf{x}) < u(\mathbf{p}) \qquad \text{for } \mathbf{x} \in \partial B_{r/2}(\mathbf{q}). \tag{7.7}$$

This is the only special case of the boundary point lemma required to prove the E. Hopf[2] strong maximum priniple. In fact, our strategy will be to prove this special case, then prove the strong maximum principle, and then go back and prove the boundary point lemma as stated using the strong maximum principle to justify the additional assumption of the special case.

Now consider $v : B_r(\mathbf{q}) \backslash B_{r/2}(\mathbf{q}) \to \mathbb{R}$ by

$$v(\mathbf{x}) = \epsilon \left(e^{-\gamma |\mathbf{x} - \mathbf{q}|^2} - e^{-\gamma r^2} \right)$$

where ϵ and γ are positive constants to be chosen below. Notice that we have

[2]There are (at least) two famous mathematicians with the last name Hopf: Heinz Hopf and Ebehard Hopf. Heinz Hopf proved that an immersed topological sphere of constant mean curvature must be a round sphere (1955). H. Hopf is also credited with a construction called the Hopf fibration. To distinguish the Hopf who proved the boundary point lemma and the strong maximum principle, I like to always use the appropriate initial.

(a) $v \in C^2\left(\overline{B_r(\mathbf{q})\backslash B_{r/2}(\mathbf{q})}\right)$,

(b) $v(\mathbf{x}) \equiv 0$ for $\mathbf{x} \in \partial B_r(\mathbf{q})$,

(c) $v(\mathbf{x}) > 0$ for $\mathbf{x} \in \partial B_{r/2}(\mathbf{q})$, and

(d)

$$Lv = 2\gamma\epsilon\left[\sum_{i,j=1}^{n} a_{ij}[\delta_{ij} + 2\gamma(x_i - q_i)(x_j - q_j)] + \sum_{j=1}^{n} b_j(x_j - q_j)\right] e^{-\gamma|\mathbf{x}-\mathbf{q}|^2}$$

where

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

denotes the Kronecker delta.

Assuming the coefficients are bounded with

$$|a_{ij}(\mathbf{x})| \le M_a \quad \text{and} \quad |b_j(\mathbf{x})| \le M_b \quad \text{for } |x - \mathbf{q}| \le r \text{ and } i, j = 1, 2, \ldots, n,$$

we may estimate the operator value Lv on $B_r(\mathbf{q})\backslash B_{r/2}(\mathbf{q})$ using the uniform ellipticity as follows

$$\begin{aligned} Lv &= 2\gamma\epsilon\left[2\gamma\sum_{i,j=1}^{n} a_{ij}(x_i - q_i)(x_j - q_j) + \sum_{j=1}^{n} a_{jj} + \sum_{j=1}^{n} b_j(x_j - q_j)\right] e^{-\gamma|\mathbf{x}-\mathbf{q}|^2} \\ &\ge 2\gamma\epsilon\left[2\gamma\lambda|\mathbf{x}-\mathbf{q}|^2 - nM_a - nM_b|\mathbf{x}-\mathbf{q}|\right] e^{-\gamma|\mathbf{x}-\mathbf{q}|^2} \\ &\ge 2\gamma\epsilon\left[\gamma\lambda r^2/2 - nM_a - nM_b r/2\right] e^{-\gamma|\mathbf{x}-\mathbf{q}|^2} \\ &= \gamma\epsilon\left[\gamma\lambda r^2 - n(2M_a + rM_b)\right] e^{-\gamma|\mathbf{x}-\mathbf{q}|^2} \\ &> 0 \end{aligned}$$

if we take

$$\gamma > \frac{n(2M_a + rM_b)}{\lambda r^2}. \tag{7.8}$$

In particular, this will imply

$$L(u - u(\mathbf{p}) + v) = Lu + Lv > 0 \qquad \text{on } B_r(\mathbf{q})\backslash B_{r/2}(\mathbf{q}).$$

Having fixed $\gamma > 0$ satisfying (7.8) and recalling our assumption (7.7) according to which $u(\mathbf{p}) - u(\mathbf{x}) > 0$ for $|\mathbf{x} - \mathbf{q}| = r/2$, we may choose $\epsilon > 0$ so that

$$v(\mathbf{x}) < u(\mathbf{p}) - u(\mathbf{x}) \qquad \text{for } |\mathbf{x} - \mathbf{q}| = r/2.$$

Therefore, $w(\mathbf{x}) = u(\mathbf{x}) - u(\mathbf{p}) + v(\mathbf{x})$ satisfies

(a) $w \in C^2(B_r(\mathbf{q})\backslash B_{r/2}(\mathbf{q})) \cap C^1\left(\overline{B_r(\mathbf{q})\backslash B_{r/2}(\mathbf{q})}\right)$, and

(b) $w(\mathbf{x}) \le 0 = w(\mathbf{p})$ for $\mathbf{x} \in \partial(B_r(\mathbf{q})\backslash B_{r/2}(\mathbf{q}))$.

