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Robert Finn

Equilibrium Capillary Surfaces

With 98 Illustrations

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To the fantasy and creativity of youth,
that they serve to build and not to destroy

Preface

Capillarity phenomena are all about us; anyone who has seen a drop of dew on a plant leaf or the spray from a waterfall has observed them. Apart from their frequently remarked poetic qualities, phenomena of this sort are so familiar as to escape special notice. In this sense the rise of liquid in a narrow tube is a more dramatic event that demands and at first defied explanation; recorded observations of this and similar occurrences can be traced back to times of antiquity, and for lack of explanation came to be described by words deriving from the Latin word "capillus", meaning hair.

It was not until the eighteenth century that an awareness developed that these and many other phenomena are all manifestations of something that happens whenever two different materials are situated adjacent to each other and do not mix. If one (at least) of the materials is a fluid, which forms with another fluid (or gas) a free surface interface, then the interface will be referred to as a *capillary surface.*

Attempts to explain observed phenomena go back at least to Leonardo da Vinci. A consistent theory capable of scientific prediction first appears however in the writings of Young and of Laplace in the early nineteenth century. The theory was later put onto a more solid foundation by Gauss, and it became the object of extensive study by some of the most imposing scientific figures of that century (although it must be remarked that very little more of major new interest was accomplished). The problem fell out of fashion during the first half of the present century; however, the impetus on the one hand of new mathematical developments on minimal surfaces, and on the other hand of the practical demands of space age technology and of medicine, have now led to renewed activity on several fronts.

Among mathematical developments, the BV theory, founded on the ideas of Caccioppoli and of de Giorgi and developed by Miranda, Giaquinta, Anzellotti, Massari, Tamanini, and others, led to the first general existence theorem for capillary surfaces (Emmer [46]). Independently the ideas of geometric measure theory were introduced and developed by Federer, Fleming, Almgren, Allard, and others, and were used effectively by Taylor [177] to prove boundary regularity.

From an engineering point of view, specific problems have been attack-ed energetically using traditional methods, chiefly that of matching expansions (due originally, incidentally, to Laplace), and also numerically with computers. In general, good results were obtained; however, in some particular situations the procedures led unexpectedly to incoherent answers.

It is this circumstance that attracted my own interest. A direct study of the underlying equations showed that a discontinuous dependence on data occurs, which is governed by the particular nonlinearity in the equations. Unconventional but simple procedures led to a precise characterization of the criterion for singular behavior, to general bounds on solutions, and to asymptotically exact information in some cases.

It turned out that various other problems also lent themselves to analogous (phenomenological) approaches. By now a number of studies have appeared by various authors, using varying methods and occasionally with striking conclusions. A common and unifying thread is appearing, which may not be evident on reading the individual papers. I hope the detailed results presented in the following chapters will be of interest in themselves, and that their juxtaposition under a single cover will help to bring the thread into visibility.

The exposition is not intended to be encyclopedic, and the omission of a particular result in no sense implies that I regard it less highly than material I have included. I have tried to illustrate by example the varying kinds of situations that may be encountered, and in each case the choice of example has been determined largely by the simple criterion of familiarity. Thus, the work of Vogel [182, 183, 184] on liquid bridges, and of Turkington [180] on exterior problems and extension to a class of nonlinear operators, each of which I hold in high regard, is omitted by circumstance and not by design.

A glance through the Contents should indicate the specific nature of the material that has been covered. Much of it refers to particular configurations that may be taken as cases of special interest in the context of the general existence (and nonexistence) results of Chapters 6 and 7. Also for these general results the exposition is not complete, my intention being to emphasize the underlying ideas and the unifying thread. Attention is directed throughout to the unexpected, in the sense of behavior that differs qualitatively from what would be predicted by usual perturbation or linearizing procedures.

In the interest of conceptual and notational simplicity all material in the text is presented for the (physical) case of two-dimensional surfaces in 3-space. Many – but not all – of the results extend without essential change to surfaces of codimension one in n-space.

The purview of this book is limited to equilibrium configurations. It is not limited to energy minimizing configurations. The equations are not cognizant of global energy relations, and can lead to interesting solutions that are not observed physically as a global entity. Some of these are studied in Chapter 4.

Time dependent situations present a different world that will require a different book, presumably by a different author. We mention however recent work by Bemelmans [9], Pukhnachev and Solonnikov [151], and Dussan V and Chow [44].

I am indebted to many colleagues and students for comments and discussions that have done much to clarify my understanding and to shape my point of view. The book has also profited immeasurably from my long collaboration with Paul Concus. As to the specific material in the text, J.B. Keller has made helpful comments with regard to Chapter 1. Chapters 6 and 7 have benefited greatly from observations by L.F. Tam. I wish especially to thank Enrico Giusti, who generously shared with me his deep insight during the course of many conversations over many years. Giusti also read Chapters 6 and 7 in detail, and his comments led to a number of improvements in the formulations and proofs of the results.

Much of the writing was done while I was visiting at Universität Bonn under the auspices of Sonderforschungsbereich 72. I owe a special debt of gratitude to Stefan Hildebrandt for his warm hospitality and for the stimulating conditions for scientific work in the Institute he directs.

The new research presented here was supported in part by the National Science Foundation and by the National Aeronautics and Space Administration.

The larger portion of the typing was done by Charlotte Crabtree, who also prepared most of the figures. I want to thank her not only for her elegant work, but also for her patience with me in the course of many rewritings and changes. I also wish to thank Anke Vogt for her excellent typing of the remainder of the material.

This book has gained in accuracy and readability from the scrupulous attention its production editor gave to layout and detail. My thanks are due also to the compositor for precise and careful work.

Finally I want to express my appreciation to Springer-Verlag for its patience and understanding while awaiting a long overdue manuscript, and for its generous attention to details of production.

Palo Alto, California
November, 1985

ROBERT FINN

Contents

IX. *Part of a Letter from Mr.* Brook Taylor, *F. R. S. to Dr.* Hans Sloane *R. S. Secr. Concerning the Ascent of Water between two Glass Planes.*

THE following Experiment seeming to be of use, in discovering the Proportions of the Attractions of Fluids, I shall not forbear giving an Account of it; tho' I have not here Conveniencies to make it in so successful a manner, as I could wish.

I fasten'd two pieces of Glass together, as flat as I could get; so that they were inclined in an Angle of about 2 Degrees and a half. Then I set them in Water, with the contiguous Edges perpendicular. The upper part of the Water, by rising between them, made this *Hyperbola*; [See *Fig.* 5.] which is as I copied it from the Glass.

I have examined it as well as I can, and it seems to approach very near to the common *Hyperbola*. But my *Apparatus* was not nice enough to discover this exactly.

The Perpendicular *Assymptote* was exactly determined by the Edge of the Glass; but the Horizontal one I could not so well discover. I am,

Sir,

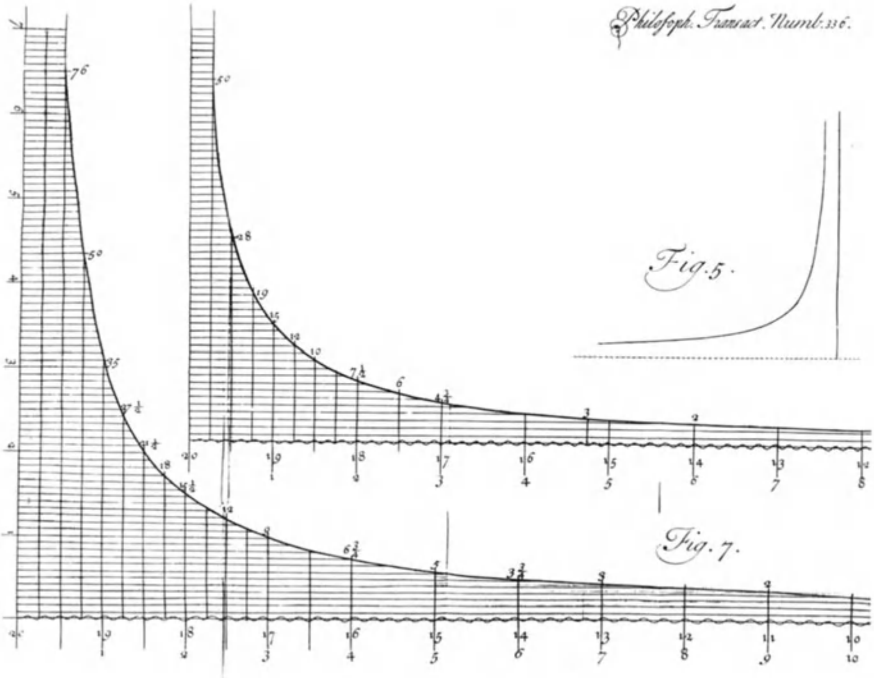
Bifrons near *Canterbury*, June 25. 1712

Your most humble Servant,

BROOK TAYLOR.

X. *An Account of an Experiment touching the Ascent of Water between two Glass Planes, in an Hyperbolick Figure. By Mr.* Francis Hauksbee, *F. R. S.*

I Took two Glass Planes, each somewhat more than 20 Inches long, of the truest Surfaces I could procure. These being held close together at one of their Ends, the other Ends were opened exactly to an Angle of 20 Minutes. In this Form they were edgeways put into a Trough of ting'd Water, which immediately aroſe between them in the Figure of the annext Scheme. *See* Fig. 7. At another time the Planes were opened to an Angle of 40 Minutes; then the Water appear'd between them, as in the Scheme with that Title. By theſe Schemes *See* Fig. 6. the Proportions of the Power of Attraction are in ſome meaſure evident to the Eye; for there may be ſeen at the ſeveral Diſtances, how many Lines (which are 12ths of Inches) the Water is elevated, and the prodigious Increaſe of them near the touching Ends. I hope the Tables are pretty accurate; for after many tryals, I find the Succeſſes to be much the ſame, according to the different Angles. This Experiment was firſt made by Mr. *Brook Taylor*, as appears by his Letter to Dr. *Hans Sloane*, R. S. Secr. but he confeſſes his *Apparatus* not nice enough to diſcover exactly the Figure which the Water made between the Planes.



Chapter 1

Introduction

1.1. Mean Curvature

In a celebrated Essay presented to the Royal Society of London in 1805, Thomas Young [189] introduced the notion of *mean curvature* of a surface; in the following year, 1806, Pierre Simon Laplace reintroduced the notion and derived for it a formal analytical expression. The original reasoning of Laplace is reproduced in the first of two supplements to the tenth book of his *Traité de mécanique céleste* [114]. The following version of his derivation is in a more modern notation and has been put into an invariant setting; the underlying ideas are however still those of Young and of Laplace.

Denoting the surface by $\mathscr{S}: \mathbf{x}(\alpha, \beta)$, we consider the curve $\mathscr{C}$ of intersection of $\mathscr{S}$ with a plane Π containing a normal vector $\mathbf{N}$ through a point p of $\mathscr{S}$ (see Fig. 1.1), and we assume $\mathscr{C}$ parametrized by its arc length s in some orientation. The curvature of $\mathscr{C}$, considered as positive when the curve is bending in the direction of $\mathbf{N}$, is

$$k = -\frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{N}}{ds}. \tag{1.1}$$

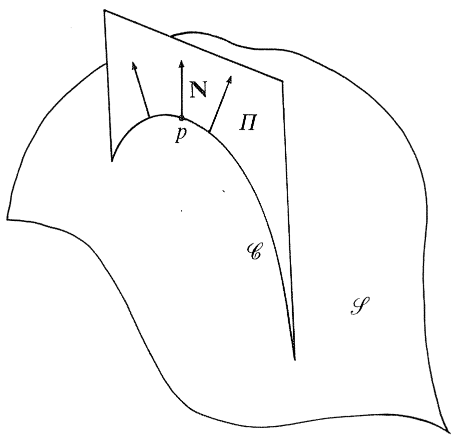


Figure 1.1. Normal section to $\mathscr{S}$ at p.

Writing

$$\dot{\alpha}=\frac{d\alpha}{ds}, \qquad \dot{\beta}=\frac{d\beta}{ds} \tag{1.2}$$

$$e=-\mathbf{x}_\alpha\cdot\mathbf{N}_\alpha, \qquad 2f=-(\mathbf{x}_\alpha\cdot\mathbf{N}_\beta+\mathbf{x}_\beta\cdot\mathbf{N}_\alpha), \qquad g=-\mathbf{x}_\beta\cdot\mathbf{N}_\beta \tag{1.3}$$

we obtain

$$k=e\dot{\alpha}^2+2f\dot{\alpha}\dot{\beta}+g\dot{\beta}^2. \tag{1.4}$$

Setting

$$E=|\mathbf{x}_\alpha|^2, \qquad F=\mathbf{x}_\alpha\cdot\mathbf{x}_\beta, \qquad G=|\mathbf{x}_\beta|^2, \tag{1.5}$$

we have also

$$\left|\frac{d\mathbf{x}}{ds}\right|^2=1=E\dot{\alpha}^2+2F\dot{\alpha}\dot{\beta}+G\dot{\beta}^2. \tag{1.6}$$

In what follows we denote with the subscripts 1 and 2 quantities that correspond to any two mutually orthogonal directions at a point on $\mathcal{S}$. We then find

$$k_1+k_2=e(\dot{\alpha}_1^2+\dot{\alpha}_2^2)+2f(\dot{\alpha}_1\dot{\beta}_1+\dot{\alpha}_2\dot{\beta}_2)+g(\dot{\beta}_1^2+\dot{\beta}_2^2) \tag{1.7}$$

$$1=E\dot{\alpha}_j^2+2F\dot{\alpha}_j\dot{\beta}_j+G\dot{\beta}_j^2, \quad j=1,2 \tag{1.8}$$

$$0=E\dot{\alpha}_1\dot{\alpha}_2+F(\dot{\alpha}_1\dot{\beta}_2+\dot{\alpha}_2\dot{\beta}_1)+G\dot{\beta}_1\dot{\beta}_2. \tag{1.9}$$

Let us write

$$2H=k_1+k_2. \tag{1.10}$$

Introducing the imaginary unit $i=\sqrt{-1}$ and setting $p=\dot{\alpha}_1+i\dot{\alpha}_2$, $q=\dot{\beta}_1+i\dot{\beta}_2$, we may write (1.7) in the form

$$2H=ep\bar{p}+f(p\bar{q}+\bar{p}q)+gq\bar{q}. \tag{1.11}$$

The remaining expressions (1.8) and (1.9) yield

$$2=Ep\bar{p}+F(p\bar{q}+\bar{p}q)+Gq\bar{q} \tag{1.12}$$

$$0=Ep^2+2Fpq+Gq^2. \tag{1.13}$$

By expressing the product and sum of the roots of (1.13) in terms of the coefficients and inserting the resulting expressions into (1.11) and (1.12), we are led to the formula

$$2H=\frac{Eg-2Ff+Ge}{EG-F^2}, \tag{1.14}$$

a result that clearly does not depend on the particular choice of (orthogonal) directions on $\mathcal{S}$. Thus the relation (1.10) has an invariant geometrical meaning; we define H to be the *mean curvature* of $\mathcal{S}$ at the point of evaluation.

1.2. Laplace's Equation

These fundamental observations of Laplace did not arise in connection with any abstract study of the geometry of surfaces. His motivation lay instead in his attempt to clarify conceptually and to characterize quantitatively the rise of liquid in a capillary tube (see Fig. 1.2). Laplace showed by an ingenious potentialtheoretic reasoning that the mean curvature H of the free surface is proportional to the pressure change across the surface; thus, by the laws of hydrostatics, there must hold $H=\frac{1}{2}\kappa u$, where u is the height of the surface above the level corresponding to atmospheric pressure and κ is a physical constant, and (1.14) then yields a differential equation for the unknown surface $\mathcal{S}$. A simplified – although not entirely convincing – version of Laplace's reasoning has become the standard presentation for engineering textbooks.

Since a conceptually preferable way to derive that relation was later given by Gauss, we do not repeat Laplace's reasoning here. For the moment, we assume the result and write the equation that results when the surface can be described as a graph $z=u(x, y)$. We have then

$$E=(1+u_x^2), \qquad F=u_x u_y, \qquad G=(1+u_y^2) \tag{1.15}$$

$$e=\frac{u_{xx}}{\sqrt{1+|Du|^2}}, \quad f=\frac{u_{xy}}{\sqrt{1+|Du|^2}}, \quad g=\frac{u_{yy}}{\sqrt{1+|Du|^2}}, \tag{1.16}$$

with $|Du|^2=u_x^2+u_y^2$, and we find from (1.14)

$$2H=a u_{xx}+2b u_{xy}+c u_{yy} \tag{1.17}$$

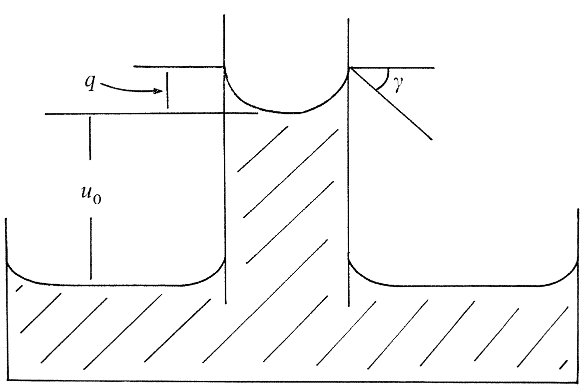


Figure 1.2. Capillary tube (symbols refer to symmetric homogeneous case).

with

$$a = \frac{1+u_y^2}{(1+|Du|^2)^{3/2}}, \qquad b = \frac{-u_x u_y}{(1+|Du|^2)^{3/2}}, \qquad c = \frac{1+u_x^2}{(1+|Du|^2)^{3/2}}. \tag{1.18}$$

We are thus led to the equation

$$a u_{xx} + 2b u_{xy} + c u_{yy} = \kappa u \tag{1.19}$$

for the height u of the surface. We note that

$$ac - b^2 = \frac{1}{(1+|Du|^2)^2} > 0$$

for any function $u(x, y)$ and thus the equation, although nonlinear, is of elliptic type for any solution. It is not uniformly elliptic; this circumstance has important consequences for the behavior of the solutions.

The equation that usually bears Laplace's name is obtained from (1.19) by setting $\kappa = 0$ and linearizing about the identically zero solution. Laplace recognized that for the present problem the linearized equation is not adequate to describe the physical surfaces.

1.3. Angle of Contact

In the same Essay of 1805, Thomas Young gave a reasoning to support the view that in an equilibrium configuration in the absence of frictional resistance to motion along the boundary walls, the fluid meets the bounding walls in a constant angle γ, depending only on the materials and in no way on the shape of the boundary or of the surface.

Like the reasoning of Laplace for the above relation $H = \frac{1}{2}\kappa u$, a version of the Young argument has become standard in engineering literature; it is however also unconvincing.

1.4. The Method of Gauss; Characterization of the Energies

It turns out that both these questions can be dealt with at once by a method proposed by Gauss [73] in 1830. Gauss based his reasoning on the principle of virtual work, according to which the energy of a mechanical system in equilibrium is unvaried under arbitrary virtual displacements consistent with the constraints. For a general (three-phase) system consisting of fluid and gas (or two fluids) and rigid bounding walls (see Fig. 1.3) the energy in question is conveniently divided into four terms:

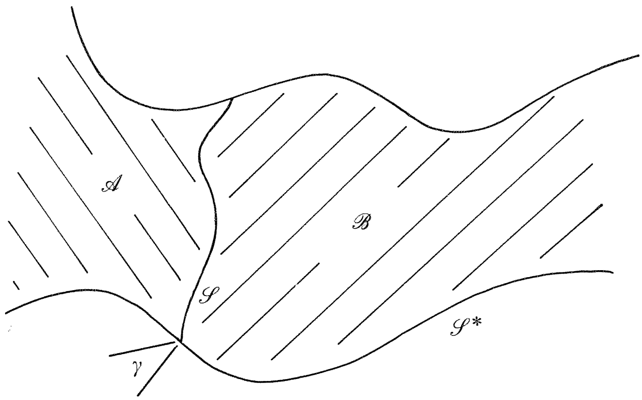


Figure 1.3. Three-phase system: fluids $\mathscr{A}$, $\mathscr{B}$; solid $\mathscr{S}^*$.

i) Free Surface Energy. If the configuration is to be in equilibrium, the elements of a fluid in the free surface separating two media must be more attracted to that fluid than to the outer medium (fluid or gas, see Fig. 1.4); otherwise the two media would mix and the surface would disappear. Thus there is a differential attraction and resultant lowering of the fluid density within the surface. The energy associated with this removal of fluid from the surface must be proportional to the surface area; we write

$$\mathscr{E}_{\mathscr{S}} = \sigma \mathscr{S}. \tag{1.20}$$

The constant σ has the dimensions force per unit length and is known as the *surface tension.*

ii) Wetting Energy. This is the adhesion energy between fluid and the (rigid) bounding walls; it is analogous to surface energy, except that fluid particles near the rigid surface can experience the larger attraction in either direction (since the walls are rigid, the surface cannot disappear

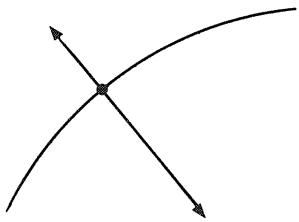


Figure 1.4. Differential attractions on free surface.

when the net attraction is toward the wall). We write

$$\mathscr{E}_W = -\sigma(\beta^* \mathscr{S}^* + \hat{\beta}^* \hat{\mathscr{S}}^*)$$

where $\mathscr{S}^*$ is the area wetted by the fluid and $\hat{\mathscr{S}}^*$ is the area in contact with the outer medium. When making the variation, only those values of $\beta^*, \hat{\beta}^*$ in an immediate neighborhood of the contact line need be taken into account. Neglecting terms that are unvaried and noting that any variation of $\mathscr{S}^*$ is the negative of that of $\hat{\mathscr{S}}^*$, we may extend $\hat{\beta}^*$ in an arbitrary (continuous) way into $\mathscr{S}^*$, and write

$$\mathscr{E}_W = -\sigma \beta \mathscr{S}^* \tag{1.21}$$

with $\beta = \beta^* - \hat{\beta}^*$. Here $\mathscr{S}^*$ can be chosen to be any portion of the wetted surface that includes a neighborhood of the contact line. The sign on the right of (1.21) is chosen so that $\beta > 0$ corresponds to a "wetting" configuration, as indicated in Fig. 1.2. We shall show that the condition $|\beta| \leq 1$ is necessary for stability of the configuration. We refer to β as the *relative adhesion coefficient* between the fluid and the bounding walls on $\mathscr{S}^*$.

iii) Gravitational Energy. We assume more generally a potential energy Υ per unit mass, depending on position within the media. The resultant energy is

$$\mathscr{E}_\Upsilon = \int \Upsilon \rho \, dx \tag{1.22}$$

where ρ is local density. The integral is formally to be taken over all of the space that is occupied by the media; however, as above, we may choose for ρ the difference between local fluid density and any continuous extension of the density of the outer medium. The domain of integration can be restricted to any region in which the variation has its support, the density exterior to the fluid being taken to be zero.

iv) Volume Constraints. For many problems (e.g., that of Fig. 6.1) the constancy of volume of fluid is a constraint that must be respected when choosing displacements. A natural way to do so is to introduce the volume $\mathscr{V}$ multiplied by a Lagrange parameter λ as a new energy term, and then to allow arbitrary displacements consistent with the constraint imposed by the rigid boundary. We write

$$\mathscr{E}_{\mathscr{V}} = \sigma \lambda \mathscr{V} \tag{1.23}$$

where the multiplier λ is to be determined.