By the preliminary strong maximum principle, Theorem 38, we have

$$w(\mathbf{x}) < w(\mathbf{p}) \qquad \text{for all } \mathbf{x} \in B_r(\mathbf{q})\backslash B_{r/2}(\mathbf{q}).$$

That is,

$$u(\mathbf{x}) < u(\mathbf{p}) - v(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in B_r(\mathbf{q})\backslash B_{r/2}(\mathbf{q}).$$

This means

$$\begin{aligned}
D_{\mathbf{n}}u(\mathbf{p}) &= \lim_{h \nearrow 0} \frac{u(\mathbf{p} + h\mathbf{n}) - u(\mathbf{p})}{h} \\
&\ge \lim_{h \nearrow 0} \frac{u(\mathbf{p}) - v(\mathbf{p} + h\mathbf{n}) - u(\mathbf{p})}{h} \\
&= -\lim_{h \nearrow 0} \frac{v(\mathbf{p} + h\mathbf{n}) - v(\mathbf{p})}{h} \\
&= -D_{\mathbf{n}}v(\mathbf{p}) \\
&= -Dv(\mathbf{p}) \cdot \mathbf{n} \\
&= 2\epsilon\gamma e^{-\gamma r^2}(\mathbf{p} - \mathbf{q}) \cdot \frac{\mathbf{p} - \mathbf{q}}{r} \\
&= 2\epsilon\gamma r e^{-\gamma r^2} \\
&> 0.
\end{aligned}$$

This completes the proof of the E. Hopf boundary point lemma under the additional assumption[3] (7.7). □

Proof of the E. Hopf Strong Maximum Principle: Say you have a connected component $\mathcal{U}_1$ of $\mathcal{U}$ with a point $\mathbf{q}_1 \in \mathcal{U}_1$ satisfying

$$u(\mathbf{q}_1) = M_1 = \max_{\overline{\mathcal{U}_1}} u$$

[3] which, incidentally, is the way E. Hopf actually stated it.

but there are also points $\mathbf{x} \in \mathcal{U}_1$ with $u(\mathbf{x}) < u(\mathbf{q}_1)$. (This is what it would mean for u to be non-constant on $\mathcal{U}_1$.). At this point the strong maximum principle follows almost immediately from the special case of the boundary point lemma if we can just find a point

$$\mathbf{q} \in U_1 = \{\mathbf{x} \in \mathcal{U}_1 : u(\mathbf{x}) < M_1\} \neq \phi$$

that is closer to a point

$$\mathbf{p} \in A = \{\mathbf{x} \in \mathcal{U}_1 : u(\mathbf{x}) = M_1\} \neq \phi$$

than it is to $\partial\mathcal{U}_1$ and such that all points in the boundary of the ball $B_r(\mathbf{q})$ with $r = |\mathbf{p} - \mathbf{q}|$ are in U_1 except, of course, $\mathbf{p} \in \partial B_r(\mathbf{q})$.

First of all, because an interior local maximum is achieved at $\mathbf{p}$, we know $Du(\mathbf{p}) = \mathbf{0}$ is the zero (gradient) vector. On the other hand, the special case of the boundary point lemma gives

$$D_{\mathbf{n}}u(\mathbf{p}) = Du(\mathbf{p}) \cdot \mathbf{n} > 0$$

where $\mathbf{n} = (\mathbf{p} - \mathbf{q})/r$. So this is a contradiction and the proof is complete.

In order to find the points $\mathbf{p}$ and $\mathbf{q}$ we need to know what it means for a set to be **connected**. (The **connected component** $\mathcal{U}_1$ of $\mathcal{U}$ is a connected set in particular.) This is the main thing. There are a few more steps, but they are not difficult. A set $\mathcal{U}_1$, say, is **connected** if you **cannot** write $\mathcal{U}_1$ as a union of **nonempty, disjoint, open sets**. This means that when you write $\mathcal{U}_1$ as a union of any two sets, then one of the three conditions (nonempty, disjoint, and open) must fail. In this case, consider the sets U_1 and A defined above. We have that U_1 and A are both nonempty. They are also obviously disjoint from one another. This means that for $\mathcal{U}_1$ to be connected, one of the sets U_1 or A must not be open. That fact that u is a continuous function implies U_1 is open. In order to review the definition of what it means to be open, let's review the justification of this fact in terms of epsilons and deltas.[4]

[4]One definition of what it means for a function to be **continuous** is that inverse images of open sets are open. Since $U_1 = u^{-1}(-\infty, M)$, the fact that U_1 is open is immediate according to this definition. In terms of epsilons and deltas, however, the definition of continuity is a bit different, and we need to remember what it means also for a set to be open, namely that for each point x in the set U_1, there should be an open ball $B_r(x)$ with $r > 0$ such that $B_r(x) \subset U_1$. This is the point of view we will take here.