For the total energy we thus have

$$\mathscr{E} = \sigma \left\{ \mathscr{S} - \beta \mathscr{S}^* + \frac{1}{\sigma} \int \Upsilon \rho \, dx + \lambda \mathscr{V} \right\}. \tag{1.24}$$

1.5. Variational Considerations

In order to apply the principle of virtual work, we introduce a virtual displacement in the form of a variation

$$\varepsilon\zeta = \varepsilon(\xi\mathbf{N} + \eta\mathbf{T}) + O(\varepsilon^2) \tag{1.25}$$

over $\mathscr{S}$. Here $\mathbf{N}$ is the unit normal on $\mathscr{S}$, directed out of the fluid; $\mathbf{T}$ is a unit tangent, defined in a closed strip Σ_δ of width δ adjoining the contact line Σ of $\mathscr{S}$ with $\mathscr{S}^*$, such that on $\Sigma, \mathbf{T}$ is orthogonal to Σ and directed out of $\mathscr{S}$. The functions ξ and η are arbitrary, subject to the conditions

$$\begin{aligned} &\xi^2 + \eta^2 \leq 1 \\ &\operatorname{supp} \eta \subset \Sigma_\delta \\ &\xi\mathbf{N} + \eta\mathbf{T} \quad \text{tangent to } \mathscr{S}^* \text{ on } \Sigma. \end{aligned} \tag{1.26a, b, c}$$

The condition (1.26c) has as consequence

$$\xi \cos\gamma - \eta \sin\gamma = 0 \tag{1.27}$$

on Σ, where γ is the angle between $\mathscr{S}$ and $\mathscr{S}^*$ on Σ, measured within the fluid (see Fig. 1.5, which indicates the configuration in a plane normal to Σ).

In terms of local coordinates α, β, the surface $\mathscr{S}$ can be represented by a function $\mathbf{x}(\alpha, \beta)$. For purposes of making the variation, we denote the unvaried surface by $\mathscr{S}_0$, and we introduce the varied surface

$$\mathscr{S}(\varepsilon)\colon \mathbf{x}(\alpha, \beta) + \varepsilon[\xi\mathbf{N} + \eta\mathbf{T}] + O(\varepsilon^2), \tag{1.28}$$

the last term being chosen so that when $\mathbf{x}(\alpha, \beta)$ lies on Σ, the varied surface lies on $\mathscr{S}^*$; thus the surfaces $\mathscr{S}$ and $\mathscr{S}^*$ continue to contact (on

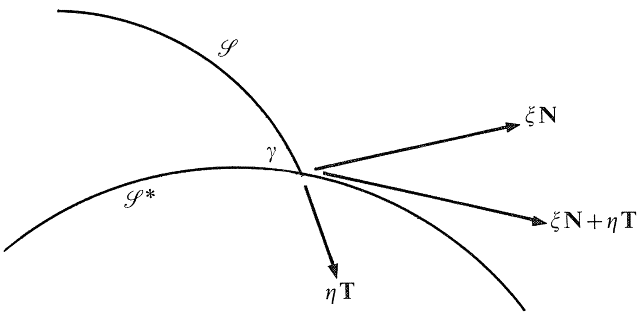


Figure 1.5. Variational constraint on contact manifold.

a varied Σ) throughout the variation, as $\varepsilon \to 0$. In view of (1.26c) the estimate $O(\varepsilon^2)$ is uniform among admissible ξ, η, for all sufficiently small $\varepsilon > 0$.

The area $\mathscr{S}$ is given by

$$\mathscr{S} = \int \sqrt{EG - F^2}\, d\alpha\, d\beta. \tag{1.29}$$

Here

$$\begin{aligned} E &= |\mathbf{x}_\alpha + \varepsilon \zeta_\alpha|^2 = E_0 + 2\varepsilon \mathbf{x}_\alpha \cdot \zeta_\alpha + O(\varepsilon^2) \\ F &= (\mathbf{x}_\alpha + \varepsilon \zeta_\alpha) \cdot (\mathbf{x}_\beta + \varepsilon \zeta_\beta) = F_0 + \varepsilon(\mathbf{x}_\alpha \cdot \zeta_\beta + \mathbf{x}_\beta \cdot \zeta_\alpha) + O(\varepsilon^2) \\ G &= |\mathbf{x}_\beta + \varepsilon \zeta_\beta|^2 = G_0 + 2\varepsilon \mathbf{x}_\beta \cdot \zeta_\beta + O(\varepsilon^2). \end{aligned} \tag{1.30}$$

We thus have, up to terms of order ε^2,

$$\sqrt{EG - F^2} = W_0 \left\{ 1 + \varepsilon [E_0 \mathbf{x}_\beta \cdot \zeta_\beta - F_0(\mathbf{x}_\alpha \cdot \zeta_\beta + \mathbf{x}_\beta \cdot \zeta_\alpha) + G_0 \mathbf{x}_\alpha \cdot \zeta_\alpha] \frac{1}{W_0^2} \right\}, \tag{1.31}$$

with $W_0 = \sqrt{E_0 G_0 - F_0^2}$, and hence

$$\dot{\mathscr{S}} = \left.\frac{\partial \mathscr{S}}{\partial \varepsilon}\right|_{\varepsilon = 0} = \int \frac{E_0 \mathbf{x}_\beta \cdot \zeta_\beta - F_0(\mathbf{x}_\alpha \cdot \zeta_\beta + \mathbf{x}_\beta \cdot \zeta_\alpha) + G_0 \mathbf{x}_\alpha \cdot \zeta_\alpha}{W_0}\, d\alpha\, d\beta. \tag{1.32}$$

Recalling the definition (1.3) of e, f, g and the relation (1.14), we may write

$$\begin{aligned} \dot{\mathscr{S}} = &-2 \int_{\mathscr{S}} \xi H\, d\mathscr{S} \\ &+ \int_{\Sigma_\delta} \frac{E_0 \mathbf{x}_\beta \cdot (\eta \mathbf{T})_\beta - F_0[\mathbf{x}_\alpha \cdot (\eta \mathbf{T})_\beta + \mathbf{x}_\beta \cdot (\eta \mathbf{T})_\alpha] + G_0 \mathbf{x}_\alpha \cdot (\eta \mathbf{T})_\alpha}{W_0}\, d\alpha\, d\beta \end{aligned}$$

where H is the mean curvature of $\mathscr{S}$, corresponding to the chosen direction for N. An integration by parts, using (1.26b), converts the integral over Σ_δ to the form

$$\begin{aligned} &\oint_\Sigma \eta \mathbf{T} \cdot \frac{1}{W_0} [(G_0 \mathbf{x}_\alpha - F_0 \mathbf{x}_\beta)\, d\beta - (-F_0 \mathbf{x}_\alpha + E_0 \mathbf{x}_\beta)\, d\alpha] \\ &- \int_{\Sigma_\delta} \eta \mathbf{T} \cdot \left\{ \left(\frac{G_0 \mathbf{x}_\alpha - F_0 \mathbf{x}_\beta}{W_0} \right)_\alpha + \left(\frac{-F_0 \mathbf{x}_\alpha + E_0 \mathbf{x}_\beta}{W_0} \right)_\beta \right\} d\alpha\, d\beta, \end{aligned}$$

which we may write in intrinsic notation (cf. [14], pp. 168–174) in the form

$$\oint_\Sigma \eta \mathbf{T} \cdot \frac{\partial \mathbf{x}}{\partial n}\, ds - \int_{\Sigma_\delta} \eta \mathbf{T} \cdot \Delta \mathbf{x}\, d\mathscr{S}. \tag{1.33}$$

Here $\partial/\partial n$ is the outer normal derivative on $\mathscr{S}_0$ at Σ, and thus $\partial \mathbf{x}/\partial n = \mathbf{T}$. For the invariant Laplacian $\Delta \mathbf{x}$ we have further (cf. [116], p. 131)

$$\Delta \mathbf{x} = 2H\mathbf{N}; \tag{1.34}$$

hence the second term in (1.33) vanishes. Since $|\mathbf{T}|=1$, we conclude

$$\dot{\mathscr{S}} = -2\int_{\mathscr{S}} \xi H\, d\mathscr{S} + \oint_{\Sigma} \eta\, ds. \tag{1.35}$$

Since $\xi\mathbf{N} + \eta\mathbf{T}$ is orthogonal to Σ and in the tangent plane to $\mathscr{S}^*$, we find immediately

$$\dot{\mathscr{S}}^* = \oint_{\Sigma} (\xi\mathbf{N} + \eta\mathbf{T}) \cdot \boldsymbol{v}\, ds \tag{1.36}$$

where $\boldsymbol{v}$ is the unit exterior normal to Σ in the tangent plane of $\mathscr{S}^*$. Since $\mathbf{N} \cdot \boldsymbol{v} = \sin\gamma$, $\mathbf{T} \cdot \boldsymbol{v} = \cos\gamma$ (see Fig. 1.5) we have

$$\dot{\mathscr{S}}^* = \oint_{\Sigma} (\xi \sin\gamma + \eta \cos\gamma)\, ds. \tag{1.37}$$

We consider next the term $\lambda\mathscr{V}$ of (1.25). Under the variation (1.28) we find

$$\mathscr{V}(\varepsilon) - \mathscr{V}_0 = \varepsilon \int_{\mathscr{S} \setminus \Sigma_\delta} \xi\, d\mathscr{S} + O(\varepsilon^2) + \mathscr{V}_\delta \tag{1.38}$$

where $\mathscr{V}_\delta$ is the change in volume due to the displacement $\varepsilon(\xi\mathbf{N} + \eta\mathbf{T})$ of Σ_δ. Since $\xi^2 + \eta^2 < 1$, $\mathscr{V}_\delta$ is contained in the tubular volume generated by balls of radius ε centered on Σ_δ; thus

$$\mathscr{V}_\delta < C\varepsilon(\varepsilon + \delta)\Sigma, \tag{1.39}$$

for some constant C, as $\varepsilon \to 0$.

An analogous discussion yields

$$\int_{\mathscr{V}(\varepsilon)} \Upsilon\rho\, dx - \int_{\mathscr{V}_0} \Upsilon\rho\, dx = \varepsilon \int_{\mathscr{S} \setminus \Sigma_\delta} \xi\, \Upsilon\rho\, d\mathscr{S} + O(\varepsilon^2) + W_\delta \tag{1.40}$$

with

$$W_\delta < CM\varepsilon(\varepsilon + \delta)\Sigma \tag{1.41}$$

where M is a bound for $|\Upsilon\rho|$ in the above-indicated tubular domain. Here ρ is to be taken as the local difference of densities between the two adjacent media.

Collecting the above evaluations, we may write from (1.25)

$$\left|\frac{1}{\sigma}\dot{\mathscr{E}}+2\int_{\mathscr{S}}\xi H\,d\mathscr{S}-\oint_{\Sigma}\eta\,ds+\beta\oint_{\Sigma}(\xi\sin\gamma+\eta\cos\gamma)\,ds\right.$$
$$\left.-\frac{1}{\sigma}\int_{\mathscr{S}\setminus\Sigma_\delta}\xi\,\Upsilon\rho\,d\mathscr{S}-\lambda\int_{\mathscr{S}\setminus\Sigma_\delta}\xi\,d\mathscr{S}\right|<C\delta\Sigma+CM\delta\Sigma. \tag{1.42}$$

We now let $\delta\to 0$. A simple way to do so is to multiply any given η by the factor $(1-\min\{d/\delta,1\})$, where d is distance from Σ; then $\operatorname{supp}\eta$ remains in Σ_δ. All hypotheses remain fulfilled in the limiting procedure, and the values of η on Σ are unvaried. Both terms on the right in (1.42) tend to zero in the limit, and we obtain finally for the limit of the rates of change of energy corresponding to the given virtual displacements

$$\dot{\mathscr{E}}^0=\lim_{\delta\to 0}\dot{\mathscr{E}}=\int_{\mathscr{S}}\xi\left(-2H+\frac{1}{\sigma}\Upsilon\rho+\lambda\right)d\mathscr{S}$$
$$+\oint_{\Sigma}[-\beta\xi\sin\gamma+\eta(1-\beta\cos\gamma)]\,ds. \tag{1.43}$$

1.6. The Equation and the Boundary Condition

According to the principle of virtual work, $\dot{\mathscr{E}}$ must vanish for any choice of ξ,η, subject to (1.26a) and to (1.27). Thus also $\dot{\mathscr{E}}^0=0$. We first observe that if $-2H+\Upsilon\rho+\lambda\neq 0$ at a point $p\in\mathscr{S}$, then by choosing $\eta\equiv 0$ and ξ to be positive and to have its support in a small enough neighborhood of p, we arrive at a contradiction. Thus

$$2H=\lambda+\frac{1}{\sigma}\Upsilon\rho \tag{1.44}$$

holds on $\mathscr{S}$, and the first integral in (1.43) must vanish, regardless of ξ. Thus the second integral must also vanish. The choice $\xi=\tau\sin\gamma$, $\eta=\tau\cos\gamma$ on Σ satisfies the admissibility conditions if $|\tau|\le 1$, and we find

$$\oint_{\Sigma}\tau(\beta-\cos\gamma)\,ds=0. \tag{1.45}$$

A repetition of the above reasoning now yields

$$\cos\gamma=\beta \tag{1.46}$$

on Σ, thus determining the "contact angle" γ in terms of the local relative adhesion coefficient β.

1.7. Divergence Structure

We can get further information by making a different choice of variation. Consider an open piece $\hat{\mathscr{S}}$ of capillary surface $\mathscr{S}$, sufficiently small that it can be represented as a graph $z=u(x, y)$ over a domain Ω. We consider only variations that have their support in Ω. Restricting attention to terms in the energy expression (1.24) that will be varied, we may write

$$\mathscr{E}=\sigma\left\{\int_{\Omega}\sqrt{1+u_x^2+u_y^2}\,d\omega+\frac{1}{\sigma}\int \Upsilon\rho\,dx+\lambda\int_{\Omega}u\,d\omega\right\}. \tag{1.47}$$

We introduce a particular virtual displacement of a candidate $u(x, y)$ for an equilibrium surface by setting $\tilde{u}(x, y; \varepsilon)=u(x, y)+\varepsilon\eta(x, y)$, where η is arbitrary, subject to smoothness conditions and to $\operatorname{supp}\eta\subset\Omega$. The principle of virtual work now implies

$$\int_{\Omega}\frac{\nabla u\cdot\nabla\eta}{\sqrt{1+u_x^2+u_y^2}}\,d\omega+\frac{1}{\sigma}\int_{\Omega}\eta\,\Upsilon\rho\,d\omega+\lambda\int_{\Omega}\eta\,d\omega=0. \tag{1.48}$$

We integrate the first term in (1.48) by parts to obtain

$$\int_{\Omega}\eta\left(-\operatorname{div}\mathsf{T}u+\frac{1}{\sigma}\Upsilon\rho+\lambda\right)d\omega=0 \tag{1.49}$$

where we have set

$$\mathsf{T}u=\frac{1}{\sqrt{1+|\nabla u|^2}}\nabla u. \tag{1.50}$$

Since η is arbitrary, we obtain

$$\operatorname{div}\mathsf{T}u=\frac{\rho}{\sigma}\Upsilon+\lambda \tag{1.51}$$

over Ω as the equation for the free surface $\mathscr{S}$ in the given coordinates. A comparison with (1.44) shows immediately that *the mean curvature of a surface can be expressed as a divergence*, a fact that can also be verified *a posteriori* by direct calculation.

1.8. The Problem as a Geometrical One

We shall see in the following sections that in many cases of general interest the constant λ in (1.44) (or (1.51)) can be determined explicitly in terms of the prescribed data. In a general case, it must be expected that its determination leads to technical difficulties or that λ will not be unique (reflecting a nonuniqueness of the surface $\mathscr{S}$). However, in every case in which λ can be determined, the problem of finding a capillary surface $\mathscr{S}$ is a purely geometric one: *to find a surface whose mean curvature is a prescribed function of position* (*see* (1.44) *or* (1.51)) *and which meets prescribed (rigid) bounding walls in a prescribed angle* γ (*see* (1.46)).

1.9. The Capillary Tube

We consider the configuration of Fig. 1.2, in which the tube has infinite height but may have arbitrary section Ω, and the container is a circular cylinder of large diameter, so that the fluid surface level at a large distance from the tube provides a reference level ($z=0$) for atmospheric pressure that does not change with perturbations of the fluid surface in the tube. Thus if the outer surface is unvaried, there will be effectively no volume constraint and we may take $\lambda=0$. We limit attention to surfaces $z=u(x,y)$ that project simply onto Ω and take for Υ the gravitational potential $\Upsilon=gz$. We assume further $\rho=$constant (incompressible fluid). Then (1.51) becomes

$$\operatorname{div} \mathsf{T}u=\kappa u \tag{1.52}$$

(in agreement with (1.19)), with

$$\kappa=\frac{\rho g}{\sigma}. \tag{1.53}$$

We refer to κ as the *capillarity constant.*

Let $\Sigma=\partial\Omega$, and let ν be exterior unit normal on Σ. Letting γ be the angle between the surface $\mathscr{S}$ and the cylinder wall $\mathscr{Z}$ over Σ, we calculate easily $\nu\cdot\mathsf{T}u=\cos\gamma$, which according to (1.46) is the physical constant β. Thus

$$\nu\cdot\mathsf{T}u=\cos\gamma \tag{1.54}$$

provides a boundary condition for u over Σ. From an analytical point of view, the problem now becomes: *to find a solution of the* (*nonlinear elliptic*) *equation* (1.52) *subject to the* (*nonlinear*) *boundary condition* (1.54).

Laplace discovered [114] that the volume of fluid lifted over Ω above the reference level $z=0$ can be determined explicitly in terms of the boundary data. Specifically:

Theorem 1.1. *Let* $u(x,y)$ *be a solution of* (1.52) *and* (1.54). *Then the volume* $\mathscr{V}$ *satisfies*

$$\mathscr{V}=\frac{1}{\kappa}\oint_{\Sigma}\cos\gamma\,ds. \tag{1.55}$$

Proof. Integration of (1.52) over Ω yields

$$\int_\Omega \operatorname{div} \mathsf{T} u \, d\omega = \kappa \mathscr{V} = \oint_\Sigma \nu \cdot \mathsf{T} u \, ds \equiv \oint_\Sigma \cos\gamma \, ds \tag{1.56}$$

by the divergence theorem and (1.54).

If the volume is constrained then (1.52) changes to

$$\operatorname{div} \mathsf{T} u = \kappa u + \lambda \tag{1.57}$$

where λ is to be determined by the constraint. The simplest such problem is that in which the tube and container coincide (Fig. 6.1) and are of homogeneous material, so that $\gamma \equiv$ const. In that case, an integration of (1.57) by parts yields

$$\lambda = \frac{1}{\Omega} (\Sigma \cos\gamma - \kappa \mathscr{V}). \tag{1.58}$$

The transformation $u = v - \lambda/\kappa$ converts (1.57) to (1.52); thus it follows from the uniqueness of the solutions to (1.52) and (1.54) that *the shape of the surface $\mathscr{S}$ is independent of the constraint.*

We shall prove the uniqueness of the solutions of the capillarity problem in Chapter 5. Existence proofs, under varying conditions of regularity of Σ, have been given by M. Emmer [46], C. Gerhardt [74, 75, 76], N. Ural'tseva [181], Finn and Gerhardt [68], J. Spruck [171], Simon and Spruck [167], E. Giusti [84, 85, 86], and – most recently – G. Lieberman [117, 118]. From the point of view of the material presented here, the proof of Emmer (the first to be given) has a special interest, as his boundary regularity condition connects closely with some remarkable properties of solutions in wedge domains, to be developed in Chapter 5. In Chapter 7 we present a general existence proof, based on ideas of Miranda and of Giusti, and to some extent on ideas of [68], in which an extended form of the Emmer condition appears. The results of Chapter 7 cover all situations encountered in the text except those of the sessile and pendent drops, which are discussed separately in Chapters 3 and 4.

For the special case of circular section with constant γ, which we develop in the following chapter, simple proofs appear in Johnson and Perko [105] and in Finn [57].

Throughout the text, we will assume $0 \le \gamma \le \pi/2$. The case $\gamma > \pi/2$ reduces to that one under the transformation $u \to -u$.

1.10. Dimensional Considerations

For many purposes, it is advantageous to write the equation (1.52) in nondimensional form. Letting a be a representative length, we set U

$=u/a$, $\mathbf{X}=\mathbf{x}/a$; (1.52) then takes the form

$$\operatorname{div} T U=\kappa a^2 U \equiv B U \tag{1.59}$$

and the boundary condition (1.54) becomes

$$\nu \cdot T U=\cos\gamma. \tag{1.60}$$

The nondimensional "Bond number" B is a measure of the "size" of the configuration. A configuration can be "large" if a or g is large, or if σ is small. However, whenever uniqueness holds, *any two capillary surfaces with the same B and γ over geometrically similar domains will themselves be geometrically similar;* thus, the nondimensional equation groups the solution surfaces into equivalence classes of geometrically similar surfaces that need not be distinguished from each other.

Notes to Chapter 1

1. §1.1. The earliest explicit mention of the sum of principal curvatures appears to occur in Meusnier [129], who showed that at each point on a surface of locally minimizing area the sum of the principal curvatures is zero. Young was apparently the first to consider the sum of the curvatures as an independent entity on which to focus attention, and to connect it with more general geometrical and physical properties of the surfaces. The name "mean curvature" was coined by Sophie Germain [78] in 1831.

2. §1.1. In an 1855 reprinting [190] of his essay [189], Young appended a long and sarcastic attack on Laplace's contributions, suggesting that Laplace had stolen from him and ridiculing Laplace's analytical methods. In fact, Laplace did use the notion of constancy of contact angle for homogeneous materials without reference, and without providing a satisfactory proof. Further, Young was able to derive many of Laplace's results by direct and ingenious geometrical reasoning without recourse to formal equations, which he professed to disdain. Apparently the only reference to Young in Laplace's work occurs at the end of the second "Supplément" (p. 498) where he attacks Young mistakenly on a relatively minor point (see the comment on p. 1018 of the Bowditch translation [114]), and remarks in general on the absence of rigor in Young's reasoning. It is certainly true that Young's reasoning was often expressed in such vague and cumbersome language as to defy comprehension. On this point the interested reader should consult the (perhaps overly) critical review by Bikerman [13]. For another point of view, see Pujado, Huh, and Scriven [152]. The case for the analytical method may indeed not have been clear at the time, but the judgment of history on the matter is unmistakable. The present volume owes very much to the influence of Laplace's writings.

3. §§1.2, 1.3. For historical remarks on the development of conceptions and of misconceptions on capillarity phenomena, the reader may wish to consult the articles of Bikerman [10, 11] and the references cited there. We mention also the fascinating book by Boys [19], the surveys by Maxwell [128] and by Minkowski [134], and the general treatise now being prepared by Emmer and Tamanini [48].

4. §1.4. Here and in the remainder of the text we use the symbols $\mathscr{S}, \Sigma, \Omega, \ldots$ interchangeably to denote either a set or its measure. Whenever the intended meaning seems unclear we revert to the customary notation $|\mathscr{S}|, |\Sigma|, \ldots$ for the measures.

5. §1.4. Our introduction of the quantities σ and β and our justification of their properties are intended to be heuristic rather than scientifically precise. There is a considerable literature devoted to establishing the existence of the energies $\mathscr{E}_S, \mathscr{E}_W$ and to proving, on the basis of hypotheses about intermolecular forces, that σ and β are constant for homogeneous materials and independent of the shape of the surfaces, volume of the fluid, and thickness of the bounding walls. For a historical overview, cf. Clairaut [27], Segner [160], Monge [141], Young [189], Laplace [114], Gauss [73], Poisson [149], Neumann [143], Poincaré [148], Minkowski [134], Bakker [6], and further references cited in those works. In that theory, it seems to be essential to assume the existence of attractive (or repulsive) forces that are very large at molecular distances and negligible at larger distances; nevertheless, in the original versions of the theory the media are assumed to be continuously distributed in space, cf. the comments on pp. 18–21 of [158]. Studies based on the molecular constitution of the materials were initiated by Gibbs [82] (see also Boltzmann [15]). Modern work in this direction, using statistical mechanical models, can be found in Davis [39], Davis and Scriven [40], Rowlinson and Widom [158] and the references cited there. See also the commemorative volume edited by Goodrich and Rusanov [94].

6. §1.4. We have assumed, with Laplace and Gauss, that the free surface is ideally thin. That is clearly not the case physically, and many authors (see, e.g., Poisson [149], Neumann [143], Maxwell [128], Rayleigh [155], Bakker [6]) have attempted to characterize the density distribution in the free surface and to estimate the surface thickness. From the macroscopic point of view adopted in the present work, the distinctions in the results do not seem of great significance. Rayleigh [156] showed by experiment that for a clean water–air interface the surface can be considered to be of negligible thickness.

7. §1.5. In the literature it is common to restrict attention to variations normal to $\mathcal{S}$. The surface must then be assumed to extend slightly across $\mathcal{S}^*$, so that contact with the walls can be maintained during the variation. Thus, parts of $\mathcal{S}$ (and of the fluid bounded by $\mathcal{S}$) may have to be discarded or new parts introduced, in order to maintain a geometrical contact. Such a procedure seems at variance with the physical principle of virtual work, which refers to a given mechanical system consisting of the same particles throughout the (virtual) displacement. A further difficulty appears in that a normal variation is singular at the boundary if $\mathcal{S}$ is tangent to $\mathcal{S}^*$.

8. §1.5. The use of the relation (1.34) is conceptually appealing but not essential for our procedure. It is reasonable to assume that the derivatives in question are bounded; if that is the case, then the corresponding integral will vanish in the limit as $\delta \to 0$.

9. §1.6. The correctness of the result (1.44) of Young and of Laplace as a description of physical reality was universally accepted from the outset; to our knowledge it has never been subjected to direct experimental test. The formula (1.46) for the contact angle was however already viewed by Laplace and by his contemporaries with some reservation, and remains still the subject of continuing controversy. For a further discussion see Chapter 8.

10. §1.9. Miranda [139, Theorem 2.2] has shown that any surface of minimizing energy over Ω must project simply onto Ω; thus it is appropriate at this point to limit attention to surfaces $z(x, y)$.

Chapter 2

The Symmetric Capillary Tube

2.1. Historical and General

The circular capillary tube in a gravity field (cf. Fig. 1.2) has served historically as one of the focal points of interest. Not a single explicit solution is known. There is an extensive literature that dates at least to the studies of Leonardo da Vinci in 1490, however the first explicit formula permitting quantitative prediction seems to be that of Laplace [114] in 1805. The achievements of Laplace provided a standard that remained unsurpassed for over 150 years. Only recently was it shown that methods already known to Laplace could be developed to obtain significantly more precise information. We describe some of those developments in this chapter. These new results have served in turn as the underlying basis for recent studies on sessile drops, that are described in Chapter 3.

We have already mentioned in §1.9 that the existence of symmetric solutions with prescribed constant γ has been proved; the uniqueness theorem (Chapter 5) then shows that the symmetric solutions are the only ones. In terms of radial distance r from the axis of symmetry, the (nondimensional) equation (1.59) takes the form

$$\left(r\frac{u_r}{\sqrt{1+u_r^2}}\right)_r = B r u \tag{2.1}$$

with $B=\kappa a^2$; the constant a is most conveniently chosen as the radius of the tube; then $r=1$ at the boundary.

The relation (2.1) admits an important geometrical interpretation: we may write it in the form

$$(r\sin\psi)_r = B r u \tag{2.2}$$

where ψ is the angle of inclination of the solution curve $u(r)$, with respect to the r-axis.

2.2. The Narrow Tube; Center Height

For the case of small B, Laplace succeeded in integrating (2.1) approximately; he obtained in particular the celebrated approximation for the height u_0 on the axis of symmetry

$$u_0 \sim \mathscr{L}(B;\gamma) \equiv \frac{2\cos\gamma}{B} - \frac{1}{\cos\gamma} + \frac{2}{3}\,\frac{1-\sin^3\gamma}{\cos^3\gamma}. \tag{2.3}$$

We point out that Laplace discovered this relation, he did not prove it, nor did he give any indication how small B must be to obtain a prescribed accuracy. Since (2.3) remains even now the basis for much of the engineering work with capillary tubes, it is remarkable that these basic questions were not addressed until 1980, when Siegel [163] gave the first proof that (2.3) is asymptotically correct. Specifically, Siegel proved

$$|u_0 - \mathscr{L}(B;\gamma)| < C(\gamma)B \tag{2.4}$$

as $B \to 0$.

We present here another proof of Siegel's result, that yields improved estimates in various ways (Finn [58, 59]). The underlying idea derives from Laplace's formula (1.55) for the volume of fluid lifted in the tube; in the symmetric nondimensional case considered, (1.55) becomes

$$\mathscr{V} = \frac{2\pi}{B}\cos\gamma. \tag{2.5}$$

We consider a solution of (2.1) with initial value $u(0) = u_0 > 0$. Using the interpretation (2.2) and integrating to a general value $r < 1$, we obtain

$$r\sin\psi = B\int_0^r \rho u(\rho)\,d\rho. \tag{2.6}$$

From (2.6) we conclude $\sin\psi > 0$, that is, $u'(\rho) > 0$. There follow immediately the inequalities

$$B\frac{u_0}{2} < \frac{\sin\psi}{r} < B\frac{u}{2}. \tag{2.7a, b}$$

We now write (2.2) in the form

$$k_l + k_m \equiv \frac{\sin\psi}{r} + (\sin\psi)_r = Bu \tag{2.8}$$

which splits the (doubled) mean curvature into a sum of latitudinal and meridional sectional curvatures k_l and k_m; k_m is exactly the planar curvature of the vertical section $u(r)$.

From (2.7b) and (2.8) we obtain

$$(\sin\psi)_r > \tfrac{1}{2} B u > \tfrac{1}{2} B u_0 > 0 \tag{2.9}$$

on the curve $u(r)$, that is, $k_m > 0$ throughout the traverse; thus $u_{rr} > 0$.

We write, from (2.2),

$$\begin{aligned}
\frac{d}{dr} k_m = (\sin\psi)_{rr} &= B u_r - \frac{1}{r}(\sin\psi)_r + \frac{1}{r^2}\sin\psi \\
&= B u_r - \frac{1}{r}\left(B u - \frac{\sin\psi}{r}\right) + \frac{1}{r^2}\sin\psi \\
&= B u_r - \frac{B}{r}\left(u - 2\frac{\sin\psi}{Br}\right) \\
&> B u_r - \frac{B}{r}(u - u_0) = \frac{B}{r}\int_0^r \rho u_{\rho\rho}\, d\rho > 0
\end{aligned} \tag{2.10}$$

by (2.7a). We have also, by (2.7b),

$$\begin{aligned}
\frac{d}{dr} k_l = \frac{d}{dr}\frac{\sin\psi}{r} &= \frac{(\sin\psi)_r}{r} - \frac{\sin\psi}{r^2} \\
&= \frac{1}{r}\left(B u - 2\frac{\sin\psi}{r}\right) > 0.
\end{aligned} \tag{2.11}$$

We have proved:

Theorem 2.1. *For any solution of* (2.1) *in* $0 < r < 1$ *with* $u(0) > 0$, *both sectional curvatures* k_l *and* k_m *are increasing functions of* r.

Letting $r \to 0$, we have $u \to u_0$, hence, by (2.7a), $k_m(0) = k_l(0) = B u_0/2$. Consider a lower circular arc $\Sigma^{(1)}$: $u^{(1)}(r)$, centered on the u-axis, with $u^{(1)}(0) = u_0$ and of radius $2/B u_0$. Since k_m is increasing and $\Sigma^{(1)}$ has constant curvature, we conclude

$$\frac{d}{dr} u^{(1)}(r) < \frac{d}{dr} u(r); \quad u^{(1)}(r) < u(r) \tag{2.12}$$

in $0 < r < 1$ (see Fig. 2.1).

Now let $\Sigma^{(2)}$: $u^{(2)}(r)$ be a lower circular arc, centered on the u-axis, with $u^{(2)}(0) = u_0$, and such that

$$\frac{d}{dr} u^{(2)}(1) = \frac{d}{dr} u(1) \tag{2.13}$$

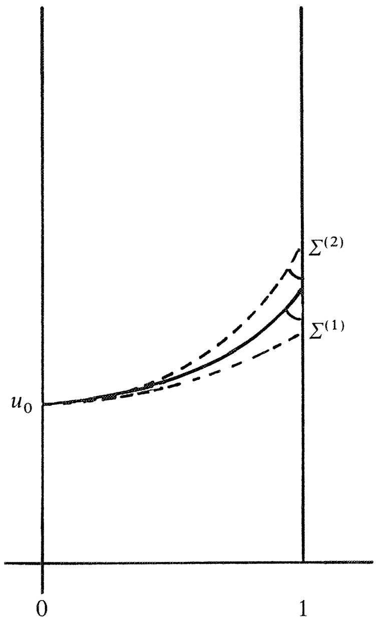


Figure 2.1. Comparison spheres.

so that $\Sigma^{(2)}$ meets the bounding walls in the same angle as does the solution surface. By (2.12)

$$\frac{d}{dr}u^{(2)}(1) > \frac{d}{dr}u^{(1)}(1) \tag{2.14}$$

from which follows the inequality $k^{(2)} > k^{(1)}$ for the two (constant) curvatures. Thus, since the curvatures $k^{(1)}$ and k_m are equal at $r=0$, there is an interval $0<r<\delta$ in which $u^{(2)}(r)>u(r)$. We choose δ to be the least upper bound of such values.

We assert that then $\delta=1$. For if not we would have $u^{(2)}(\delta)=u(\delta)$ and hence $\psi^{(2)}(\delta)\le\psi(\delta)$ so that

$$\begin{aligned}\int_0^\delta (\sin\psi-\sin\psi^{(2)})_r\,dr &= \sin\psi(\delta)-\sin\psi^{(2)}(\delta)\\ &= g(\delta)\ge 0\end{aligned} \tag{2.15}$$

and hence there would be a value $\hat{r}$, $0<\hat{r}<\delta$, such that

$$k_m(\hat{r})=(\sin\psi)_r>(\sin\psi^{(2)})_r=k^{(2)}. \tag{2.16}$$

But k_m is increasing; thus,

$$0<\int_\delta^1 (k_m-k^{(2)})\,dr=\sin\psi(1)-\sin\psi^{(2)}(1)-g(\delta) \tag{2.17}$$

contradicting (2.14).

Collecting the above results, we find:

The circular arcs $\Sigma^{(1)}, \Sigma^{(2)}$ lie respectively below and above the solution curve, and hence the associated rotation surfaces lift respectively smaller and larger volumes of fluid.

These volumes can be determined explicitly in terms of u_0 and the respective radii. Comparison of $\mathscr{V}^{(2)}$ with the actual volume lifted, as given by (2.5), yields

$$\frac{2}{B}\cos\gamma<u_0+R_2-\frac{2}{3}[R_2^3-(R_2^2-1)^{3/2}]\equiv F(u_0;R_2), \tag{2.18}$$

where

$$R_2=\frac{1}{\sin\psi(1)}=\frac{1}{\cos\gamma} \tag{2.19}$$

is the radius of $\Sigma^{(2)}$. Insertion of this value into (2.18) yields the inequality (cf. (2.3)) $u_0>\mathscr{L}(B;\gamma)$.

The same procedure applied to $\Sigma^{(1)}$ yields

$$\frac{2}{B}\cos\gamma>F(u_0;R_1), \qquad R_1=\frac{2}{Bu_0}. \tag{2.20}$$

One verifies easily that $\partial F/\partial u_0>0$, all $u_0>0$. Since (2.20) is satisfied by the u_0 of the solution surface, we certainly have $u_0<u_0^+$, which is the unique solution of the relation

$$\frac{2}{B}\cos\gamma=F\left(u_0;\frac{2}{Bu_0}\right). \tag{2.21}$$

However, (2.20) is not satisfied when the value $(2/B)\cos\gamma$ is substituted for u_0, and thus $u_0^+<(2/B)\cos\gamma$. We have proved:

Theorem 2.2. *For any choice of B, γ, there holds*

$$\mathscr{L}(B;\gamma)<u_0<u_0^+<\frac{2}{B}\cos\gamma. \tag{2.22a, b}$$

This result establishes the asymptotic correctness of the Laplace formula to $O(1)$, uniformly in γ, as $B\to 0$. More precisely, one obtains by formal expansion in (2.20)

$$u_0^+ = \mathscr{L}(B;\gamma) + \frac{B}{6}\frac{\cos\gamma}{(1+\sin\gamma)^4}(1+2\sin\gamma) + O(B^2) \tag{2.23}$$

uniformly in γ. We observe further from (2.22):

Corollary 2.2. *The Laplace expression* (2.3) *provides in every case a strict lower bound for the center height* u_0.

2.3. The Narrow Tube; Outer Height

Let us shift the circular arc $\Sigma^{(2)}$ vertically downward until it meets the solution curve (tangentially) at $(1, u_1)$. We then have, at the point $(1, u_1)$ of contact,

$$k_l = \sin\psi_1 = k^{(2)}. \tag{2.24}$$

We have also, from (2.8),

$$k_l + k_m = B u_1 \tag{2.25}$$

so that

$$k_m = B u_1 - \sin\psi_1. \tag{2.26}$$

By (2.7b), $B u_1 > 2\sin\psi_1$ and thus

$$k_m > \sin\psi_1 = k^{(2)} \tag{2.27}$$

at the contact point. Thus, the (displaced) arc $\Sigma^{(2)}$ lies locally below the solution curve.

If the two curves were to contact again in the interval $(0, 1)$ there would be a point $r_c \in (0, 1)$ such that $u^{(2)} < u$ in $(r_c, 1)$ and such that $u^{(2)}(r_c) = u(r_c)$. There would follow $\psi^{(2)}(r_c) \le \psi(r_c)$, and hence

$$\begin{aligned} \int_{r_c}^{1} (k^{(2)} - k_m)\,dr &= \int_{r_c}^{1} \frac{d}{dr}(\sin\psi^{(2)} - \sin\psi)\,dr \\ &= \sin\psi(r_c) - \sin\psi^{(2)}(r_c) \ge 0. \end{aligned} \tag{2.28}$$

Hence there would be a point $\hat{r}$ in the interval, such that $k^{(2)}(\hat{r}) > k_m(\hat{r})$. But k_m is increasing, thus $k^{(2)} > k_m$ throughout $(0, \hat{r})$ and hence also throughout $(0, r_c) \subset (0, \hat{r})$. Thus

$$0 < \int_0^{r_c} (k^{(2)} - k_m)\,dr = \sin\psi^{(2)}(r_c) - \sin\psi(r_c) \le 0 \tag{2.29}$$

by (2.28). This contradiction shows that in its new position, $\Sigma^{(2)}$ lies below the solution curve and hence lifts a smaller volume of fluid. The formal expression of that inequality now yields

$$u_1 < \mathscr{F}(B;\gamma) \equiv \frac{2}{B}\cos\gamma - \frac{\sin\gamma}{\cos\gamma} + \frac{2}{3}\,\frac{1-\sin^3\gamma}{\cos^3\gamma}. \tag{2.30}$$

This relation can be viewed as a counterpart, for the boundary height u_1, of the inequality (2.22a) for u_0.

To obtain a lower bound for u_1, we return to the relation (2.8)

$$k_l + k_m = Bu$$

from which

$$k_m(1) = Bu_1 - \cos\gamma. \tag{2.31}$$

Since k_m is increasing, it follows that a lower circular arc of radius

$$\rho_1 = \frac{1}{Bu_1 - \cos\gamma}, \tag{2.32}$$

tangent to the solution curve at $(1, u_1)$, lies strictly above that curve. We continue the arc till it becomes horizontal, then extend it to the axis as a

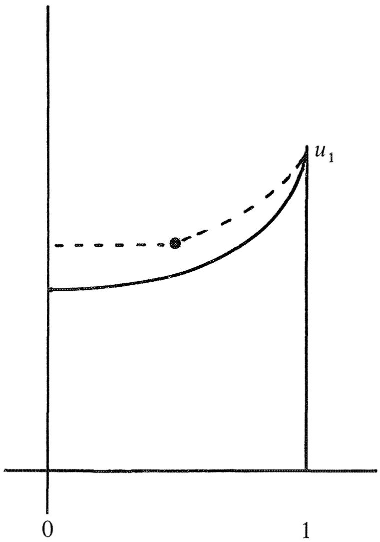


Figure 2.2. Upper comparison surface.

straight segment (Fig. 2.2). The volume inequality now yields

$$\begin{aligned}\frac{2}{B}\cos\gamma < u_1 &- (1-\rho_1)^2\rho_1(1-\sin\gamma)\\ &- (1-\rho_1)\rho_1^2\left(\frac{\pi}{2}-\gamma-\sin\gamma\cos\gamma\right)\\ &- \rho_1^3\frac{1-\sin\gamma}{3}(2+\sin\gamma)\equiv G(u_1;\rho_1).\end{aligned}\tag{2.33}$$

The right side of (2.33) is increasing in u_1, in any interval in which $u_1 > (2/B)\cos\gamma$. This latter inequality holds for the actual u_1 (see (2.7)) and (2.33) fails at $(2/B)\cos\gamma$. Thus $u_1 > u_1^-$, which is the unique solution in the interval $u_1 > (2/B)\cos\gamma$ of the relation

$$\frac{2}{B}\cos\gamma = G(u_1;\rho_1) \tag{2.34}$$

with ρ_1 determined by (2.32).

Collecting the above results, we have

Theorem 2.3. *For any choice of B, γ, there holds*

$$\frac{2}{B}\cos\gamma < u_1^- < u_1 < \mathscr{F}(B;\gamma). \tag{2.35a, b}$$

2.4. The Narrow Tube; Estimates Throughout the Trajectory

The following observations are due to Siegel [164]. In deriving them, we anticipate the estimate (2.65a) on meniscus height to be given later. Since the arc $\Sigma^{(2)}$ lies above or below the solution, according to whether it contacts at $r=0$ or $r=1$, we have

$$\begin{aligned}u_1 + \sqrt{\frac{1}{\cos^2\gamma}-1} &- \sqrt{\frac{1}{\cos^2\gamma}-r^2}\\ &< u(r) < u_0 + \frac{1}{\cos\gamma} - \sqrt{\frac{1}{\cos^2\gamma}-r^2}.\end{aligned}\tag{2.36a, b}$$

In (2.36a) we write

$$u_1 = u_0 + (u_1 - u_0) > \mathscr{L}(B;\gamma) + \frac{1-\sin\gamma}{\cos\gamma} - f(\gamma)B \tag{2.37}$$

by (2.22a) and (2.65a). Here

$$f(\gamma)=\frac{1-\sin\gamma}{\cos^2\gamma}\left(\frac{1}{3}\frac{1-\sin^3\gamma}{\cos^3\gamma}-\frac{1}{2}\frac{\sin\gamma}{\cos\gamma}\right). \tag{2.38}$$

Similarly, in (2.36b) we may write

$$u_0=u_1-(u_1-u_0)<\mathscr{F}(B;\gamma)-\frac{1-\sin\gamma}{\cos\gamma}+f(\gamma)B \tag{2.39}$$

by (2.35b) and (2.65a). There follows immediately

Theorem 2.4. *If we set*

$$\mathscr{S}(r)\equiv\mathscr{S}(B;\gamma;r)=2\frac{\cos\gamma}{B}+\frac{2}{3}\frac{1-\sin^3\gamma}{\cos^3\gamma}-\sqrt{\frac{1}{\cos^2\gamma}-r^2} \tag{2.40}$$

then there holds for any B, γ, $r\in[0,1]$

$$\mathscr{S}(r)-f(\gamma)B<u(r)<\mathscr{S}(r)+f(\gamma)B. \tag{2.41}$$

Theorem 2.4 provides an easily applicable bound on both sides, with explicit error estimates. However, some accuracy has been lost (e.g., the term $f(\gamma)B$ is needed only on one side at $r=0, 1$). The accuracy can be improved somewhat by using (2.70a) to estimate u_1-u_0, with u_0 estimated from (2.21); however, the calculations then become rather cumbersome.

2.5. Height Estimates for Tubes of General Size

The estimates we have considered till now, although valid for any B, do not provide useful information when B is large. It is however possible to give explicit bounds for solutions of (2.1), that are useful for every B, and which in some cases are at once exact asymptotically, both as $B\to 0$ and as $B\to\infty$.

Since, as we have shown, $k_m>0$ on a solution curve of (2.2) through $u_0>0$, we may introduce the inclination angle $\psi=\tan^{-1}u'(r)$ as independent variable. From (2.2) we have

$$\frac{\sin\psi}{r}+(\sin\psi)_r\equiv\frac{\sin\psi}{r}+(\cos\psi)\,\psi_r=Bu \tag{2.42}$$

and thus

$$\frac{dr}{d\psi}=\frac{r\cos\psi}{Bru-\sin\psi},\qquad \frac{du}{d\psi}=\frac{r\sin\psi}{Bru-\sin\psi}, \tag{2.43a, b}$$

a system of two (nonlinear) first-order equations for r and u in terms of the parameter ψ.

From (2.7) we obtain the inequalities

$$\tfrac{1}{2}Bru < Bru - \sin\psi < Br(u - \tfrac{1}{2}u_0) \qquad (2.44\text{a, b})$$

which when inserted into (2.43b) lead to two differential inequalities in separated form. Simple quadratures then yield [55, 57]

$$\frac{u_0}{2} + \left[\frac{2}{B}(1-\cos\psi) + \frac{u_0^2}{4}\right]^{1/2} < u < \left[\frac{4}{B}(1-\cos\psi) + u_0^2\right]^{1/2}. \qquad (2.45\text{a, b})$$

We shall prove in Chapter 3 that the solutions of (2.43) can be extended to the entire interval $0 < \psi \leq \pi$. The inequality (2.45b) continues to hold on that interval; however, (2.45a) does not.

We note that the radial distance r does not appear explicitly in (2.45). We obtain easily from those relations

$$\frac{2}{B}(1-\cos\psi) < u^2 - u_0^2 < \frac{4}{B}(1-\cos\psi); \qquad (2.46\text{a, b})$$

thus, *the difference of squares of the maximum and minimum heights in a circular capillary tube is bounded above and below by positive constants, independent of the radius of the tube.*

Using (2.7b) in (2.43a), we obtain

$$\frac{dr}{d\psi} < r\cot\psi \qquad (2.47)$$

which implies

$$\frac{d}{dr}k_l = \frac{d}{dr}\frac{\sin\psi}{r} > 0, \qquad (2.48)$$

a new and simpler proof of a result established previously by formal calculation.

We now write (2.43b) in the form

$$\left(Bu - \frac{\sin\psi}{r}\right)du = \sin\psi\, d\psi$$

and integrate in two steps, obtaining

$$\int_{u_0}^{A}\left(Bu - \frac{\sin\psi}{\rho}\right)du + \int_{A}^{u}\left(Bu - \frac{\sin\psi}{\rho}\right)du = 1-\cos\psi. \qquad (2.49)$$

We estimate the first integrand using (2.7b); in the second we use (2.48). We obtain

$$u(\psi)<\frac{\sin\psi}{Br}+\left\{\frac{2}{B}(1-\cos\psi)+\left(\Lambda-\frac{\sin\psi}{Br}\right)^2-\left(\frac{\Lambda^2-u_0^2}{2}\right)\right\}^{1/2}. \quad (2.50)$$

The right side of (2.50) is minimized by $\Lambda=2\sin\psi/Br$, which according to (2.7) lies interior to the interval of integration. Inserting that Λ, we obtain the result of Siegel [163]:

Theorem 2.5. *Throughout the traverse of any solution curve of* (2.43), *there holds in the range* $0<\psi\leqq\pi/2$

$$u(\psi)<\frac{\sin\psi}{Br}+\left\{\frac{2}{B}(1-\cos\psi)-\left(\frac{\sin\psi}{Br}\right)^2+\frac{u_0^2}{2}\right\}^{1/2}. \quad (2.51)$$

Using (2.7a) we obtain from (2.51)

Corollary 2.5. *On any solution curve of* (2.43) *there holds in the range* $0<\psi\leqq\pi/2$

$$u(\psi)<\frac{\sin\psi}{Br}+\left\{\frac{2}{B}(1-\cos\psi)+\left(\frac{\sin\psi}{Br}\right)^2\right\}^{1/2}. \quad (2.52)$$

We note that (2.52) can be shown to give asymptotically exact information, both as $B\to 0$ and as $B\to\infty$.