If $\mathbf{q} \in U_1$, then $u(\mathbf{q}) < M_1$, and there is some $\delta > 0$ such that $B_\delta(\mathbf{q}) \subset \mathcal{U}_1$ and

$$|u(\mathbf{x}) - u(\mathbf{q})| < \epsilon = M_1 - u(\mathbf{q}) \qquad \text{for } \mathbf{x} \in B_\delta(\mathbf{q}).$$

This implies $u(\mathbf{x}) - u(\mathbf{q}) \leq |u(\mathbf{x}) - u(\mathbf{q})| < M_1 - u(\mathbf{q})$ which means $u(\mathbf{x}) < M_1$ and $\mathbf{x} \in U_1$. Thus, $B_\delta(\mathbf{x}) \subset U_1$ and U_1 is open.

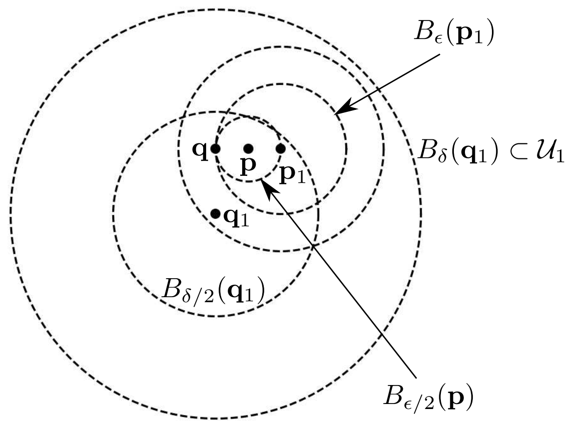

Figure 7.4: Interior points $\mathbf{q}$ and $\mathbf{p}$ with $u(\mathbf{p}) < u(\mathbf{q}) = M_1$ for which the E. Hopf boundary point lemma (special case) applies on $B_r(\mathbf{p})$ with $r = \epsilon/2$.

Finally, then, we know from the definition of connectedness that the set A must not be open. This means there is at least one point $\mathbf{q}_1 \in A$ for which **every ball** $B_\delta(\mathbf{q}_1)$ contains points $\mathbf{x} \in \mathcal{U}_1 \backslash A = U_1$. In particular, if we take $B_\delta(\mathbf{q}_1) \subset \mathcal{U}_1$, then there is a point $\mathbf{p}_1 \in B_{\delta/2}(\mathbf{q}_1) \cap U_1 \subset \mathcal{U}_1$. This point $\mathbf{p}_1$ is closer to $\mathbf{q}_1$ than it is to $\partial\mathcal{U}_1$. In particular, we can take the point $\mathbf{q} \in A$ closest to $\mathbf{p}_1$. We know such a point exists because A is a closed set. We also know

$$|\mathbf{q} - \mathbf{p}_1| \leq \operatorname{dist}(A, \mathbf{p}_1) = \min\{|\mathbf{x} - \mathbf{p}_1| : \mathbf{x} \in A\} = |\mathbf{q}_1 - \mathbf{p}_1| = \epsilon > 0.$$

There may be other points in $A \cap \partial B_\epsilon(\mathbf{p}_1)$, but we do know $B_\epsilon(\mathbf{p}_1) \subset \mathcal{U}_1 \cap U_1$. In particular, setting $r = \epsilon/2$ and taking $\mathbf{p} = (\mathbf{q} + \mathbf{p}_1)/2$, we obtain the required conditions on points $\mathbf{q} \in A \cap \mathcal{U}_1$ and $\mathbf{p} \in U_1$:

(i) $u(\mathbf{q}) = M_1$ and

(ii) $u(\mathbf{x}) < M_1$ for $\mathbf{x} \in \overline{B_r(\mathbf{p})} \backslash \{\mathbf{q}\}$.

Construction of the points $\mathbf{p}$ and $\mathbf{q}$ with the various open balls mentioned above is illustrated in Figure 7.4.