Let us now write (2.52) in the form

$$u(\psi)<\frac{\sin\psi}{Br}(1+p) \quad (2.53)$$

with

$$p=\sqrt{1+B\left(\frac{r}{k}\right)^2}, \qquad k=\cos\frac{\psi}{2}. \quad (2.54)$$

We note that r/k is increasing in ψ. We return to (2.49), this time using (2.7a) in the first integral. As to the second integral, we observe that (2.53) implies

$$Bu-\frac{\sin\psi}{\rho}<Bu\frac{p}{1+p}.$$

We are led to the inequality

$$\left(\Lambda-\frac{u_0}{2}\right)^2-\frac{u_0^2}{4}+\frac{p}{1+p}(u^2-\Lambda^2)>\frac{2}{B}(1-\cos\psi). \quad (2.55)$$

The best dividing point is now seen to be $\Lambda=(u_0/2)(1+p)$; from (2.7a) and (2.45a) and the defining relation (2.54) for p, we have for that choice

$$\begin{aligned} u_0<\Lambda=\frac{u_0}{2}\left(1+\sqrt{1+B\left(\frac{r}{k}\right)^2}\right)=\frac{u_0}{2}+\sqrt{\frac{u_0^2}{4}+\frac{B}{2}\frac{u_0^2 r^2}{1+\cos\psi}} \\ <\frac{u_0}{2}+\sqrt{\frac{u_0^2}{4}+\frac{2}{B}(1-\cos\psi)}<u, \end{aligned} \tag{2.56}$$

and thus Λ lies interior to the interval of integration. We thus obtain [57]:

Theorem 2.6. *In the range* $0<\psi\leq\pi/2$ *there holds*

$$u(\psi)>\sqrt{\frac{p+1}{p}}\left\{\frac{2}{B}(1-\cos\psi)+\frac{p+1}{4}u_0^2\right\}^{1/2}. \tag{2.57}$$

This estimate complements (2.51).

We now place (2.53) into (2.43a), obtaining

$$\frac{1}{k}\frac{\sqrt{k^2+Br^2}}{r}\,dr>\cot\psi\,d\psi, \tag{2.58}$$

an inequality that is preserved if k is replaced by any smaller value. We integrate over the interval $(0, r)$, replacing k by its value at r. Using

$$\lim_{\psi\to 0}\frac{r(\psi)}{\sin\psi}=\frac{2}{Bu_0} \tag{2.59}$$

which follows from (2.7), we are led [57] to:

Theorem 2.7. *In the range* $0<\psi\leqq\pi/2$ *there holds*

$$u_0>\frac{\sin\psi}{Br}(1+p)\,e^{1-p}. \tag{2.60}$$

This result provides a new lower bound for u_0, that still contains useful information for all B.

For large B a better estimate can be obtained by observing from (2.53) that

$$\frac{Bru}{\sin\psi}-1<p=\sqrt{1+B\left(\frac{r}{k}\right)^2}<\frac{1}{k}\sqrt{1+Br^2}, \tag{2.61}$$

so that (2.43a) yields

$$\frac{\sqrt{1+Br^2}}{r}\,dr > \frac{\cos\psi\sqrt{1+\cos\psi}}{\sqrt{2}\sin\psi}\,d\psi, \tag{2.62}$$

an inequality that can be integrated explicitly. We find after some manipulation [57]

Theorem 2.8. *In the range* $0<\psi\le\pi/2$ *there holds*

$$\frac{r}{1+\sqrt{1+Br^2}}\,e^{\sqrt{1+Br^2}-1} > \frac{4\sqrt{2}\,e^{2\left(\cos\frac{\psi}{2}-1\right)}\sin\frac{\psi}{2}}{1+\sqrt{2}\cos\frac{\psi}{2}}\,\frac{\psi}{Bu_0}. \tag{2.63}$$

The relations (2.22a), (2.60), and (2.63) provide estimates from below for u_0 with varying ranges of utility. If $\psi=\pi/2$ (i.e., $\gamma=0$) then (2.60) is of no interest; in this case, below the "crossover point" $B=2.889$ the best result is obtained from (2.22a), while for larger B (2.63) is preferable. However, we note that $\mathscr{L}(B;\gamma)<0$ when $B>8$, regardless of γ, and that for any fixed $B>0$ (2.60) gives a better result than (2.63) when ψ is close enough to zero. Thus, each of the three relations will under appropriate situations become the estimate of choice.

Finally, we introduce (2.57) into (2.43a). After some manipulation, we obtain [57]

Theorem 2.9. *In the range* $0<\psi\le\pi/2$ *there holds*

$$\frac{\sin\psi}{\sqrt{B}\,u_0} > A\,\frac{e^{\sqrt{B}r}}{(\sqrt{B}\,r)^{1/2}} \tag{2.64}$$

for values $\sqrt{B}\,r$ *sufficiently large.*

The constant A can be estimated explicitly, although the details become cumbersome (see [57]).

2.6. Meniscus Height; Narrow Tubes

We denote the (nondimensional) change of height of the surface, from the point of symmetry to the contact line with the wall, by $q=u_1-u_0$ (see Fig. 1.2). The estimates we have already found for u_0 and u_1 yield corresponding estimates for q, but the results obtained that way are not very precise.

Theorem 2.10. *There holds always*

$$\frac{1-\sin\gamma}{\cos\gamma}-f(\gamma)B<q<\frac{1-\sin\gamma}{\cos\gamma} \tag{2.65a, b}$$

where

$$f(\gamma)=\frac{1-\sin\gamma}{\cos^2\gamma}\left(\frac{1}{3}\,\frac{1-\sin^3\gamma}{\cos^3\gamma}-\frac{1}{2}\,\frac{\sin\gamma}{\cos\gamma}\right). \tag{2.66}$$

Proof. The relation (2.65b) follows immediately from the observation that the circular arc $\Sigma^{(2)}$, when in contact with the solution curve at $(1, u_1)$ (see §2.3), lies entirely below the curve. To obtain the other inequality, we write from (2.45a)

$$q>\frac{2(1-\sin\gamma)}{Bu_1}$$

and from (2.35b)

$$u_1<\frac{2}{B}\cos\gamma\left[1-\frac{B}{2}\left(\frac{\sin\gamma}{\cos^2\gamma}-\frac{2}{3}\,\frac{1-\sin^3\gamma}{\cos^4\gamma}\right)\right].$$

The result thus follows from the inequality

$$(1-x)^{-1}>1+x, \qquad x<1.$$

2.7. Meniscus Height; General Case

We may obtain universally applicable estimates by exploiting remarkable monotonicity properties that are implicit in the estimates (2.51) and (2.57). Noting that $\gamma=(\pi/2)-\psi_1$, we set

$$p_1=\sqrt{1+\frac{2B}{1+\sin\gamma}} \tag{2.67}$$

$$\mathsf{F}(u_0;B;\gamma)=\sqrt{\frac{p_1+1}{p_1}}\left\{\frac{2}{B}(1-\sin\gamma)+\frac{p_1+1}{4}u_0^2\right\}^{1/2}-u_0 \tag{2.68}$$

$$\mathsf{G}(u_0;B;\gamma)=\frac{\cos\gamma}{B}+\left\{\frac{2}{B}(1-\sin\gamma)-\frac{\cos^2\gamma}{B^2}+\frac{u_0^2}{2}\right\}^{1/2}-u_0. \tag{2.69}$$

We then have, by (2.51) and (2.57),

$$\mathsf{F}(u_0;B;\gamma)<q<\mathsf{G}(u_0;B;\gamma). \tag{2.70a, b}$$

Now observe that $\mathsf{F}_{u_0}(0; B; \gamma) = -1$. Further, at any zero of F_{u_0} there holds

$$\frac{B u_0}{2} = 4\sqrt{\frac{p_1}{p_1 - 1}} \cos\gamma > 4\cos\gamma.$$

Thus $\mathsf{F}_{u_0} < 0$ throughout the range $0 \le u_0 \le 4(2\cos\gamma/B)$ (cf. (2.7a)).

We next calculate

$$\mathsf{G}_{u_0} = \lambda\left[u_0 - 2\sqrt{\frac{2}{B}(1-\sin\gamma) - \frac{\cos^2\gamma}{B^2} + \frac{u_0^2}{2}}\right]$$

for some $\lambda > 0$, from which we conclude $\mathsf{G}_{u_0}(\infty; B; \gamma) < 0$. At any zero of G_{u_0}, there holds

$$u_0^2 = \frac{4\cos^2\gamma}{B^2} - \frac{8}{B}(1 - \sin\gamma) \equiv \Phi(B; \gamma).$$

But from (2.22a) we find

$$\begin{aligned} u_0^2 &> \frac{4\cos^2\gamma}{B^2} - \frac{4}{B}\left(1 - \frac{2}{3}\,\frac{1-\sin^3\gamma}{\cos^2\gamma}\right) \\ &= \frac{4\cos^2\gamma}{B^2} - \frac{4}{B}\,\frac{1+2\sin\gamma}{3(1+\sin\gamma)}(1-\sin\gamma) \\ &> \frac{4\cos^2\gamma}{B^2} - \frac{4}{B}(1-\sin\gamma) > \Phi(B; \gamma). \end{aligned}$$

Thus, $\mathsf{G}_{u_0} < 0$ whenever $u_0 > \mathscr{L}(B; \gamma)$. We have proved [58, 59]:

Theorem 2.11. *The inequality* (2.70a) *continues to hold if* u_0 *is replaced by any upper bound not exceeding* $4(2\cos\gamma/B)$; *the inequality* (2.70b) *continues to hold if* u_0 *is replaced by any lower bound not less than* $\mathscr{L}(B; \gamma)$.

The relations (2.70), in conjunction with the appropriate bounds for u_0, give in all cases results preferable to those obtainable by other methods we have indicated. The improvement over the results given by (2.65) is not very significant if $B \ll 1$, and (2.65) is certainly easier to apply. Also, since u_0 vanishes exponentially for large B (see (2.64)), the u_0 terms in (2.70) can be neglected when $B \gg 1$. It is in the intermediate range $0.1 < B < 10$ that (2.70) finds its greatest interest. That is especially the case since the known (formal) asymptotic expansions do not apply well in that range.

2.8. Comparisons with Earlier Theories

There is a modest literature on procedures for estimating the surface parameters in symmetric capillary tubes. All previous methods are based on formal asymptotic expansions for small or large B; the error bounds appearing in [57] are apparently the first in the literature. In Figs. 2.3–

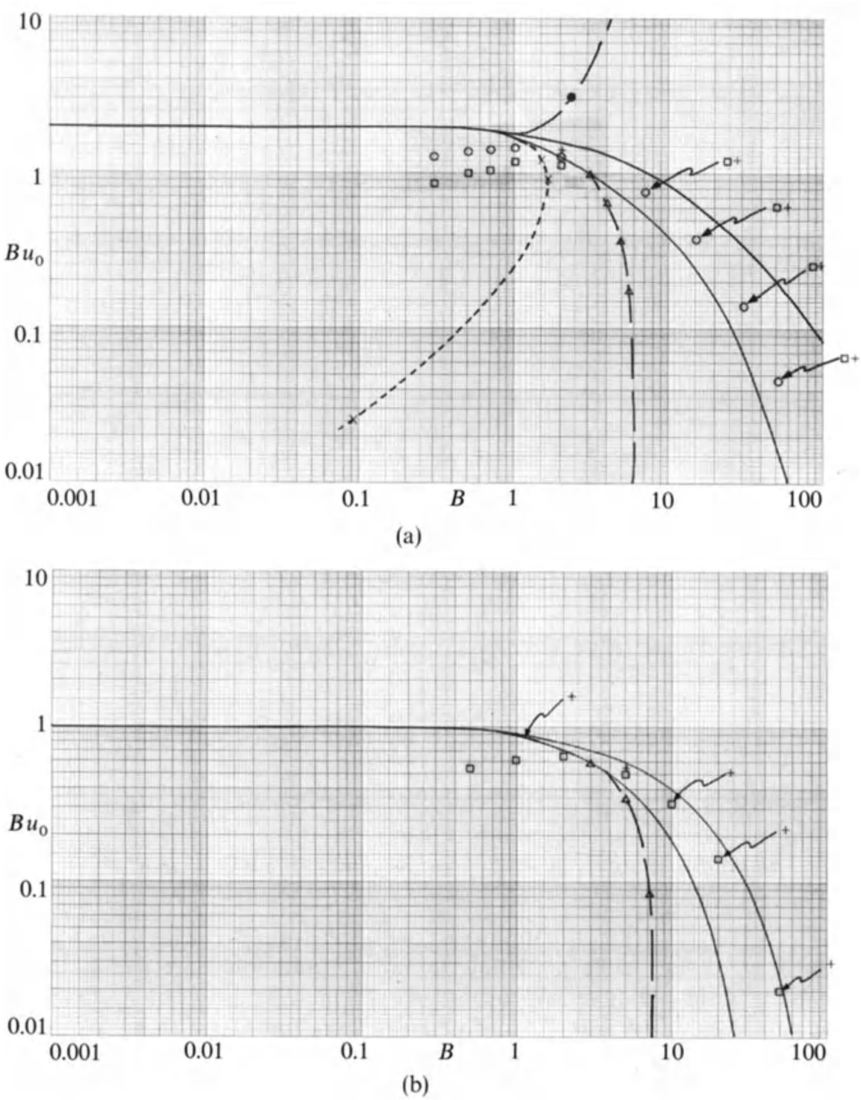


Figure 2.3. Center height. (a) $\gamma=0$. (b) $\gamma=\pi/3$. △: Laplace, Concus (inner); ●: Poisson (inner); ×: Rayleigh (inner); □: Concus (outer); ○: Rayleigh (outer); +: Computer (Concus).

2.5 are shown as solid lines the calculated upper and lower bounds for u_0, u_1, and q as functions of B, for several γ, using the estimates of the preceding sections. The results are given nondimensionally in the range $B=0.001$ to $B=100$. They are compared (where the earlier results apply) with calculations from asymptotic formulas due to Laplace [114], to Poisson [149], to Rayleigh [157], and to Concus [28] for small B, with calculations from formulas of Rayleigh [157] and of Concus [28] for

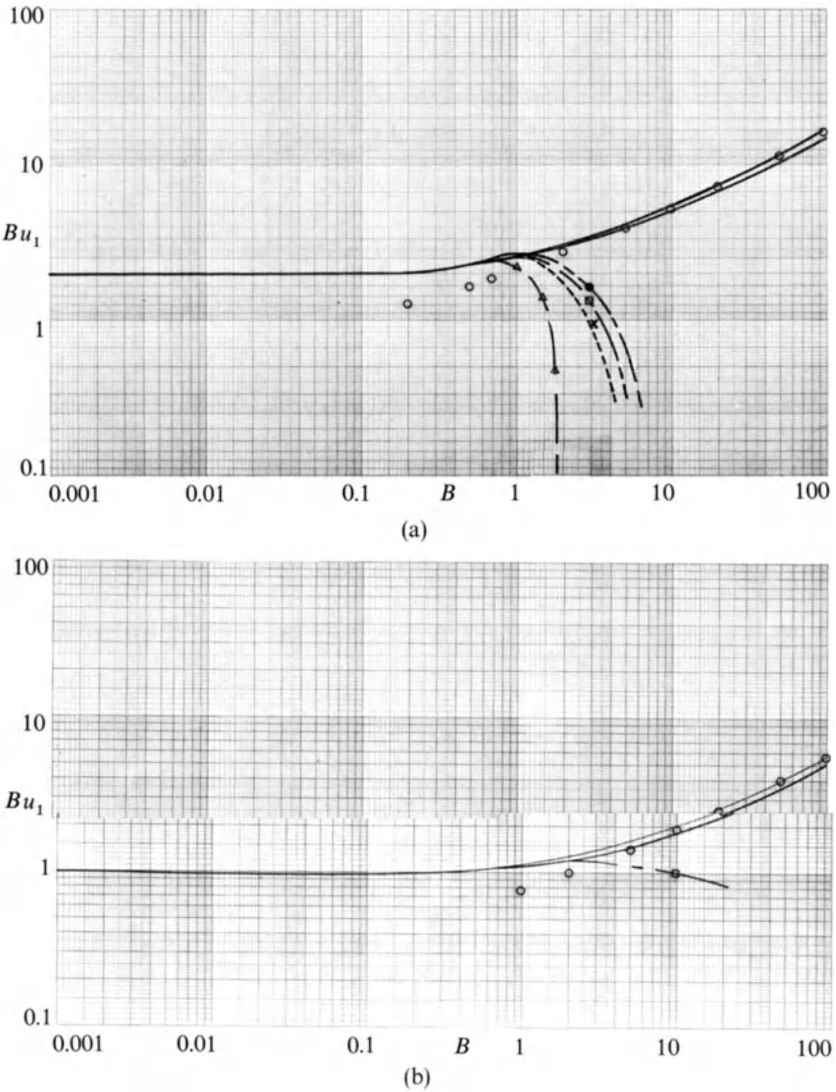
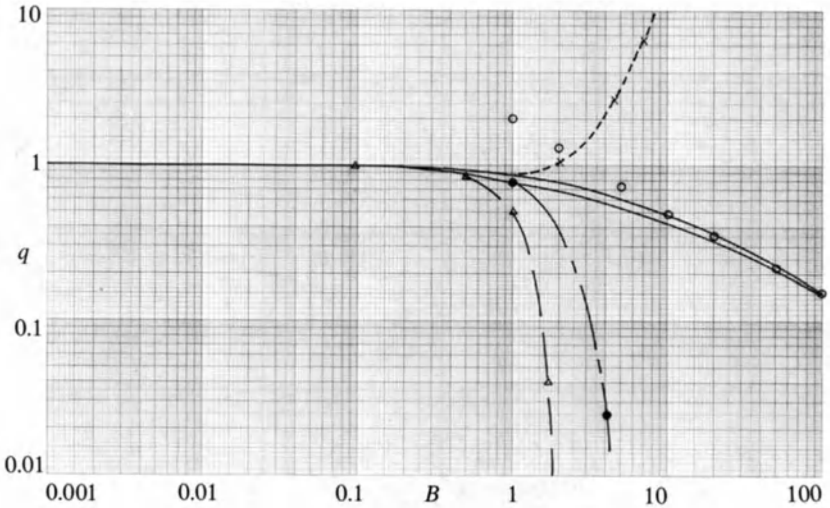
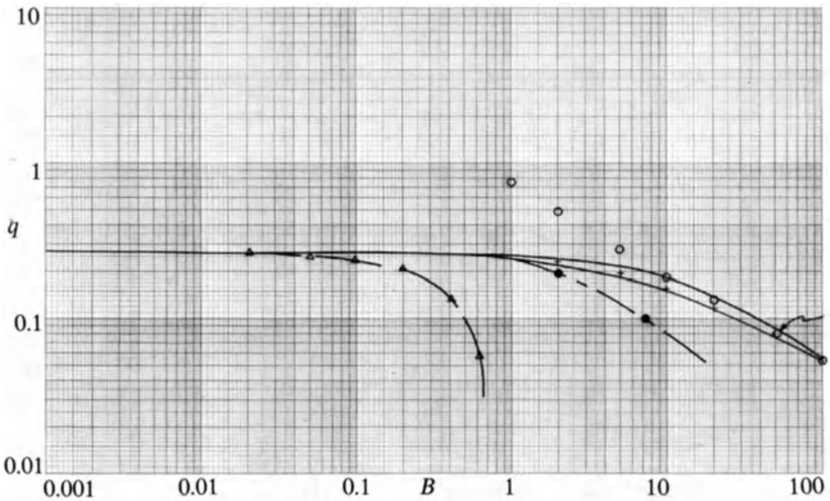


Figure 2.4. Outer height. (a) $\gamma=0$. (b) $\gamma=\pi/3$. △: Laplace (inner); ●: Poisson (inner); ×: Rayleigh (inner); □: Concus (inner); ○: Concus (outer).

large B, and with the results of computer calculations by Concus for some particular cases. The scale is logarithmic, so that changes in linear height indicate relative changes in the values. Figure 2.3 shows Bu_0. The large relative change between upper and lower bounds at large B is probably not of major importance, as u_0 is exponentially small in B when B is large. Nevertheless, the precision of both the Rayleigh and



(a)



(b)

Figure 2.5. Meniscus height. (a) $\gamma=0$. (b) $\gamma=\pi/3$. $\triangle$: Laplace (inner); $\bullet$: Poisson. Concus (inner); $\times$: Rayleigh (inner); $\circ$: Concus (outer); $+$: Computer (Concus).

Concus "outer" expansions is striking. Particularly notable is the fact that these expansions (especially that of Rayleigh) continue to give good information down to values of B small enough that the "inner" expansions start to have meaning. Even so, there is an interval of B in which none of the formal expansions give results lying within the error bounds. It should be noted that the Rayleigh expansions were given only for the case $\gamma = 0$.

Figure 2.4 shows Bu_1; we note that Bu_1 does not tend to zero with increasing B, but that $\sqrt{B}\,u_1$ is asymptotic to the (normalized) rise height $\sqrt{2(1-\sin\gamma)}$ of a capillary surface at an isolated vertical plate. Finally, Fig. 2.5 shows the corresponding estimates for q.

One sees from the figures that, in addition to providing strict upper and lower bounds, the procedures outlined above also yield new information in the (intermediate) range $B=1$ to $B=10$, where the earlier asymptotic expansions do not always apply well.

A computational procedure for calculating meniscus shapes and height parameters under prescribed conditions was developed by Concus and Pereyra [37].

Notes to Chapter 2

1. Theorem 2.1. It can be shown that if $B>0$ (as is here assumed) then every solution of (2.1) in $0<r<1$ can be extended continuously to $r=0$. Thus $u_0 = u(0)$ is defined. Further, there holds always $\lim_{r\to 0} u'(r) = 0$. Since the negative of any solution is again a solution, we may always assume $u(0) \geq 0$. *If* $u(0)=0$ then the solution vanishes identically.

2. (2.46a, b). It might appear from the form of the relations that the estimate depends on the radius a, as a appears in $B = \kappa a^2$. However, the (nondimensional) height u in the formulas is actual height divided by a, hence the radius is a common factor and can be cancelled. The estimate does depend, of course, on κ.

It should be remarked that the indicated property is not shared by the height difference $u - u_0$, nor, in fact, by any power of u other than two.

3. §2.2, 2.5. The estimates given here, although they are asymptotically exact both for small and large B and provide useful information in all cases, do not in their present form yield an iterative procedure for determining a solution with arbitrary accuracy. Brulois [21] has developed a procedure for determining upper and lower bounds of various functionals associated with a solution, which are asymptotically exact to any order in B. Some of the estimates we have given can be obtained from the initial steps of his procedure. Brulois also showed the convergence for all $r \leq 1$ of the formal power expansions used in the local existence proof (cf. §4.2).

4. §§2.5–2.7. The estimates presented here are explicit and in closed form. Siegel [163] has given (nonexplicit) asymptotic estimates on order of magnitude in which the principal terms yield more precise limiting information.