This completes the proof of the E. Hopf strong maximum principle. □

Proof of the E. Hopf boundary point lemma: Under the general hypotheses of the boundary point lemma (Theorem 43) we may apply the strong maximum principle to conclude $u(\mathbf{x}) < M = u(\mathbf{p})$ for $\mathbf{x} \in B_r(\mathbf{q})$ unless u is constant on $B_r(\mathbf{q})$. In the former case the ball $B_{r/2}((\mathbf{q}+\mathbf{p})/2)$ satisfies the hypotheses of the special case of the boundary point lemma with the additional assumption $u(\mathbf{x}) < M$ on $\partial B_{r/2}((\mathbf{q}+\mathbf{p})/2)\backslash\{\mathbf{q}\}$. This allows us to conclude $D_{\mathbf{n}}u(\mathbf{p}) = Du(\mathbf{p}) \cdot \mathbf{n} > 0$. □

Exercise 7.13 *Formulate versions of the strong maximum principle and the Hopf boundary point lemma for ordinary differential operators applied to functions on an interval. Prove the latter and use it to prove the former. This should give you a (different?) solution for Exercise 7.5.*

7.14 The Comparison Principle for Quasilinear PDE

The results above apply to subsolutions of second order **linear** elliptic operators. The mean curvature operator, of course, is not linear. But it is quasilinear having the form

$$Mu = \sum_{i,j=1}^{n} a_{ij} D_i D_j u + \sum_{j=1}^{n} b_j D_j u + cu$$

where the coefficients a_{ij}, b_j and c are functions of $\mathbf{x} \in \mathcal{U}$, $\mathbf{u} \in \mathbb{R}$, and $Du \in \mathbb{R}^n$. That is, $a_{ij} = a_{ij}(\mathbf{x}, u, Du)$, $b_j = b_j(\mathbf{x}, u, Du)$, and $c = c(\mathbf{x}, u, Du)$ for $i, j = 1, \ldots, n$.

Exercise 7.15 *Identify the coefficients in the mean curvature operator $\mathcal{M}$ and show that $\mathcal{M}$ is uniformly elliptic on any domain $\mathcal{U}$ with $u \in C^1(\overline{\mathcal{U}})$ in the sense that there is a constant $\lambda > 0$ for which*

$$\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, u(\mathbf{x}), Du(\mathbf{x})) p_i p_j \geq \lambda |\mathbf{p}|^2 \qquad \textit{for every } \mathbf{p} \in \mathbb{R}^n.$$

The mean curvature operator also has **divergence form**, meaning it can be expressed as

$$Mu = \sum_{j=1}^{n} D_j \left[A_j(\mathbf{x}, u, Du)\right] + + \sum_{j=1}^{n} b_j D_j u + cu \tag{7.9}$$

where $A = (A_1, A_2, \ldots, A_n)$ is a vector valued function.

Exercise 7.16 *Assume the vector valued function A in the divergence form operator is differentiable and expand the second order terms in terms of derivatives of (the component functions of) A. Under what conditions on A is the divergence form operator M uniformly elliptic?*

Theorem 44 *(comparison principle) If $M : C^2(\mathcal{U}) \to C^0(\mathcal{U})$ is a uniformly elliptic divergence form operator with the form (7.9) with $A \in C^1(\overline{\mathcal{U}} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n)$, and $u, v \in C^2(\overline{\mathcal{U}})$ satisfy*

$$Mu \geq Mv \quad \text{on } \mathcal{U} \qquad \text{and} \qquad u\Big|_{\partial\mathcal{U}} \leq v\Big|_{\partial\mathcal{U}},$$

Then $u \leq v$ on $\mathcal{U}$. Furthermore, if $\mathcal{U}$ is connected, then either $u < v$ on $\mathcal{U}$ or $u \equiv v$ on $\mathcal{U}$.

There is also a version of the boundary point lemma:

Theorem 45 *(boundary comparison principle) Given the hypotheses of Theorem 44 if $\mathcal{U}$ satisfies an interior sphere condition at $\mathbf{p} \in \partial\mathcal{U}$, that is, there is some ball $B_r(\mathbf{q}) \subset \mathcal{U}$ with $\mathbf{p} \in \partial B_r(\mathbf{q})$, then*

$$D_{\mathbf{n}} u(\mathbf{p}) = Du(\mathbf{p}) \cdot \mathbf{n} > Dv(\mathbf{p}) \cdot \mathbf{n} = D_{\mathbf{n}} v(\mathbf{p})$$

where $\mathbf{n} = (\mathbf{p} - \mathbf{q})/r$.

7.17 The Concus-Finn Comparison Principle

If u and v satisfy the hypotheses of Theorem 44 and the operator is the **capillary operator**

$$Mu = \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) - \kappa u$$

then even more is true.