Chapter 3

The Symmetric Sessile Drop

3.1. The Correspondence Principle

We consider a connected drop of liquid of prescribed volume $\mathscr{V}$ resting on a horizontal plane Π in a vertical gravity field directed toward Π, and we suppose the plane to be of homogeneous material so that the contact angle γ will be constant, $0 \leq \gamma \leq \pi$. Wente has proved [186] that under these conditions any equilibrium surface is generated by an interval of disks centered on a line segment orthogonal to Π, so we may restrict attention to that case (see Fig. 3.1). According to (1.51), at a point on the free surface $\mathscr{S}$ where the fluid is below $\mathscr{S}$, the height $u(x, y)$ of $\mathscr{S}$ above Π will satisfy

$$\operatorname{div} \mathsf{T}u = \kappa u + \lambda \tag{3.1}$$

for some constant (Lagrange multiplier) λ. (Where the liquid lies above $\mathscr{S}$, the sign of $\operatorname{div} \mathsf{T}u$ must be reversed.) Since a representative length is not yet apparent, we have written (3.1) in dimensional form.

The transformation $u = -v - (1/\kappa)\lambda$ changes (3.1) to

$$\operatorname{div} \mathsf{T}v = \kappa v, \tag{3.2}$$

which is exactly the equation for the free surface in a capillary tube. Thus, to every solution of (3.1) with center height u_0 there corresponds a

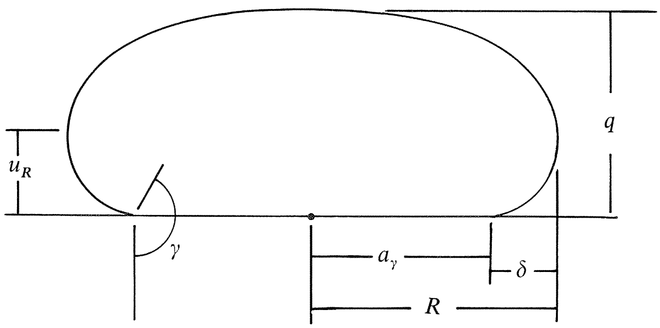


Figure 3.1. Sessile drop.

solution of (3.2) with center height $v_0 = -(u_0 + (1/\kappa)\lambda)$. Since, as indicated in §§4.2 and 4.3, the symmetric solutions of (3.2) are uniquely determined by the center height, we find that to every symmetric sessile drop there corresponds a unique capillary surface, and the two surfaces are (locally) geometrically congruent. Conversely, to every symmetric capillary surface there corresponds a sessile drop, determined up to an additive constant. If we identify any two such surfaces when one is a continuation of the other, we obtain the correspondence principle [57]: *there is a 1–1 correspondence between sessile drops and capillary surfaces.*

The set of all symmetric capillary surfaces is, however, completely determined as a one-parameter family in terms of the center height u_0. We thus conclude from the correspondence principle that *the set of all symmetric sessile drops can be described by a one parameter family of curves.* This property will be basic in what follows.

3.2. Continuation Properties

In view of the preceding remarks, we focus attention on a symmetric capillary surface, with center height $u_0 > 0$. Following (2.43), we write the equation in terms of the inclination angle ψ, but now in dimensional form:

$$\frac{dr}{d\psi} = \frac{r\cos\psi}{\kappa r u - \sin\psi}, \qquad \frac{du}{d\psi} = \frac{r\sin\psi}{\kappa r u - \sin\psi}. \tag{3.3a, b}$$

Theorem 3.1. *For any initial value u_0, the (unique) solution of (3.3) can be continued as a solution, with monotonely increasing $u(\psi)$, throughout the range $0 < \psi \le \pi$. The function $r(\psi)$ increases to a maximum $r(\pi/2) = R$, and then decreases to a value $a = \lim_{\psi \to \pi} r(\psi) > 0$.*

Proof. From the form of the equation in (2.1), we conclude as before

$$\kappa\frac{u_0}{2} < \frac{\sin\psi}{r} < \kappa\frac{u}{2}, \qquad 0 < \psi \le \frac{\pi}{2}, \tag{3.4a, b}$$

from which we see that the solution curve becomes vertical ($\psi = \pi/2$) at a value $r = R$ satisfying $R < 2/\kappa u_0$. Although the original equation (2.1) becomes singular at $r = R$, the system (3.3) does not. In fact, we find from (3.4b)

$$\Delta \equiv \kappa r u - \sin\psi > \sin\psi \tag{3.5}$$

and thus, from (3.3), that both u and r tend to finite positive limits as $\psi \to \pi/2$. Thus, the solution (r, u) of (3.3) can be continued as a solution of that system past the value $\psi = \pi/2$ corresponding to $r = R$.

The stated monotonicity properties up to and across $\psi = \pi/2$ are evident from the equation (3.3). On any continuation into an interval in

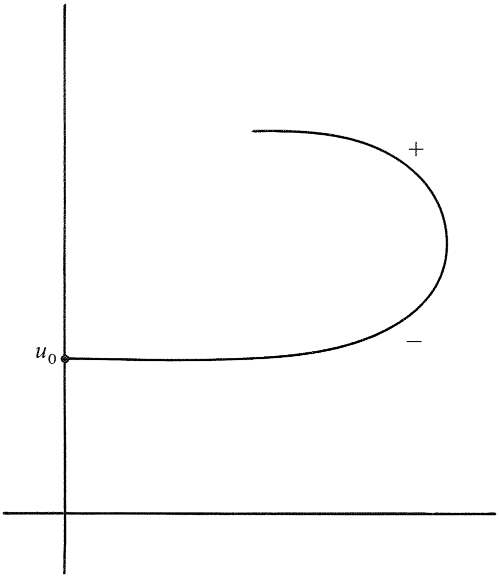


Figure 3.2. Continued capillary section.

which $\pi/2 < \psi < \pi$, we find again from (3.3a)

$$(r \sin \psi)_r = \kappa r u, \tag{3.6}$$

which is identical to the original relation (2.2) that holds on $0 < \psi < \pi/2$; see Fig. 3.2. Integrating from r to R on the top $(+)$ and on the bottom $(-)$ and subtracting, we get

$$r(\sin \psi^- - \sin \psi^+) = \kappa \int_r^R (u^+ - u^-)\, \rho\, d\rho. \tag{3.7}$$

For r close enough to R, we have $u^+ > u^-$, hence $\sin \psi^- > \sin \psi^+$, hence (3.5) continues to hold, hence by (3.3b) u^+, u^- are both increasing in ψ, hence $\sin \psi^- > \sin \psi^+$, $u^+ > u^-$ as long as continuation is possible.

On the continuation, we cannot have $r \to 0$, as (3.7) would yield in that case $0 > 0$. Thus, there exists $a = \lim_{\psi \to \pi} r(\psi) > 0$, as was to be proved.

In what follows, we use γ to denote the value of ψ at which the bounding free surface meets Π. Thus, γ is the contact angle in the sense originally introduced. In terms of the original problem, we have proved:

Corollary 3.1. *Corresponding to any value of the parameter u_0, there is exactly one sessile drop making boundary angle γ, for any γ in $0 < \gamma \leq \pi$.*

In the following sections, we study the properties of sessile drops by fixing γ and examining the dependence of the volume $\mathscr{V}$ on u_0.

3.3. Uniqueness and Existence

Theorem 3.2 ([57]). *Let $\mathscr{V}>0$ be arbitrary, and $0<\gamma\leq\pi$. There is exactly one symmetric simply connected sessile drop with boundary angle γ and volume $\mathscr{V}$.*

Lemma 3.2. *$\mathscr{V}$ is a continuously differentiable function of u_0.*

The lemma does not follow from standard continuous dependence theorems, as the equation is singular at $r=0$. However, the proof is not basically difficult – it can be obtained for example by exploiting monotonicity properties and simple growth estimates for the solutions – and we do not include details here.

Proof of Theorem 3.2. We first prove existence. Since, by (3.4), $R<2/\kappa u_0$, we have $R\to 0$ as $u_0\to\infty$.

As shown in Chapter 2, a ball of radius R contains the lower part of the surface; since $\sin\psi^+<\sin\psi^-$ the same ball contains also the upper part, for any $\gamma\leq\pi$. Thus, $\mathscr{V}<4/3\,\pi R^3$, which $\to 0$ as $u_0\to\infty$.

Letting $u_0\to 0$, we find by (2.60) that $r(\gamma;u_0)\to\infty$ for any fixed $\gamma\leq\pi/2$, and by (2.45a) that $u(\gamma;u_0)>\sqrt{2/\kappa(1-\cos\gamma)}$. Since the surface is convex, there follows $\mathscr{V}\to\infty$. If $\gamma>\pi/2$, then $\mathscr{V}(\gamma;u_0)>\mathscr{V}(\pi/2;u_0)$ so the same conclusion holds.

Thus, for fixed γ, every value $\mathscr{V}$ between 0 and ∞ is attained, and existence follows.

To prove uniqueness, it will suffice to prove

$$\dot{\mathscr{V}}\equiv\frac{\partial\mathscr{V}(\psi;u_0)}{\partial u_0}<0 \tag{3.8}$$

for all $u_0>0$ and each fixed ψ in $0<\psi\leq\pi$.

We obtain easily from (3.6) that

$$\mathscr{V}=\pi r\left(ru-\frac{2}{\kappa}\sin\psi\right) \tag{3.9}$$

and thus

$$\dot{\mathscr{V}}=\pi\left(\frac{2}{\kappa}\dot{r}\Delta+r^2\dot{u}\right). \tag{3.10}$$

We note

$$\dot{\mathscr{V}}(0;u_0)=0. \tag{3.11}$$

We calculate

$$\frac{d\dot{\mathscr{V}}}{d\psi}=\pi\,\frac{r^2\sin\psi}{\Delta^2}\{(2\Delta-\sin\psi)\dot{r}-\kappa r^2\dot{u}\}. \tag{3.12}$$

We shall show $\dot{r}<0$, $\dot{u}>0$. We obtain from (3.3)

$$\frac{d\dot{r}}{d\psi}=-\frac{\kappa r^2\dot{u}+\dot{r}\sin\psi}{\Delta^2}\cos\psi \tag{3.13}$$

$$\frac{d\dot{u}}{d\psi}=-\frac{\kappa r^2\dot{u}+\dot{r}\sin\psi}{\Delta^2}\sin\psi \tag{3.14}$$

with

$$\dot{r}(0)=0, \qquad \dot{u}(0)=1. \tag{3.15}$$

For $0<r<R$ we have

$$r\sin\psi=\kappa\int_0^r \rho\, u\, d\rho \tag{3.16}$$

and writing $u=u(\rho;u_0)=u(\rho(\psi;u_0);u_0)$,

$$\left.\frac{\partial u}{\partial u_0}\right]_\psi=\left.\frac{\partial u}{\partial\rho}\right]_{u_0}\dot{\rho}+\left.\frac{\partial u}{\partial u_0}\right]_\rho=\dot{\rho}\tan\psi+\left.\frac{\partial u}{\partial u_0}\right]_\rho \tag{3.17}$$

so that, from (3.16)

$$-\dot{r}\Delta=\kappa\int_0^r(\dot{u}-\dot{\rho}\tan\psi)\,\rho\,d\rho. \tag{3.18}$$

Using (3.15) we see that for each $u_0>0$ there is an initial interval $\mathscr{I}$: $0<\psi<\delta\le\pi$, in which $\dot{r}<0$.

In $\mathscr{I}$ there holds by (3.14)

$$\frac{d\dot{u}}{d\psi}>-\frac{\kappa r^2\dot{u}\sin\psi}{\Delta^2}. \tag{3.19}$$

We have by (3.5)

$$\frac{r^2\sin\psi}{\Delta^2}<\frac{r^2}{\sin\psi}, \tag{3.20}$$

the right side of (3.20) tending to zero as $\psi\to 0$ by (3.4a). Since initially $\dot{u}>0$ we find, for sufficiently small ψ,

$$\dot{u}>\exp\left\{-\kappa\int_0^\psi\frac{\rho^2(\psi)}{\sin\psi}\,d\psi\right\}. \tag{3.21}$$

We conclude immediately $\dot{u}>0$ and also (3.21) continues to hold, throughout $\mathscr{I}$. Hence, from (3.12)

$$\frac{d\dot{\mathscr{V}}}{d\psi}<0 \quad \text{in } \mathscr{I}. \tag{3.22}$$

Suppose there were a positive value $\hat{\psi}\leq\pi$ at which $\dot{r}=0$. There would then be a smallest such value, which we again label with $\hat{\psi}$. Since (3.22) would hold on $0<\psi\leq\hat{\psi}$, and since $\dot{\mathscr{V}}(0)=0$, we would have $\dot{\mathscr{V}}(\hat{\psi})<0$. But by (3.21) $\dot{u}(\hat{\psi})>0$, hence by (3.10) $\dot{\mathscr{V}}(\hat{\psi})=\pi r^2\dot{u}(\hat{\psi})>0$. This contradiction establishes that $\dot{r}<0$ on $0<\psi\leq\pi$. Thus by (3.12), (3.5), and (3.21) $d\dot{\mathscr{V}}/d\psi<0$ on $0<\psi<\pi$, hence $\dot{\mathscr{V}}<0$ on $0<\psi\leq\pi$, for any choice of $u_0>0$. Uniqueness is thus proved.

3.4. The Envelope

The family of solutions of (3.6) parametrized by u_0 admits an envelope, determined by the condition

$$\begin{vmatrix} u'(\psi;u_0) & r'(\psi;u_0) \\ \dot{u}(\psi;u_0) & \dot{r}(\psi;u_0) \end{vmatrix}=0$$

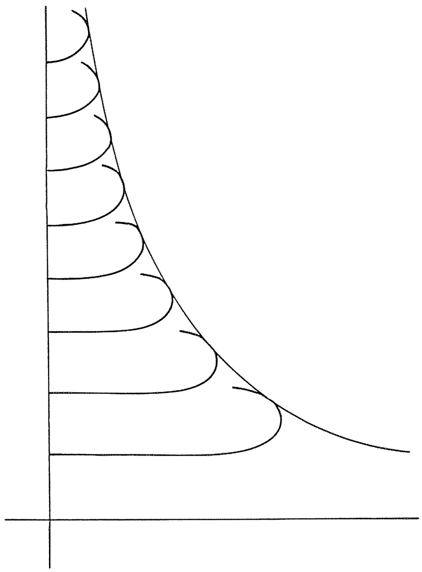


Figure 3.3. The envelope.

or, equivalently, in view of (3.3),

$$F(\psi) \equiv \dot{u} \cos \psi - \dot{r} \sin \psi = 0.$$

Since, as shown above, $\dot{u} > 0$, $\dot{r} < 0$ on $0 < \psi \leq \pi$, there can be no envelope points on $0 < \psi \leq \pi/2$. We have $F(\pi/2) > 0$, $F(\pi) \leq 0$, $F'(\psi) = -\dot{u} \sin \psi - \dot{r} \cos \psi < 0$ on $\pi/2 \leq \psi \leq \pi$. Thus on each integral curve there is exactly one contact point ψ^c with the envelope, and $\pi/2 < \psi^c < \pi$. The envelope is illustrated in Fig. 3.3.

3.5. Comparison Theorems

We wish to relate the volume $\mathscr{V}$ of a drop with the radius of the wetted disk and the boundary angle γ. General estimates can be obtained from the explicit formula (3.9) in conjunction with the bounds developed in Chapter 2 (see [57]). The following estimates, based on direct geometrical comparison, are simpler and yield results that are more precise for small drops. We set $r_\gamma = r(\gamma; u_0)$, $u_\gamma = u(\gamma; u_0)$, and observe that these quantities are related to u_0 and to the maximal radius R through the relations (2.7) among others.

Theorem 3.3 ([60]). *In the range* $0 < \gamma \leq \pi/2$, *there holds*

$$\mathscr{V}(\gamma) < \pi \left(\frac{r_\gamma}{\sin \gamma}\right)^3 \int_0^\gamma \sin^3 \theta \, d\theta. \tag{3.23a}$$

If $\pi/2 \leq \gamma \leq \pi$, *there holds*

$$\mathscr{V}(\gamma) < \pi R^3 \int_0^\gamma \sin^3 \theta \, d\theta. \tag{3.23b}$$

Theorem 3.4 ([60]). *In the range* $0 < \gamma \leq \pi/2$, *there holds*

$$\mathscr{V}(\gamma) > \pi \left(\frac{2}{\kappa u_\gamma}\right)^3 \int_0^\gamma \sin^3 \theta \, d\theta. \tag{3.24}$$

Proof of (3.23a). We introduce a lower circular arc v, centered on the z-axis, and tangent to the solution curve at (r_γ, u_γ). The latitudinal curvatures at (r_γ, u_γ) of the rotation surfaces defined by u, v then coincide and are equal to $k_l(\gamma) = \sin \gamma / r_\gamma$. We have however

$$2H = k_m(\gamma) + k_l(\gamma) = \kappa u_\gamma$$

and thus

$$k_m(\gamma) = \kappa u_\gamma - \frac{\sin \gamma}{r_\gamma} > \frac{\sin \gamma}{r_\gamma} = k_l(\gamma)$$

by (3.4b). Since on the arc v the planar curvature $k = \text{const.} = k_l(\gamma)$, we conclude there is an interval $\hat{r} < r < r_\gamma$, in which $u > v$. Were the two curves to intersect in $0 < r \leq \hat{r}$, we would be led, as in the proof of (2.30), to a contradiction. Thus $u > v$ in $0 < r < r_\gamma$ and it follows that the volume of the spherical cap exceeds $\mathscr{V}$. This statement is expressed by the inequality (3.23a).

Proof of (3.23b). We first prove

Lemma 3.3. *On the curve $u(\psi; u_0)$, k_m increases over the entire traverse $0 < \psi \leq \pi$; k_l increases until the vertical point $\psi = \pi/2$, and then decreases until $\psi = \pi$.*

The statement was proved for the range $0 < \psi \leq \pi/2$ in Chapter 2. We have, for all ψ in $(\pi/2, \pi)$,

$$\frac{d}{d\psi} k_l = \frac{d}{d\psi}\left(\frac{\sin\psi}{r}\right) = \frac{\cos\psi}{r} - \frac{\sin\psi}{r^2} r_\psi = \frac{\cos\psi}{r} \frac{\kappa r u - 2\sin\psi}{\kappa r u - \sin\psi} < 0$$

on $\pi/2 < \psi < \pi$ by (3.3a) and (3.5) (see the remarks following (3.7)). Since $du/d\psi > 0$ on $0 < \psi < \pi$, there follows $dk_m/d\psi > 0$ on $\pi/2 < \psi < \pi$, which completes the proof.

Corollary. *The entire trajectory of the solution curve lies interior to the closed semicircle determined by the arc v, with a single point of boundary tangency; see Fig.* 3.4.

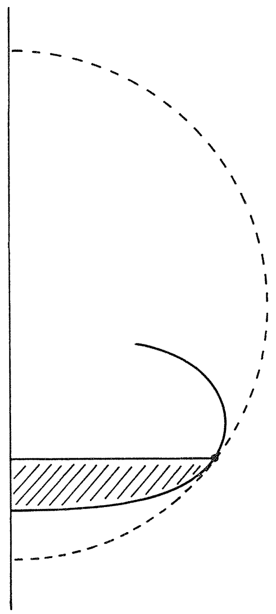


Figure 3.4. Inclusion property.

To prove (3.23b), we choose the point of contact to be (R, u_R), and denote by φ the inclination angle of the curve v, which we extend upwards. By Lemma 3.3, we have $\sin\varphi^+ > \sin\psi^+$ for each $r<R$, from which follows $v^+ > u^+$ at each fixed r, and $v(\gamma) > u(\gamma)$. The consequent inequality of volumes is now expressed by (3.23b).

Proof of (3.24). We now introduce the lower circular arc, parametrized in terms of its inclination angle φ,

$$r = \frac{2}{\kappa u_\gamma} \sin\varphi$$

$$v = u_0 + \frac{2}{\kappa u_\gamma}(1-\cos\varphi), \qquad 0 \le \varphi \le \frac{\pi}{2},$$

which passes through $(0, u_0)$ and has curvature $\kappa u_\gamma/2 = (\sin\varphi)_r = (1/2r)\,(r\sin\varphi)_r$. Using (3.6) we find

$$r(\sin\varphi - \sin\psi) = \kappa \int_0^r \rho(u_\gamma - u)\,d\rho > 0$$

if $u(r) < u_\gamma$; thus $v > u$ in the subinterval of $0 < r < r_\gamma$ in which v is defined (Fig. 3.5). If we can show $v(\gamma) < u_\gamma$, then the arc v will determine, up to the inclination angle $\varphi = \gamma$, a smaller volume than does u, which in turn would imply the inequalities (3.24).

By (2.45a) we have

$$\begin{aligned} u - u_0 &> \left[\frac{2}{\kappa}(1-\cos\gamma) + \frac{u_0^2}{4}\right]^{1/2} - \frac{u_0}{2} \\ &= \frac{1}{\left[\frac{2}{\kappa}(1-\cos\gamma) + \frac{u_0^2}{4}\right]^{1/2} + \frac{u_0}{2}} \cdot \frac{2}{\kappa}(1-\cos\gamma) \\ &> \frac{2}{\kappa u_\gamma}(1-\cos\gamma) = v - u_0 \end{aligned}$$

by a second use of (2.45a). Thus, the spherical segment up to the height $v(\gamma; u_0)$ yields a smaller volume than does the solution surface, as was to be shown.

The relation (3.23b) can be strengthened, as more is known about the curvature at (R, u_R) than was used in the derivation. We have, in fact

$$k_m(R) = \kappa u_R - \frac{1}{R} > \frac{1}{R}$$

by (3.4b).

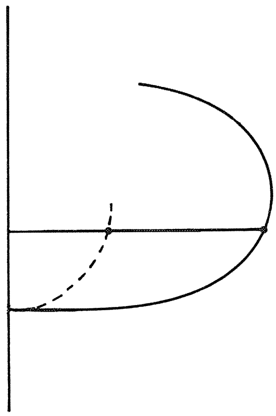


Figure 3.5. Comparison sphere.

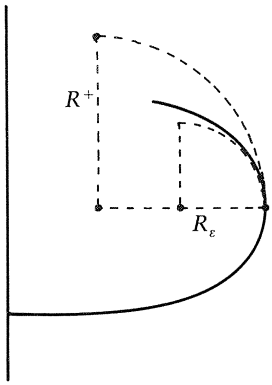


Figure 3.6. Comparison circles.

Setting

$$R^+ = \frac{R}{\kappa R u_R - 1} \tag{3.25}$$

and using Lemma 3.3, we conclude that the (upper) circular arc v^+ of radius R^+, tangent to the solution curve at (R, u_R) as in Fig. 3.6, lies everywhere above the solution. Further, the inclination angle φ^+ of v^+ satisfies

$$\sin\psi < \sin\varphi^+$$

at each fixed r on $a \le r \le R$. An examination of the geometry then leads immediately to the result:

In the range $\pi/2 \le \psi \le \pi$, there holds

$$\begin{aligned} \mathscr{V}(\gamma) < \frac{2\pi}{3} R^3 + \pi R^{+^3} \int_{\pi/2}^{\gamma} \sin^3\theta \, d\theta \\ + \pi R^+ (R - R^+)^2 \sin\left(\gamma - \frac{\pi}{2}\right) \\ + \pi R^{+^2} (R - R^+) \left(\gamma - \frac{\pi}{2} - \frac{1}{2}\sin 2\gamma\right). \end{aligned} \tag{3.23c}$$

In the defining relation (3.25) for R^+, u_R can be estimated by (2.57); the u_0 that appears in (2.57) must in turn be estimated from below. That can be accomplished with the aid of (2.22a) or (2.63). The former relation should be used below the "crossover point" $\kappa R^2 \sim 2{,}889$, the latter relation for larger values of κR^2, see the remarks following (2.63).

In connection with the material of the coming sections, (3.23c) leads to estimates that are marginally improved over those that follow from (3.23b).

We now ask the question: *suppose a small amount of liquid is removed symmetrically from a sessile drop. Does the resulting drop lie strictly interior to the original one?* We consider here only the case $0<\gamma\leq\pi/2$. We parametrize the solutions once more by u_0, and consider solutions $u=u(r;u_0)$ and $u^\delta=u(r;u_0+\delta)$, with $\delta>0$. Let $v^\delta(r;u_0)=u^\delta-\delta$; then $v^\delta(0;u_0)=u(0;u_0)$ and

$$(r\sin\psi^\delta)_r=\kappa r u^\delta$$

$$(r\sin\psi)_r=\kappa r u.$$

Thus

$$r(\sin\psi^\delta-\sin\psi)=\kappa\int_0^r \rho\cdot(u^\delta-u)\,d\rho,$$

from which we conclude $(u^\delta-u)$ is increasing in r. It follows that if the curve u^δ is moved rigidly downward a distance δ, it will lie above the curve u except at the single point $(0;u_0)$ of contact. To show the inclusion property corresponding to an angle γ, it thus suffices to show that for any $\delta>0$ there holds $v^\delta<u$ at the points where the angle γ is achieved. We have, for given γ,

$$(v^\delta-u)]_\gamma=\int_0^\delta(\dot{u}-1)\,du_0$$

with $\dot{u}=\partial/\partial u_0\; u(\gamma;u_0)$. Since $\dot{u}(0;u_0)=1$, there holds

$$\dot{u}=\int_0^\gamma\frac{d\dot{u}}{d\psi}\,d\psi+1;$$

it will thus suffice to show that $d\dot{u}/d\psi<0$ in $0<\psi<\gamma$, for all u_0 exceeding the given one. We shall determine sufficient conditions for that inequality.

We observe first, by (3.4),

$$\Delta\sim\frac{\kappa}{2}ru_0\quad\text{as } r\to 0$$

and thus, by (3.16) and (3.15),

$$\dot{r}\sim-\frac{1}{u_0}r\quad\text{as } r\to 0.$$

Writing (3.18) in the form

$$f(\psi)=\dot{r}\sin\psi+\kappa r^2\dot{u}=\kappa\left[\dot{r}ru+r^2\dot{u}+\int_0^r(\dot{u}-\dot{\rho}\tan\psi)\,\rho\,d\rho\right]$$

and using $u(r)\to u_0$, $\dot u(r)\to 1$, $\dot\rho(r)\to 0$, we find $f(\psi)\sim\frac{1}{2}\kappa r^2>0$, and thus there is an interval $\mathscr{I}_\delta$: $0<\psi<\delta$, in which, by (3.14), $d\dot u/d\psi<0$.

At a first zero, $\hat\psi$, of $d\dot u/d\psi$, there would hold $f(\hat\psi)=0$, $f'(\hat\psi)\le 0$. We calculate

$$f'(\psi)=\frac{\dot r\cos\psi}{\Delta}(\kappa r u-3\sin\psi). \tag{3.26}$$

We showed in §3.3 that $\dot r<0$ on $0<\psi\le\pi$. Hence the inclusion property will follow on $\mathscr{I}_\gamma$ whenever we can show that

$$g(\psi)\equiv\kappa r u-3\sin\psi<0 \tag{3.27}$$

on $\mathscr{I}_\gamma$, for the given surface and for all surfaces with smaller volume.

Theorem 3.5 ([57]). *Suppose* $0<\gamma\le\pi/2$, *and*

$$u_0^2>\frac{4}{3\kappa}(1-\cos\gamma)\frac{2+\cos\gamma}{1+\cos\gamma}. \tag{3.28}$$

Then every liquid drop with contact angle γ, *corresponding to an initial value* $\hat u_0>u_0$, *can be translated rigidly so as to lie strictly interior to the drop corresponding to* u_0.

Proof. It suffices to show that if u_0 satisfies (3.28), then $g(\psi;u_0)<0$ on $\mathscr{I}_\gamma$. From (3.28) and (3.4a) follow

$$\kappa r^2<\frac{4\sin^2\psi}{\kappa u_0^2}<3\frac{(1+\cos\psi)^2}{2+\cos\psi} \tag{3.29}$$

while from the bound (2.30) follows

$$\kappa r u<2\sin\psi+\kappa r^2\frac{\sin\psi}{3(1+\cos\psi)^2}(2+\cos\psi). \tag{3.30}$$

The result is obtained by placing (3.29) into (3.30).

From a physical point of view, it is desirable to have a criterion in terms of volume.

Corollary 3.5. *Suppose* $0<\gamma\le\pi/2$, *and suppose the volume* $\mathscr{V}$ *of a drop making contact angle* γ *satisfies*

$$\frac{\mathscr{V}}{\sin\gamma}<\pi\left(\frac{3}{\kappa}\right)^{3/2}\frac{(1+\cos\gamma)^3}{(5+4\cos\gamma)^{3/2}}\frac{1}{\sin^4\gamma}\int_0^\gamma\sin^3\theta\,d\theta. \tag{3.31}$$

Then every drop with smaller volume and making the same contact angle can be translated rigidly so as to lie strictly interior to the given one.

Proof. From (3.24), (2.45b), and Theorem 3.5.

We complement the above results by showing that when $\mathscr{V}/\sin\gamma$ is sufficiently large the inclusion property fails. Specifically, we have [57]:

Theorem 3.6. *There exists a universal constant* $\mathsf{V}<\infty$ *such that, to any drop for which* $\kappa^{3/2}\mathscr{V}/\sin\gamma>\mathsf{V}$, *there exists another drop with the same* γ *and larger* $\mathscr{V}$, *in which the given one cannot be enclosed.*

Proof. We consider the drops again in "capillary" representation, parametrized by u_0. Thus, the height of the drop is $q=u_\gamma-u_0$. From (3.23a) and (3.4a) we have

$$\left(\frac{2\sin\gamma}{\kappa u_0}\right)^3>r_\gamma^3>\frac{1}{\pi}\frac{\mathscr{V}}{\sin\gamma}\frac{\sin^4\gamma}{\int_0^\gamma \sin^3\theta\,d\theta} \tag{3.32}$$

so that, setting $\omega=\kappa^{3/2}\mathscr{V}/\sin\gamma$,

$$\lim_{\omega\to\infty}\sqrt{\kappa}\,r_\gamma=\infty,\qquad \lim_{\omega\to\infty}\sqrt{\kappa}\,u_0=0 \tag{3.33}$$

uniformly in γ. Hence from (2.52) and (2.57) we find there exists

$$\sqrt{\kappa}\,q_\infty=\lim_{\omega\to\infty}\sqrt{\kappa}\,q(\gamma;u_0)=\sqrt{2(1-\cos\gamma)} \tag{3.34}$$

uniformly in γ, $0<\gamma\le\pi/2$. It will thus suffice to show that when $\kappa^{3/2}\mathscr{V}/\sin\gamma$ is sufficiently large, there holds $q(\gamma)>q_\infty$. From (2.57) we now obtain

$$u_\gamma>\sqrt{\frac{p+1}{p}}\sqrt{\frac{2}{\kappa}(1-\cos\gamma)}. \tag{3.35}$$

We have, with $k=\cos\frac{1}{2}\gamma$, $\tau=\frac{\sqrt{\kappa}\,r}{k}$, $p=\sqrt{1+\tau^2}$,

$$\frac{p+1}{p}=1+\frac{1}{p}>1+\frac{1}{1+\tau},$$

and thus

$$\begin{aligned}\sqrt{\frac{p+1}{p}}&>1+\frac{1}{2}\frac{1}{1+\tau}-\frac{1}{8}\left(\frac{1}{1+\tau}\right)^2\\&>1+\frac{1}{2}\tau^{-1}(1-\tau^{-1})-\frac{1}{8}\tau^{-2}\end{aligned}$$

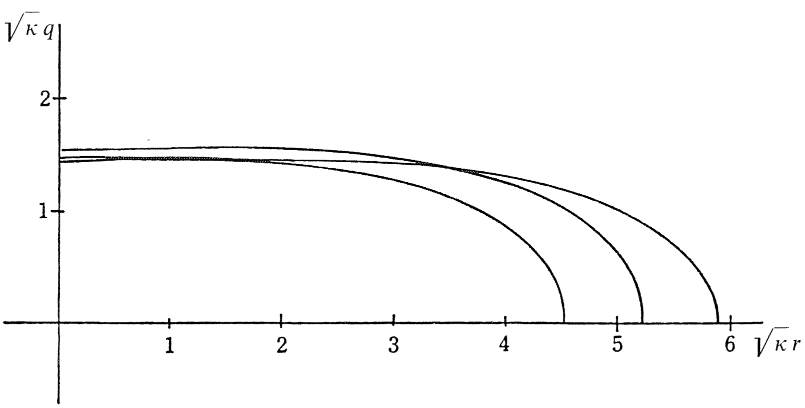


Figure 3.7. Nonmonotonicity of height, $\gamma=\pi/2$.

so that

$$\sqrt{\frac{p+1}{p}}>1+\frac{1}{2}\frac{k}{\sqrt{\kappa}\,r}-\frac{5}{8}\left(\frac{k}{\sqrt{\kappa}\,r}\right)^2. \tag{3.36}$$

Thus we find from (3.34), (3.35), and (3.36)

$$\sqrt{\kappa}\,q(\gamma)>\sqrt{\kappa}\,q_\infty+\left(\frac{1}{2\sqrt{\kappa}\,r}-\frac{\sqrt{\kappa}\,u_0}{\sin\gamma}\right)\sin\gamma-\frac{5k}{8\kappa r^2}\sin\gamma. \tag{3.37}$$

Finally, from (2.64),

$$\frac{\sqrt{\kappa}\,u_0}{\sin\gamma}<A(\sqrt{\kappa}\,r)^{1/2}\,e^{-\sqrt{\kappa}r} \tag{3.38}$$

for a fixed constant A and all $\sqrt{\kappa}\,r$ sufficiently large. The result now follows immediately from (3.37) and (3.38).

Figure 3.7 shows the result of computer calculations of three drop sections, with $\gamma=\pi/2$. The calculations were performed by Heidi Bjørstad.

3.6. Geometry of the Sessile Drop; Small Drops

Since the sessile drop is determined uniquely by $\mathscr{V}$ and γ, there must be a relation connecting $\mathscr{V}$, γ, and $a=$radius of wetted disk. We consider first the case of small drops, and we show that, for any fixed γ in $0<\gamma<\pi$, the (normalized) free surface tends uniformly as $\mathscr{V}\to 0$, together with its sectional curvatures, to a spherical cap. If $\gamma=\pi$, the uniformity still holds for the surface and its unit normals, but fails for the curvature.

The case $\gamma=\pi$ is singular also in other respects. Although they provide some information, the bounds we have given in preceding sections do not suffice for our purpose here. If $\gamma\leq\pi/2$, the results we shall obtain are a special case of preceding results. We therefore consider the case $\pi/2<\gamma\leq\pi$, for which, in the "capillary" representation, the drop has the appearance indicated in Fig. 3.2.

We intend to compare, for fixed r in $a<r<R$, the meridional curvatures k_m^+, k_m^- on the upper and lower parts of the curve. Since estimates on k_m^- have already been given in Chapter 2, corresponding estimates will follow for k_m^+.

On each vertical section we have

$$(r\sin\psi)_r=\kappa r u;$$

hence

$$r(\sin\psi^- - \sin\psi^+)=\kappa\int_r^R (u^+ - u^-)\,\rho\,d\rho. \tag{3.39}$$

From (3.39) we find, as in §3.3, that $u^+>u^-$, $\sin\psi^->\sin\psi^+$ in the entire range $a\leq r<R$. As was shown in §3.5, the entire configuration lies interior to a ball of radius R, as indicated in Figs. 3.3 and 3.5. Thus, $u^+-u^-<2\sqrt{R^2-r^2}$, and hence

$$\begin{aligned} 0<k_l^- - k_l^+ \equiv \frac{\sin\psi^-}{r}-\frac{\sin\psi^+}{r} &< \frac{2\kappa}{r^2}\int_r^R \sqrt{R^2-\rho^2}\,\rho\,d\rho \\ &= \frac{2\kappa}{3}\,\frac{(R^2-r^2)^{3/2}}{r^2}. \end{aligned} \tag{3.40}$$

Since $k_m+k_l=\kappa u$, we obtain now

$$0<k_m^+ - k_m^- < 2\kappa\sqrt{R^2-r^2}+\frac{2\kappa}{3}\,\frac{(R^2-r^2)^{3/2}}{r^2}. \tag{3.41}$$

Set $\lambda=r/R$, $B_R=\kappa R^2$. Given $\varepsilon>0$, we see from (3.41) that, in the range $a\leq r\leq R$, there will hold

$$0\leq k_m^+ - k_m^- \leq \varepsilon/R \tag{3.42}$$

whenever

$$\lambda^2\geq\frac{2\sqrt{2}}{3}\,\frac{B_R}{\varepsilon}. \tag{3.43}$$

When these conditions are satisfied, we find

$$k_m^+(r)\leq\frac{\varepsilon}{R}+k_m^-(r)<\frac{\varepsilon}{R}+k_m(R) \tag{3.44}$$

by Lemma 3.3. But

$$k_m(R)+\frac{1}{R}=\kappa u_R, \tag{3.45}$$

and by (2.30)

$$\kappa u_R<\frac{2}{R}+\frac{2}{3}\kappa R.$$

Thus, writing

$$R_\varepsilon=\frac{1}{1+\varepsilon+\frac{2}{3}\kappa R^2}R \tag{3.46}$$

we find

$$k_m^+<\frac{1}{R_\varepsilon} \tag{3.47}$$

on the portion of $a\le r\le R$ for which (3.43) holds. It follows that if we can choose λ such that (3.43) and

$$\lambda R\le R-R_\varepsilon(1-\sin\gamma) \tag{3.48}$$

can hold simultaneously, the circular arc $v_\varepsilon(r)$ of radius R_ε that contacts the solution curve at (R, u_R) as shown in Fig. 3.6 will lie under the upper part u^+, at least until the inclination angle φ_ε of $v_\varepsilon(r)$ increases to γ; further, at all such r there holds

$$\sin\varphi_\varepsilon<\sin\psi^+. \tag{3.49}$$

It follows that the radius $a_\gamma=r(\gamma)$ of a wetted disk satisfies

$$a_\gamma<R-R_\varepsilon(1-\sin\gamma) \tag{3.50a}$$

and consequently

$$u_\gamma=u(\gamma)>u_R-R_\varepsilon\cos\gamma. \tag{3.50b}$$

Setting $t=\sqrt{\frac{2}{3}B_R}$, one sees easily that the requirements are equivalent to

$$2^{1/4}\frac{1}{\sqrt{\varepsilon}}t\le\frac{\sin\gamma+\varepsilon+t^2}{1+\varepsilon+t^2}.$$

For technical reasons, it is convenient to choose for ε the unique value for which

$$2^{1/4}\frac{1}{\sqrt{\varepsilon}}t=\frac{\sin\gamma+\varepsilon+t^2}{1+\varepsilon+t^2}. \tag{3.51}$$

We note that the choice (3.51) implies

$$\frac{\sin\gamma+\varepsilon}{1+\varepsilon}\varepsilon^{1/2}<2^{1/4}t<(\sin\gamma+\varepsilon)\varepsilon^{1/2} \tag{3.52}$$

and thus the asymptotic relations, as $R\to 0$,

$$\varepsilon\sim\sqrt{2}\left(\frac{\kappa}{3}\right)^{1/3}R^{2/3} \tag{3.53a}$$

if $\gamma=\pi$, and

$$\varepsilon\sim\frac{\sqrt{2}}{\sin^2\gamma}\frac{2}{3}\kappa R^2 \tag{3.53b}$$

if $\gamma\neq\pi$.

We have proved:

Lemma 3.4. *If* ε *is chosen to satisfy* (3.51), *then* (3.47) *holds when* $\psi>\pi/2$, *for all* r *in the interval* $R-R_\varepsilon(1-\sin\gamma)\le r\le R$. *There hold also* (3.49) *and* (3.50a, b). *In particular, if* $\gamma=\pi$, *we find* $a<R-R_\varepsilon$.

Finally we observe, as in the proof of Theorem 3.3, that the entire configuration lies interior to a semicircle of radius R, tangent at (R, u_R) to the solution curve. Since k_m increases, and in consideration of the geometrical meanings of (3.49) and (3.50a, b), we may write:

Lemma 3.5. *There hold, in* $\pi/2\le\psi\le\pi$, $\pi/2\le\gamma\le\pi$,

$$R^{-1}<k_m^+<R_\varepsilon^{-1} \tag{3.54a}$$

$$\frac{r-(R-R_\varepsilon)}{R_\varepsilon}<\sin\psi^+<\frac{r}{R} \tag{3.54b}$$

$$R\sin\gamma<a_\gamma<R-R_\varepsilon(1-\sin\gamma) \tag{3.54c}$$

$$u_R+\sqrt{R_\varepsilon^2-(r-R+R_\varepsilon)^2}<u^+<u_R+\sqrt{R^2-r^2}. \tag{3.54d}$$

We note the left side of (3.54c) provides information only if $\gamma<\pi$. That should not be surprising in view of the following remarks.

At the point of symmetry $\psi=0$, there holds

$$k_m(0)=k_l(0)=\frac{1}{2}\kappa u_0>\frac{1}{R}-\frac{\kappa}{6}R$$

by (2.22a). At the vertical point there holds $k_l=R^{-1}$, $k_m=\kappa u_R-R^{-1}<R^{-1}+\frac{2}{3}\kappa R$ by (2.30). Since k_m is increasing, we find

$$-\frac{1}{6}\kappa R<k_m-\frac{1}{R}<\frac{2}{3}\kappa R$$

uniformly in the initial range $0 \leq \psi \leq \pi/2$. Further, Theorems 3.3 and 3.4, together with (2.30), yield the inequalities

$$\frac{R}{1+\frac{2}{3}\kappa R^2} < \left(\frac{3}{2}\frac{\mathscr{V}_R}{\pi}\right)^{1/3} < R.$$

Thus, uniformly on $0 \leq \psi \leq \pi/2$ we have

$$\lim_{\mathscr{V}_R \to 0} R\, k_m(\psi) = 1.$$

It follows that, under the normalization given by the transformation $u = UR + u_R$, $r = \rho R$, the initial arcs converge uniformly as $\mathscr{V}_R \to 0$, together with their unit normals and curvatures, to a unit quarter circle (and the corresponding free surface to a lower unit hemisphere). Lemma 3.4 shows that the same uniformity extends on the upper arc at least back to the value $r = R - R_\varepsilon$ in the original coordinates, or – equivalently – to the value $\rho = \sqrt{2}(\frac{1}{3}\kappa)^{1/3} R^{2/3}$, which tends to zero. It follows that given any angle $\gamma < \pi$, the convergence to spherical shape as $\mathscr{V}_\gamma \to 0$ is uniform up to angle γ. *The uniformity cannot however extend to the angle* π. To see that, we need only note that $k_l(\pi) = 0$ on any solution, hence $k_m(\pi) = \kappa u(\pi)$, whereas $k_m(\gamma) \sim \frac{1}{2}\kappa u(\gamma) \sim \frac{1}{2}\kappa u(\pi)$ for any $\gamma < \pi$.

We return to this point in Theorem 3.10 with the aid of an explicit formula for the volume. At present, we note that Lemma 3.4 enables us to extend Theorem 3.4 to angles $\gamma > \pi/2$:

Theorem 3.7. *In the range* $\pi/2 \leq \gamma \leq \pi$ *the volume* $\mathscr{V}$ *of a drop satisfies*

$$\begin{aligned}\mathscr{V} > {} & \frac{2\pi}{3}\left(\frac{2}{\kappa u_R}\right)^3 + \pi R_\varepsilon^3 \int_{\pi/2}^{\gamma} \sin^3\theta\, d\theta \\ & + \pi R_\varepsilon (R - R_\varepsilon)^2 \sin\left(\gamma - \frac{\pi}{2}\right) \\ & + \pi (R - R_\varepsilon) R_\varepsilon^2 \left(\gamma - \frac{\pi}{2} - \frac{1}{2}\sin 2\gamma\right).\end{aligned} \tag{3.55}$$

Here ε *and* R_ε *are determined by* (3.51) *and* (3.47); u_R *can be estimated by* (2.30), (2.51), *or* (2.52).

Proof. By (3.49) and (3.50) and by the choice of γ, $\mathscr{V}$ satisfies

$$\mathscr{V} > \mathscr{V}_R + \pi \int_0^{R_\varepsilon \sin(\gamma - \pi/2)} \left(R - R_\varepsilon + \sqrt{R_\varepsilon^2 - h^2}\right)^2 dh, \tag{3.56}$$

where $\mathscr{V}_R$ is the volume below the point (R, u_R). The result thus follows from Theorem (3.4) and a formal integration in (3.56).

Lemma 3.6. *In the "capillary" representation, there holds, in the range* $\pi/2 \leq \psi \leq \pi$,

$$
\begin{aligned}
u_R - R_\varepsilon \cos\psi < u(\psi) &< \frac{1}{\kappa R} + \left\{\frac{2}{\kappa}(1-\cos\psi) - \frac{1}{\kappa^2 R^2} + \frac{u_0^2}{2}\right\}^{1/2} \\
&< \frac{1}{\kappa R} + \left\{\frac{2}{\kappa}(1-\cos\psi) + \frac{1}{\kappa^2 R^2}\right\}^{1/2}.
\end{aligned}
\tag{3.57a, b, c}
$$

Proof. According to the above construction, we have, for any r in $a \leq r \leq R$,

$$u > u_R + R_\varepsilon \sin\left(\varphi_\varepsilon - \frac{\pi}{2}\right). \tag{3.58}$$

By (3.49) we have $\varphi_\varepsilon > \psi$, hence $\sin(\varphi_\varepsilon - \pi/2) > -\cos\psi$, which establishes the left side of (3.57). To prove the remaining estimates, we place Lemma 3.3 (applied to $k_l = r^{-1}\sin\psi$) and the estimate (2.7b) into (2.43b) to obtain

$$\left(\kappa u - \frac{1}{R}\right) du < \sin\psi\, d\psi \tag{3.59}$$

and thus, integrating from $\pi/2$ to ψ,

$$u < \frac{1}{\kappa R} + \sqrt{\frac{-2}{\kappa}\cos\psi + \left(u_R - \frac{1}{\kappa R}\right)^2}, \tag{3.60}$$

so that the result follows from (2.51) and (2.7a).

We note that the estimates on the right of (3.57) are exact asymptotically both for small and large R. For small R the second of these estimates can be improved somewhat by using (2.30) instead of (2.51) as a bound for u_R.

Theorem 3.8. *In the range* $\pi/2 < \gamma \leq \pi$ *the height* q *of a drop satisfies*

$$q > \sqrt{\frac{2}{\kappa}}\sqrt{\frac{p-1}{p+1}} - R_\varepsilon \cos\gamma, \qquad p = \sqrt{1 + 2\kappa R^2}. \tag{3.61}$$

Here ε *and* R_ε *are determined as functions of* R *by* (3.51) *and* (3.47) *and* R *can be estimated in terms of* $\mathscr{V}$ *by* (3.23b).

Proof. According to the above construction we have, for any r in $a \leq r \leq R$,

$$q > u_R - u_0 + R_\varepsilon \sin\left(\varphi_\varepsilon - \frac{\pi}{2}\right). \tag{3.62}$$

By (3.49) we have $\varphi_\varepsilon > \psi$, hence $\sin(\varphi_\varepsilon - \pi/2) > -\cos\psi$. By (2.57)

$$u_R - u_0 > \sqrt{\frac{p+1}{p}} \left\{ \frac{2}{\kappa} + \frac{p+1}{4} u_0^2 \right\}^{1/2} - u_0. \tag{3.63}$$

As in §2.6, the right side of (3.63) is decreasing in u_0 in the range $0 < u_0 < 2/\kappa R$; hence, by (3.4a),

$$u_R - u_0 > \sqrt{\frac{p+1}{p}} \sqrt{\frac{2}{\kappa} + \frac{p+1}{4} \frac{4}{\kappa^2 R^2}} - \frac{2}{\kappa R} = \sqrt{\frac{2}{\kappa}} \sqrt{\frac{p-1}{p+1}},$$

which completes the proof.

We remark that a sharper (if less explicit) estimate can be obtained by using the bound $u_0 < u_0^+$ of (2.22b) in place of (2.7), as above. This was done in the preparation of Fig. 3.8 (see below).

Theorem 3.9. *In the range* $\pi/2 < \gamma \le \pi$, $\kappa R^2 \le 6$, *there holds*

$$q < \left\{ \frac{2}{\kappa}(1 - \cos\gamma) + \frac{1}{\kappa^2 R^2} - \frac{2}{3\kappa} + \frac{1}{18} R^2 \right\}^{1/2} - \frac{1}{\kappa R} + \frac{R}{3}. \tag{3.64}$$

If $\kappa R^2 > 6$, *then*

$$q < \frac{1}{\kappa R} + \sqrt{\frac{2}{\kappa}(1 - \cos\gamma) - \frac{1}{\kappa^2 R^2}}. \tag{3.65}$$

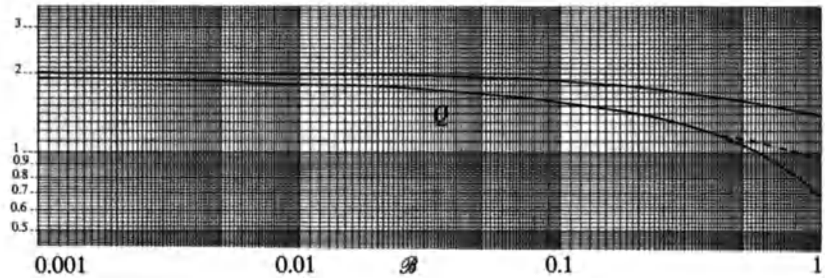
The right side of (3.64) *is increasing in R, and thus* (3.64) *continues to hold if R is estimated by* (3.55). *The right side of* (3.65) *decreases in R when* $\kappa R^2 > (1 - \cos\gamma)^{-1}$, *and hence R can be estimated by* (3.23b) *in that range.*

Proof. From Lemma 3.6 we have

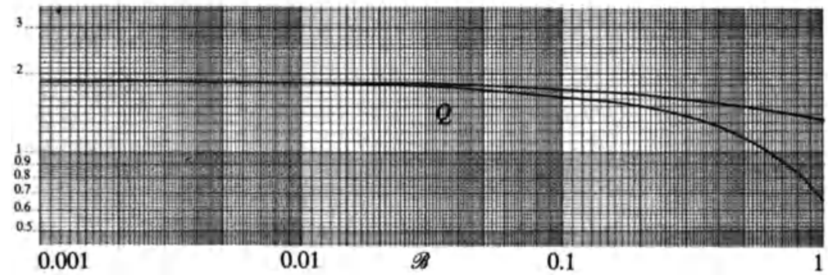
$$u - u_0 < \frac{1}{\kappa R} + \left\{ \frac{2}{\kappa}(1 - \cos\gamma) - \frac{1}{\kappa^2 R^2} + \frac{u_0^2}{2} \right\}^{1/2} - u_0. \tag{3.66}$$

As in §2.6 we find that the right side of (3.66) decreases in u_0; thus we may apply (2.22a) to obtain (3.64) whenever $\mathscr{L}(B;\gamma)$ is nonnegative. If $\mathscr{L}(B;\gamma) < 0$ we set $u_0 = 0$ in (3.66) to obtain (3.65). Formal differentiation establishes the stated monotonicity properties.

Figures 3.8 (a) and (b) show upper and lower bounds for q as function of $\mathscr{V}$, for small drops, for the cases $\gamma = \pi$ and $\gamma = 5\pi/6$. The figures are plotted in terms of the nondimensional variables $Q = q/R$ and $\mathscr{B} = \kappa \mathscr{P}^2$, where $\mathscr{P}$ is the radius of the ball of equivalent volume, $4\pi\mathscr{P}^3 = 3\mathscr{V}$. The greatly improved accuracy in the second case for small $\mathscr{B}$ reflects the



(a)



(b)

Figure 3.8. Drop height, small drops. (a) $\gamma=\pi$. (b) $\gamma=\frac{5}{6}\pi$.

nonuniformity at $\gamma=\pi$ in convergence of ε to zero (see (3.53a, b)). It should be pointed out that in the expressions

$$Q>\sqrt{\frac{2}{B_R}\frac{p-1}{p+1}}-\sqrt{\frac{B_{R_\varepsilon}}{B_R}}\cos\gamma \tag{3.61a}$$

$$Q<\left\{\frac{2}{B_R}(1-\cos\gamma)+\frac{1}{B_R^2}-\frac{2}{3B_R}+\frac{1}{18}\right\}^{1/2}-\frac{1}{B_R}+\frac{1}{3}, \tag{3.65a}$$

corresponding to (3.61) and (3.65), the sense of monotonicity in R is the reverse of what occurs in the original (dimensional) inequalities. Thus, when expressing the results in terms of $\mathscr{B}$, (3.23b) (instead of (3.55)) must be used in (3.65a) and (3.55) (instead of (3.23b)) must be used in (3.61a).

We recall that in the case $0<\gamma\leqq\pi/2$, upper and lower bounds for q were given in Chapter 2.

We now wish to estimate the radius a_γ of a wetted surface in terms of volume and contact angle. We present the discussion in terms of the non-dimensional variables $B_R=\kappa R^2$, $\mathscr{B}=\kappa\mathscr{P}^2$, and $B=\kappa a_\gamma^2$. The formula (3.9) for the volume of a drop now takes the form

$$\tfrac{4}{3}\mathscr{B}^{3/2}=B\sqrt{\kappa}\,u-2\sqrt{B}\sin\gamma. \tag{3.67}$$

We have from Lemma 3.6

$$u > u_R - R_\varepsilon \cos\gamma > \frac{2}{\kappa R} - R_\varepsilon \cos\gamma \tag{3.68}$$

by (3.4b), and thus

$$\tfrac{4}{3}\mathscr{B}^{3/2} > \frac{2B}{\sqrt{B_R}} - \frac{B\sqrt{B_R}\cos\gamma}{\varepsilon+1+B_R} - 2\sqrt{B}\sin\gamma. \tag{3.69}$$

A disagreeable but formal calculation, using that ε as defined by (3.51) satisfies $\varepsilon'(t)>0$, shows that the right side of (3.69) decreases in B_R. From Theorem 3.7 we may estimate $\sqrt{B_R} < F(\mathscr{B})$. We have proved:

Theorem 3.10. *The diameter of the wetted disk is bounded above in terms of the drop volume by the relation*

$$\tfrac{4}{3}\mathscr{B}^{3/2} > \frac{2B}{F(\mathscr{B})} - \frac{BF(\mathscr{B})\cos\gamma}{\varepsilon+1+F^2(\mathscr{B})} - 2\sqrt{B}\sin\gamma. \tag{3.70}$$

Here ε is to be determined by (3.51).

We remark that

$$F(\mathscr{B}) \sim \left(\frac{4}{\cos^3\gamma - 3\cos\gamma + 2}\right)^{1/3} \mathscr{B}^{1/2} \tag{3.71}$$

as $\mathscr{B}\to 0$; thus, from (3.70)

$$B < \left(\frac{4\sin^3\gamma}{\cos^3\gamma - 3\cos\gamma + 2}\right)^{2/3} \mathscr{B} + o(\mathscr{B}) \tag{3.72}$$

when $\gamma \neq \pi$. If $\gamma = \pi$, then

$$B < \tfrac{2}{3}\mathscr{B}^2 + o(\mathscr{B}^2), \tag{3.73}$$

thus indicating a nonuniformity in the rate at which the wetted disk contracts to zero.

To obtain an estimate in the other direction, we return to (3.67) and apply Lemma 3.6 to obtain

$$\tfrac{4}{3}\mathscr{B}^{3/2} < \frac{B}{\sqrt{B_R}}\left[1+\sqrt{1+2B_R(1-\cos\gamma)}\right] - 2\sqrt{B}\sin\gamma. \tag{3.74}$$

We may use Theorem 3.3 to estimate B_R in terms of $\mathscr{B}$; setting

$$\tfrac{4}{3}\mathfrak{s}^3(\gamma) = \int_0^\gamma \sin^3\theta\, d\theta \tag{3.75}$$

we obtain $B < B_R\,\mathfrak{s}(\gamma)$, and hence from (3.74):

Theorem 3.11. *The diameter of the wetted disk is bounded below in terms of the drop volume by*

$$\tfrac{4}{3}\mathscr{B}^2 < B[\mathfrak{s}(\gamma)+\sqrt{\mathfrak{s}^2(\gamma)+2\mathscr{B}(1-\cos\gamma)}]-2\sqrt{B\mathscr{B}}\sin\gamma. \tag{3.76}$$

From (3.76) we find immediately the asymptotic relations, as $\mathscr{B}\to 0$,

$$B > \frac{\sin^2(\gamma)}{\mathfrak{s}^2(\gamma)}\mathscr{B}+o(\mathscr{B}) \tag{3.77}$$

when $\gamma \neq \pi$, and

$$B > \tfrac{2}{3}\mathscr{B}^2 + o(\mathscr{B}^2) \tag{3.78}$$

when $\gamma=\pi$. In conjunction with (3.72) and (3.73) we thus have

Theorem 3.12 ([60]). *Asymptotically as $\mathscr{B}\to 0$, there holds*

$$B \sim \frac{\sin^2\gamma}{\mathfrak{s}^2(\gamma)}\mathscr{B} \tag{3.79a}$$

if $\gamma \neq \pi$, and

$$B \sim \tfrac{2}{3}\mathscr{B}^2 \tag{3.79b}$$

if $\gamma=\pi$.

These estimates confirm the nonuniformity in the rate of contraction of the wetted disk, which was indicated above.

The above results are illustrated graphically in Fig. 3.9. Note the striking effect of the change in γ from π to $\frac{5}{6}\pi$, so that a factor of ten was re-

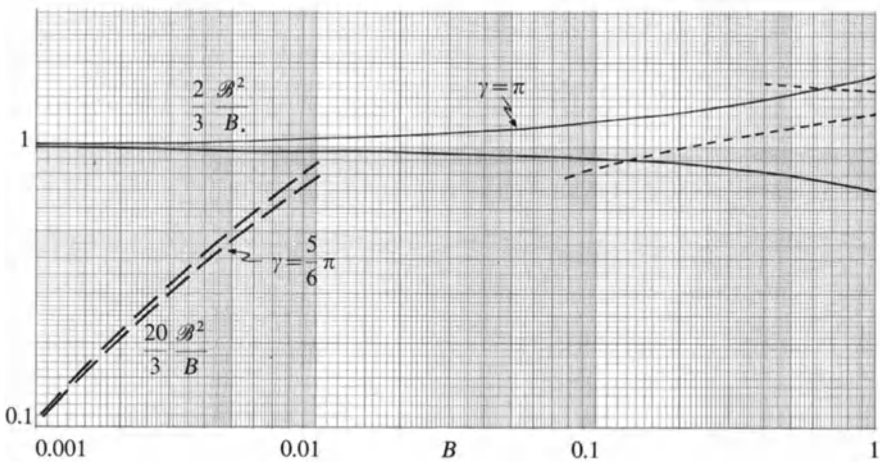


Figure 3.9. Volume-wetted area relationship; small drops.

quired to keep the function corresponding to the latter value on scale. In this respect see also §8.7.

In the special case $\gamma = \pi$, the relation (3.76) can be simplified (see §8.6).

It is of interest to note that the estimates (3.73), (3.78), and (3.79) refer to wetted surface radii in the order of R^2, even though the curvature estimates used in the derivations were justified only for radius values of order larger than $R^{4/3}$. That was possible because the curvature estimates were used only at a subsidiary point in the derivations, where information at smaller values of r was not required.

3.7. Geometry of the Sessile Drop; Larger Drops

Although the above results are correct in all cases, it is preferable to adopt other methods in the case of large drops. The crucial initial step is an estimate for the radius a of wetted surface in terms of outer radius R; that estimate also has an independent interest, although it does not seem to have been studied previous to the paper of Finn [57] where initial (inexact) estimates appear.

Our starting point is the relation

$$\frac{dr}{d\psi} = \frac{\cos\psi}{\kappa u - \dfrac{\sin\psi}{r}}, \tag{3.3a}$$

in which we seek to estimate the denominator on the right. We observe first from Lemma (3.6)

$$\begin{aligned} \sqrt{\kappa}\, u - \frac{\sin\psi}{\sqrt{\kappa}\, r} &< \sqrt{2(1-\cos\psi) - \frac{1}{B_R} + \frac{\kappa u_0^2}{2}} + \frac{1}{\sqrt{\kappa} R} - \frac{\sqrt{1-\cos^2\psi}}{\sqrt{\kappa}\, r} \\ &< \sqrt{2(1-\cos\psi)}\left\{\sqrt{1+\lambda\left(\kappa\frac{u_0^2}{2} - \frac{1}{B_R}\right)} + \frac{1-\sqrt{1+\cos\gamma}}{\sqrt{2B_R}}\right\} \\ &= \sqrt{2(1-\cos\psi)}\, P \end{aligned} \tag{3.80a}$$

where

$$\lambda = \begin{cases} \dfrac{1}{2} & \text{if } \kappa u_0^2 > \dfrac{2}{B_R} \\[2ex] \dfrac{1}{2(1-\cos\gamma)} & \text{if } \kappa u_0^2 < \dfrac{2}{B_R}. \end{cases} \tag{3.81}$$

Alternatively, one may write

$$\sqrt{\kappa}\,u-\frac{\sin\psi}{\sqrt{\kappa}\,r}<\sqrt{2(1-\cos\psi)}\left\{\sqrt{1+\frac{1}{2}\frac{1}{B_R}}+\frac{1-\sqrt{1+\cos\gamma}}{\sqrt{2B_R}}\right\}$$
$$=\sqrt{2(1-\cos\psi)}\,P, \tag{3.80b}$$

which is less precise but may be simpler to apply.

Lemma 3.7. *On the interval $\pi/2\le\psi\le\pi$ there holds*

$$u(\psi)>\frac{\sin\psi}{\kappa r}+\left\{-\frac{2}{\kappa}\cos\psi+\left(u_R-\frac{\sin\psi}{\kappa r}\right)^2\right\}^{1/2}. \tag{3.82}$$

Proof. Use Lemma 3.3 (applied to k_l) in (2.43b), and integrate from $\pi/2$ to ψ.

From (3.4b), Lemma 3.3, and Lemma 3.7 we find

$$u(\psi)>\frac{\sin\psi}{\kappa r}+\left\{-\frac{2}{\kappa}\cos\psi+\left(u_R-\frac{1}{\kappa R}\right)^2\right\}^{1/2}. \tag{3.83}$$

From (3.4b) and (2.45b) we have

$$\frac{2}{\kappa R}<u_R<\sqrt{\frac{4}{\kappa}+u_0^2}. \tag{3.84}$$

From these relations and from Theorem 2.6 we conclude that

$$\kappa u_R^2>\frac{p+1}{p}\left(2+\frac{p+1}{4}\kappa u_0^2\right)>\frac{2}{\kappa R^2},$$

with $p=\sqrt{1+2\kappa R^2}$, and thus from (3.83) and (2.57)

$$\sqrt{\kappa}\,u-\frac{\sin\psi}{\sqrt{\kappa}\,r}>\sqrt{2(1-\cos\psi)+(A^2-2)} \tag{3.85}$$

with

$$A=\sqrt{\frac{p+1}{p}}\sqrt{2+\frac{p+1}{4}\kappa u_0^2}-\frac{1}{\sqrt{\kappa}\,R}. \tag{3.86}$$

We may thus write, on the interval $\pi/2\le\psi\le\gamma$,

$$\sqrt{\kappa}\,u-\frac{\sin\psi}{\sqrt{\kappa}\,r}>\sqrt{2(1-\cos\psi)}\sqrt{1+\lambda(A^2-2)}$$
$$=\sqrt{2(1-\cos\psi)}\,Q \tag{3.87}$$

where

$$\lambda = \begin{cases} \dfrac{1}{2(1-\cos\gamma)} & \text{if } A^2 > 2 \\ \dfrac{1}{2} & \text{if } A^2 < 2. \end{cases} \tag{3.88}$$

The inequality (3.87) is to be understood in the sense that u_0 can be replaced by any lower bound for which $A > 0$ continues to hold. In this respect both the bounds (2.60) and (2.63) are admissible.

We now place the above estimates into (2.43a) and integrate between $\pi/2$ and γ. We obtain [60]:

Theorem 3.13. *In the interval $\pi/2 \le \gamma \le \pi$, there holds*

$$\begin{aligned} &Q\sqrt{\kappa}(R-a_\gamma) \\ &< \sqrt{2} - \log(1+\sqrt{2}) - 2\cos\frac{\gamma}{2} + \log\frac{1+\cos\gamma/2}{\sin\dfrac{\gamma}{2}} \\ &< P\sqrt{\kappa}(R-a_\gamma). \end{aligned} \tag{3.89a, b}$$

In particular

$$\lim_{R\to\infty} \sqrt{\kappa}(R-a_\gamma) = \sqrt{2} - \log(1+\sqrt{2}) - 2\cos\frac{\gamma}{2} + \log\frac{1+\cos\dfrac{\gamma}{2}}{\sin\dfrac{\gamma}{2}}. \tag{3.90}$$

These relations take a particularly simple form when $\gamma = \pi$. We have then

$$\lim_{R\to\infty} \sqrt{\kappa}(R-a) = \sqrt{2} - \log(1+\sqrt{2}). \tag{3.91}$$

The estimate of Theorem 3.13 is illustrated in Fig. 3.10.

An estimate above for the height q of the drop is obtained immediately from (3.66), together with either of the estimates (2.60) or (2.63). To bound q below, we may write

$$q = u - u_0 = (u - u_R) + (u_R - u_0) > f(u_R) + g(u_0)$$

with

$$f(u_R) = -\left(u_R - \frac{\sin\gamma}{\kappa a_\gamma}\right) + \sqrt{-\frac{2}{\kappa}\cos\gamma + \left(u_R - \frac{\sin\gamma}{\kappa a_\gamma}\right)^2} \tag{3.92}$$

(Lemma 3.7) and

$$g(u_0) = \sqrt{\frac{p+1}{p}}\sqrt{\frac{2}{\kappa} + \frac{p+1}{4}u_0^2} - u_0 \tag{3.93}$$

from (2.57).

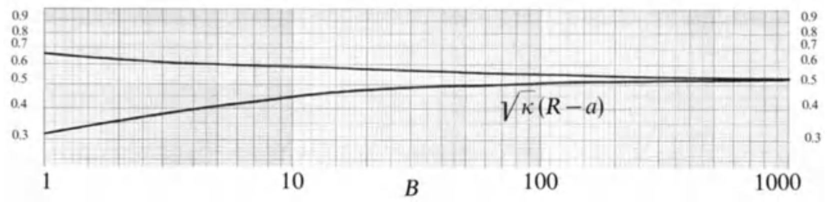


Figure 3.10. Drop overhang; $\gamma=\pi$.

We verify directly that f decreases in u_R and in a_γ. As in §2.6 we find that g decreases in u_0, in the range $u_0<4/\kappa R$. We have found:

Theorem 3.14. *In the interval* $\pi/2\leq\gamma\leq\pi$, *there holds*

$$q<\frac{1}{\kappa R}+\left\{\frac{2}{\kappa}(1-\cos\gamma)-\frac{1}{\kappa^2R^2}+\frac{u_0^2}{2}\right\}^{1/2}-u_0 \tag{3.94}$$

where u_0 *can be estimated by* (2.60) *or* (2.63). *There holds also*

$$\begin{aligned} q> &-\left(u_R-\frac{\sin\gamma}{\kappa a_\gamma}\right)+\sqrt{-\frac{2}{\kappa}\cos\gamma+\left(u_R-\frac{\sin\gamma}{\kappa a_\gamma}\right)^2} \\ &+\sqrt{\frac{p+1}{p}}\sqrt{\frac{2}{\kappa}+\frac{p+1}{p}u_0^2}-u_0 \end{aligned} \tag{3.95}$$

with $p=\sqrt{1+2\kappa R^2}$. *Here* u_R *can be estimated from* (2.51) *or* (2.52), a_γ *can be estimated by* (3.89b), *and* u_0 *can be estimated by* (3.4a), *or by* (2.64).

Since, as follows from $\lim_{R\to\infty}\sqrt{\kappa}\,u_R=\sqrt{2}$, we find immediately from (2.7a), (3.90), (3.23b), and (2.70a, b):

Corollary 3.14. *For any* γ *in* $0<\gamma\leq\pi$, *there holds*

$$\lim_{\mathscr{V}\to\infty}\sqrt{\kappa}\,q=\sqrt{2(1-\cos\gamma)}. \tag{3.96}$$

We note for later reference that the right side of (3.94) decreases in R, and that the right side of (3.95) decreases in R in the range $\sqrt{\kappa}R>1/(1-\cos\gamma)$.

In order to express the above bounds in terms of $\mathscr{V}$, we return to the formula (3.9). We have immediately:

Theorem 3.15. *The relation*

$$\mathscr{V}=\pi\left(a_\gamma^2 u-\frac{2}{\kappa}a_\gamma\sin\gamma\right), \tag{3.97}$$

in conjunction with the inequalities (3.57), (3.89), (3.94), *and* (3.95), *yields a set of parametric inequalities from which* a, q, *and* $\mathscr{V}$ *can be estimated above and below in terms of* R.

Alternatively the relations can be used to estimate a_γ, q, and R in terms of the "physical" parameter $\mathscr{V}$. In that connection, we note that the right side of (3.96) is increasing in a_γ.

We may also obtain analogues of Theorems 3.10 and 3.11 suitable for large drops; to do so, we estimate u in terms of R by (3.57c) or (3.82) (together with (2.57) and (2.60) or (2.63)), then R in terms of a_γ, using (3.89), and insert the result into (3.96). In this case there is no nonuniformity. For the case $\gamma=\pi$, we carry out the procedure in detail in Chapter 8. The results are shown in Fig. 3.11 and also in Fig. 3.9, where they are seen to improve the "small drop" estimate over part of the range.

As an example, we apply our estimates to a configuration studied by Laplace (1830), who considered a drop of mercury on a glass table, with diameter of wetted disk equal to 10 cm. Using the values $\kappa=400/13$, $\gamma=136.8$ degrees, Laplace calculated, by a method of matching expansions, $q \sim 0.3397$ cm. Laplace gave no estimate for R.

The above bounds, applied to the same configuration, yield

$$\begin{aligned} 5.0326 &< R < 5.0334 \\ 0.3387 &< q < 0.3416, \end{aligned} \tag{3.98}$$

thus confirming Laplace's calculations.

From (3.94) and (3.95) we obtain additionally the estimates

$$8.245 < \mathscr{V} < 8.318. \tag{3.99}$$

We remark that the above values for κ and γ differ significantly from those in current usage; they may however have conformed to the mercury samples available at the time, when techniques for purification were less developed than they are at present.

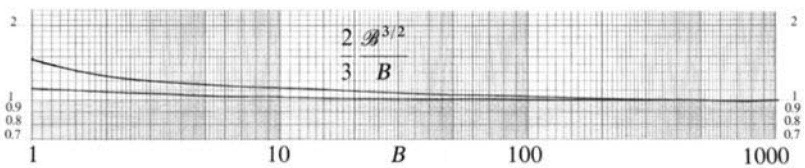


Figure 3.11. Volume-wetted area relationship; large drops.

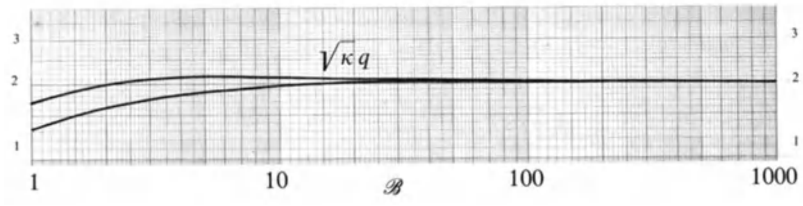


Figure 3.12. Drop height, large drops; $\gamma = \pi$.

The bounds for $\sqrt{\kappa}\, q$ indicated in Theorems 3.14 and 3.15 are plotted in terms of $\mathscr{B}$ in Fig. 3.12. The lower bound (for q/R as function of B) is extended onto Fig. 3.8(a) and is seen to improve the "small drop" estimate over part of the range. Thus, the two estimates taken together provide useful information for configurations of all sizes.

The upper curve in Fig. 3.12 indicates the nonmonotonicity of drop height with volume; see Theorem 3.6.

Notes to Chapter 3

1. §3.1. The existence (locally) of a solution to (2.2) with prescribed u_0 was first proved by Lohnstein [119], who proved the convergence of a formal power series (an earlier "proof" given by Lasswitz [115] is not correct in all details). In connection with the "pendent drop" equation, a simplified version of Lohnstein's proof (due to Wente) is given in Chapter 4. The same method applies to the case considered here.

2. §3.3. An explicit determination of u_0 in terms of λ (for given $\mathscr{V}$, γ) and conversely presents technical difficulties. Some estimates appear in [57, 60]. In [57] a "reciprocity principle" is established, relating the quantity $\sin\psi/u_0$ (for the capillary tube) to the quantity $\mathscr{V}/\sin\psi$ (for the sessile drop). The correspondence becomes asymptotically exact in both limits, as the quantities in question tend to zero or to infinity.

3. §3.3. The global existence of a sessile drop on a horizontal plate with prescribed volume was first proved by Gonzalez [90], who obtained it as a configuration of minimizing energy. The method yields also the symmetry of any such configuration, but not its uniqueness. The uniqueness proof of §3.3 applies to any symmetric equilibrium configuration, and does not assume it to be minimizing. Independently, Wente [186] proved that any equilibrium configuration is symmetric. Putting all these results together, we find that *there is exactly one equilibrium configu-*

ration, that it is symmetric (in the sense it can be generated by an interval of disks centered on an axis orthogonal to the base plane) and that it provides a strict minimum for potential energy. A particular consequence is that the drop is statically stable, that is, any change in shape will increase its potential energy.

4. The problems considered here can be much more difficult if the supporting plane is not of homogeneous material; one must then impose conditions to ensure that the liquid does not flow out to infinity. Hypotheses under which existence can be assured have been introduced by Giusti [88] and by Caffarelli and Spruck in a work now in preparation. Uniqueness can in general not be expected; Finn has given an example of a continuum of distinct solutions.

5. We have noted in §1.9 that if a capillary tube is closed at the bottom and if there is enough liquid to cover the base, the shape of the free surface will be identical to that obtained when the open tube is dipped into the liquid. In this sense the configurations studied in Chapters 2 and 3 are both limiting cases for the situation studied by Giusti in [88], of liquid on a supporting surface $\mathscr{S}$ that rises to infinity (although neither case is included in Giusti's hypotheses). In these two special cases uniqueness proofs are provided by Theorems 3.2 and 5.1. For general configurations (even symmetric ones) uniqueness clearly fails; it would thus be of interest to characterize surfaces $\mathscr{S}$ for which it will hold. For example, if $\mathscr{S}$ is rotationally symmetric and convex, then uniqueness should be expected; however, that has not yet been proved.

Chapter 4

The Pendent Liquid Drop

4.1. Mise en Scène

If, in the configurations considered in Chapter 3, the direction of the gravity field is reversed, we obtain the problem of a drop suspended from a horizontal plate. The equation of the free surface now becomes, from (1.44),

$$2H = -\kappa u + \lambda. \tag{4.1}$$

The theorem of Wente [186] shows that any solution is symmetric and generated by horizontal disks about a vertical line, whose heights fill out the range $u_0 < u < u_\Pi$; thus, as in (3.6), (4.1) can be written in the form

$$(r \sin \psi)_r = -\kappa r u \tag{4.2}$$

after suitable choice of coordinate origin, in any neighborhood in which the generating curve projects simply onto the r-axis.

The behavior of solutions of (4.2) differs basically from that of the solutions of (3.6). In general, it cannot be expected that solutions corresponding to prescribed boundary data exist. There is, however, a local theorem, which goes back to Lohnstein [119]. The proof in the following section is (essentially) due to Wente [187]. The underlying idea is the same as that of Lohnstein, but the details are simpler.

4.2. Local Existence

Theorem 4.1. *For any prescribed $u_0 \le 0$, there exists $R_0 = R(u_0)$, such that a solution $u(r; u_0)$ of* (4.2) *exists in $0 < r < R_0$, with $\lim_{r \to 0} u(r; u_0) = u_0$.*

Proof. By a similarity transformation, we may normalize (4.2) so that $\kappa = 1$. We write $v = \sin \psi$, then (4.2) takes the form

$$v' + (v/r) + u = 0. \tag{4.3}$$

We restrict attention to the interval $0 \leq \psi < \pi/2$. By differentiation of (4.3), u can be eliminated and we find

$$(r\,v')' + \left(\frac{r}{W} - \frac{1}{r}\right) v = 0, \qquad W = (1 - v^2)^{1/2}. \tag{4.4}$$

Whenever existence holds, we find by integration of (4.2) that $\lim_{r\to 0}(v/r) = -\frac{1}{2}u_0$, and hence, by (4.3), $\lim_{r\to 0} v' = -\frac{1}{2}u_0$. We thus seek a solution of (4.4) such that $v(0)=0$, $v'(0) = -\frac{1}{2}u_0$. One sees easily the equivalence of this problem with the original one.

We write $v(r) = \sum_1^\infty a_n r^n$, $a_1 = -\frac{1}{2}u_0$, and we rewrite (4.4) in the form $L(v) = M(v)$, where

$$L(v) \equiv (r\,v')' - v/r = \sum_2^\infty (n^2 - 1)\,a_n r^{n-1}$$

and

$$\begin{aligned} M(v) &\equiv -r\,v/W = -r\left(v + \sum_2^\infty c_n v^n\right) \\ &= -\{a_1 r^2 + (a_2 + c_2 a_1^2) r^3 + \cdots \\ &\qquad + P_n(a_1, \ldots, a_{n-1};\, c_1, \ldots, c_{n-1}) r^n + \cdots\} \end{aligned}$$

where P_n is a polynomial of degree $(n-1)$ in $a_1, \ldots, a_{n-1}$ and is affine in $c_1, \ldots, c_{n-1}$ with positive coefficients. Here $c_1 = 1$. The coefficients $a_2, a_3, \ldots$ can be calculated formally from the relations

$$(n^2 - 1)\,a_n = P_{n-1}(a_1, \ldots, a_{n-2};\, c_1, \ldots, c_{n-2}).$$

Since the series for $M(v)$ has radius of convergence 1, we have $|c_n|\,\rho^n < c < \infty$ for every ρ with $0 < \rho < 1$.

Now set $L_1(v) = v'(r) - v'(0) \equiv v'(r) + \frac{1}{2}u_0$, and set $M_1(v) = C\,r[(v/\rho)/(1 - (v/\rho))]$. We consider the auxiliary problem $L_1(w) = M_1(w)$ for a function $w = \sum_1^\infty b_n r^n$, with $w'(0) = |a_1|$. We have

$$L_1(w) = \sum_2^\infty n\,b_n r^n$$

$$M_1(w) = C\,r \sum_1^\infty \left(\frac{w}{\rho}\right)^n.$$

If $w > 0$, there follows

$$\begin{aligned} M_1(w) &> r(|c_1|\,w + |c_2|\,w^2 + \cdots) \\ &= \sum_2^\infty P_n(b_1, \ldots, b_{n-1};\, |c_1|, \ldots, |c_n|)\,r^n. \end{aligned}$$

Also, $nb_n < (n^2-1)b_n$ when $b_n > 0$, $n > 1$. Setting $b_1 = |a_1|$, we thus obtain, successively, $b_n > |a_n|$, all $n > 1$, and thus the series for w will majorize the series for v.

The stated problem for w admits, however, a convergent series solution with the required properties. We have

$$w' + \frac{1}{2}u_0 = Cr\left[\left(\frac{w}{\rho}\right)\Big/\left(1-\frac{w}{\rho}\right)\right].$$

The initial value problem with $w(0)=0$ is nonsingular at $r=0$ and thus can be solved by a convergent power series; for this solution, $w'(0) = \frac{1}{2}u_0 = a_1 > 0$. Clearly, w is increasing and thus remains positive for all $r > 0$ in the circle of convergence. We conclude that the series for v is also convergent, as was to be shown.

4.3. Uniqueness

The above proof establishes uniqueness among solutions that are analytic at $r=0$; in fact, uniqueness holds quite generally.

Theorem 4.2. *For any u_0, there is at most one solution $u(r;u_0)$ of* (4.2) *in* $0 < r < R(u_0)$, *such that* $\lim_{r\to 0} u(r;u_0) = u_0$.

Proof. Let $u(r;u_0)$, $v(r;u_0)$ be solutions, with corresponding inclination angles ψ, φ, and let $s = \sin\psi$, $t = \sin\varphi$. Then

$$u_r - v_r = \frac{1}{(1-\xi^2)^{3/2}}(s-t), \tag{4.5}$$

where ξ lies between s and t. From (4.2) we find

$$s = \sin\psi = -\frac{\kappa}{r}\int_0^r \rho u(\rho;u_0)\,d\rho; \tag{4.6}$$

thus, given $\varepsilon > 0$, there exists $r_0(\varepsilon)$ such that

$$s < \frac{1+\varepsilon}{2}\kappa|u_0|r < \frac{1}{2}\sqrt{2} \tag{4.7}$$

on $0 < r < r_0(\varepsilon)$. We may assume $t < \frac{1}{2}\sqrt{2}$ on the same interval, so that $(1-\xi^2) > 1/2$. Thus from (4.5) and (4.6)

$$u_r - v_r = -\frac{1}{(1-\xi^2)^{3/2}}\frac{\kappa}{r}\int_0^r (u-v)\rho\,d\rho$$

so that

$$|u-v| \leq 2\sqrt{2}\,\kappa \int_0^r \frac{1}{\tau} \int_0^\tau |u-v|\,\rho\,d\rho$$

on $0<r<r_0$, by (4.7). Let $M=\max_{0\leq r\leq r_0}|u-v|$, then $M\leq\frac{1}{2}\sqrt{2}\,\kappa M r_0^2$. If now r_0 is chosen sufficiently small, we obtain a contradiction unless $u\equiv v$ on $0<r<r_0$. Thus the two solutions coincide on some initial interval; their identity throughout the traverse then follows from standard uniqueness theorems.

4.4. Global Behavior; General Remarks

To prove the existence of the pendent drop solutions and to characterize their shape, it is necessary to examine the continuations of the solutions just found. A cursory investigation indicates that for large $|u_0|$ neither r nor ψ can be adopted as parameter; for that reason we introduce arc length s along the trajectory as independent variable, obtaining the system

$$\begin{aligned}\frac{dr}{ds}&=\cos\psi\\ \frac{du}{ds}&=\sin\psi\\ \frac{d\psi}{ds}&=-\kappa u-\frac{\sin\psi}{r}\end{aligned}\qquad(4.8\,\mathrm{a,b,c})$$

(see the remark preceding (4.9)).

From the point of view of classical theory, one expects a solution of (4.8) to be determined by the initial data $r(0)=0$; $\psi(0)=0$; $u(0)=u_0$; however, as we have just seen, because of the singularity of the system at $s=0$ the condition $\psi(0)=0$ is superfluous. As with the sessile drop, this circumstance yields an important simplification for the problem of characterizing all solutions. It suffices to describe the one parameter family determined by u_0, and it is that approach we shall adopt.

These solutions have remarkable continuation properties. Although they cannot in general be continued indefinitely as solutions of (4.2), we shall show that for any u_0 the function $u(r;u_0)$ can be extended as a parametric solution of (4.8) for all s, yielding a curve without limit sets or double points. The resulting capillary rotation surface does not again contact the axis $r=0$ (although it can come perilously close to that axis) and in fact for large s, r increases beyond all bounds, with the trajectory again projecting simply onto the r-axis. This behavior seems noteworthy in that for large values of s the associated rotation surfaces, when considered as interfaces bounding drops pendent from horizontal planes through those points, are always physically unstable. (If $|u_0|$ is large, instability occurs already for small s.)

Numerical evidence suggests there are very few solutions of (4.8) without double points. In that respect the solutions we shall study—and especially the singular solution discussed in §4.13—appear to have an isolated character. We take up that matter in §4.14.

The central difficulty in the general study of the solutions of (4.2) lies in the failure of the maximum principle. We replace that principle here by a simple geometrical one, the conceptual content of which is that if surfaces S_1 and S_2 contact at p, and if the normal sections of S_1 at p in two orthogonal directions have (planar) curvatures exceeding those for S_2, then there is a neighborhood of p in which S_1 does not again meet S_2. The analytical formulation (Lemma 4.11) of that principle in the situation encountered here yields a global result and also encompasses cases in which the curvatures at p are respectively equal. For the case of large $|u_0|$, it is the key to success of the method.

The comparison technique just mentioned has proved effective also in other contexts, and has led in particular to new information on the behavior of solutions of (4.2) near isolated singular points (see, e.g., [54]).

In order to simplify notation in what follows, we will assume throughout the remainder of this chapter that (4.2) and (4.8) have been normalized (by a similarity transformation) so that $\kappa=1$. Thus, (4.2) becomes

$$(r\sin\psi)_r \equiv \left(\frac{r u_r}{\sqrt{1+u_r^2}}\right)_r = -ru. \tag{4.9}$$

We may also assume $u_0<0$. The case $u_0>0$ can be transformed to that one by a simple change of sign, while if $u_0=0$ the unique solution of (4.9) is $u\equiv 0$.

4.5. Small $|u_0|$

Theorem 4.3. *If in the initial value problem for* (4.9) *there holds* $u_0\geq -2$, *then the function* $u(r;u_0)$ *can be continued as a* (*nonparametric*) *solution of that equation for all* $r>0$. *It has an infinity of zeros. For any two successive extrema at* r_a, r_b *of* $u(r;u_0)$ *there holds* $|u(r_b)|<|u(r_a)|<\sqrt{2}$. *Between any two extrema occurs exactly one inflection, which appears on a* (*monotone*) *curve segment approaching the r-axis in the sense of increasing s. At any two successive points* α, β *at which* $|u(\alpha)|=|u(\beta)|$, *there holds* $|du/dr|_\alpha>|du/dr|_\beta$. *Asymptotically as* $u_0\to 0$, *the ratio* u/u_0 *tends uniformly on compact sets to the Bessel function* $\mathrm{J}_0(r)$, *and the first zero occurs at* $r\approx 2.405$.

The proof of Theorem 4.3 will follow from the lemmas of this section.

Lemma 4.1. *Let $u(r)$ satisfy* (4.9) *in* $0<r<R$, *and suppose* $\lim_{r\to 0} u(r) = u_0 < 0$. *Then there is an interval* $0<r<r_p$ *in which* $u(r)$ *is increasing, and*

$$-\frac{u}{2} < \frac{\sin\psi}{r} < -\frac{u_0}{2} \tag{4.10a, b}$$

in that interval. The interval includes any initial segment $0<r<R$ *in which* $u(r)<0$.

Proof. The proof follows immediately from the relation

$$\sin\psi = -\frac{1}{r}\int_0^r \rho u \, d\rho \tag{4.11}$$

which under the hypotheses is equivalent to (4.9).

Lemma 4.2. *Under the hypotheses of Lemma* 4.1, *suppose* $u<0$ *in* $0<r<R$. *If* $u_0 \ge -2$, *then there exists* $\psi(R) = \lim_{r\to R}\psi(r)$, *and* $0<\psi R) < \pi/2$.

Proof. The existence of $\psi(R)$ follows from (4.11) and Lemma 4.1. Let us write (4.9) in the form

$$r^{-1}\sin\psi + (\sin\psi)_r \equiv r^{-1}\sin\psi - (\cos\psi)_u = -u \tag{4.12}$$

(which is permissible since $u_r>0$) and integrate from u_0 to $u(R)$. Using (4.10a) we find

$$\cos\psi(R) > 1 - \tfrac{1}{4}(u_0^2 - u^2(R)),$$

which implies $\psi(R)<\pi/2$. We obtain $0<\psi(R)$ as a consequence of (4.11), for any $R<\infty$. This restriction is however superfluous as follows from

Lemma 4.3. *If* $u(r)$ *satisfies* (4.9) *and* $u(r)<0$ *in* $0<a\le r<R$, *and if* $\sin\psi(a)>0$, *then*

$$R < a\exp\left\{\frac{-u(a)}{a\sin\psi(a)}\right\}. \tag{4.13}$$

Proof. By (4.9), $r\sin\psi \ge a\sin\psi(a)$, and therefore $\sin\psi>0$. Thus

$$\frac{du}{dr} = \tan\psi > \sin\psi \ge a r^{-1}\sin\psi_a \tag{4.14}$$

on the interval, and the result follows on an integration.

We have also

Corollary 4.3. *If* $u(r)$ *satisfies* (4.9) *and* $u(r)<0$ *on* $0<r<R$ *and if there exists* $u_0=\lim_{r\to 0} u(r)<0$, *then* $R<2e^{1/2}\approx 3.297$.

Proof. From (4.9) follows $\sin\psi>0$ as above. From (4.13) and (4.10a) we find $R<a\exp\{2/a^2\}$, which is minimized for $a=2$.

We conclude from the Corollary that the solution of the indicated initial value problem for (4.9) increases until a value $R<\infty$ at which either $u(R)=0$ or $\sin\psi=1$. Under the conditions of Lemma 4.2, the former case must occur.

Lemma 4.3 can be strengthened, as follows from the following two lemmas.

Lemma 4.3(a). *Let* $u(r)$ *satisfy* (4.9) *and* $u(r)<0$ *on* $0<a\le r<R$, *and suppose* $\sin\psi(a)>0$, $u(R)=0$. *Then*

$$\left(1-\frac{a^2}{2}\right)\ln\frac{R}{a}+\frac{R^2-a^2}{4}+\frac{1}{2}\left(a\ln\frac{R}{a}\right)^2<-\frac{u(a)}{a\sin\psi(a)}. \tag{4.15}$$

Proof. From (4.9) we find

$$r\sin\psi(r)-a\sin\psi(a)=-\int_a^r \rho u\,d\rho \tag{4.16}$$

from which follows $\sin\psi>0$ on $a\le r<R$. From (4.14) we obtain

$$-u(r)>(a\sin\psi(a))\ln\frac{R}{r}.$$

Placing this result into (4.16) and integrating, we obtain

$$\frac{r}{a\sin\psi(a)}\frac{du}{dr}>1+\frac{1}{2}r^2\ln\frac{R}{r}-\frac{1}{2}a^2\ln\frac{R}{a}+\frac{r^2-a^2}{4},$$

since $du/dr>\sin\psi$. An integration from a to R now yields (4.15).

Lemma 4.3(b). *Let* $a_0\approx 1.443$ *be the (unique) positive solution of*

$$\left(1-\frac{a^2}{2}\right)\frac{\sqrt{4-a^2}}{a^2}+\frac{a^2}{4}\left[\exp\left(\frac{2}{a^2}\sqrt{4-a^2}\right)-1\right]-\frac{4}{a^2}+\frac{1}{2}=0. \tag{4.17}$$

Let $R_0\approx 2.8065$ *be the unique solution in the range* $r>a_0$ *of*

$$\left(1-\frac{a_0^2}{2}\right)\ln\frac{r}{a_0}+\frac{r^2-a_0^2}{4}-\frac{1}{2}\left(a_0\ln\frac{r}{a_0}\right)^2-\frac{2}{a_0^2}=0.$$

If $u(r)$ satisfies (4.9) *and $u(r)<0$ on $0<r<R$, and if there exists $u_0 = \lim_{r\to 0} u(r) < 0$, then $R<R_0$.*

Proof. We may assume $u(R)=0$. In view of (4.10a) we may write from (4.15), whenever $0<a<R$,

$$F(R;a) \equiv \left(1-\frac{a^2}{2}\right) \ln\frac{R}{a} + \frac{R^2-a^2}{4} + \frac{1}{2}\left(a \ln\frac{R}{a}\right)^2 - \frac{2}{a^2} < 0.$$

We verify easily $\partial F/\partial R>0$, $F(a;a)<0$, and $F(\infty;a)=\infty$; hence there is a unique $\hat{r}>a$ for which $F(\hat{r};a)=0$. This relation, together with $\partial\hat{r}/\partial a=0$, determines a unique value $a=a_0$ as solution of (4.17). There holds $\partial\hat{r}/\partial a \gtrless 0$ according as $a \gtrless a_0$, and thus $\hat{r}$ attains a minimum R_0 at $a=a_0$. Since $R<\hat{r}$ always (by Lemma 4.3a), there holds $R<R_0$.

We have thus established that if $u_0 \geq -2$ the solution curve can be continued as an increasing function from its initial point $(0, u_0)$ until it meets the r-axis at a point $r=a_1$. To study the further trajectory, we observe (because of Lemma 4.1) that the curve can be continued at least locally across the axis as a solution of (4.9), and we compare its inclination at a given height h with the inclination of the initial branch at an equal negative height.

Lemma 4.4. *If the curve can be continued monotonically to a height h above the r-axis, then its inclination at that height is less than the inclination of the initial branch at the height $-h$, that is*

$$\left.\frac{du}{dr}\right|_h < \left.\frac{du}{dr}\right|_{-h}.$$

Proof. We integrate (4.12) with respect to u between the heights $-h$ and h, obtaining

$$\cos\psi|_h - \cos\psi|_{-h} = \int_{-h}^{h} r^{-1} \sin\psi \, du > 0.$$

Lemma 4.5. *Under the above conditions, in the indicated interval the curve is strictly convex downward when $u>0$, and $u_{rr} < -u(1+u_r^2)^{3/2}$.*

Proof. From (4.12)

$$(\sin\psi)_r \equiv \frac{u_{rr}}{(1+u_r^2)^{3/2}} = -u - r^{-1} \sin\psi < -u.$$

From Lemmas 4.4 and 4.5 we conclude that the curve can be continued as a solution of (4.9) until a strict local maximum is attained at a height $u_M < |u_0|$. More precisely, we may now state

Lemma 4.6. *Let $\hat{\psi}$ be the value of ψ at $u=0$. There holds*

$$u_M^2 < 2(1-\cos\hat{\psi}) < \tfrac{1}{2}u_0^2. \tag{4.18}$$

As a corollary, we find that $u_M < \sqrt{2}$ always. To prove the result, we integrate (4.12) from $u=0$ to $u=u_M$ to obtain

$$\int_0^{u_M} \frac{\sin\psi}{r}\,du - 1 + \cos\hat{\psi} = -\frac{1}{2}u_M^2,$$

which contains the first inequality of (4.18). To obtain the second inequality we integrate (4.12) from u_0 to 0 and use (4.10a); thus

$$\int_{u_0}^{0} \frac{\sin\psi}{r}\,du + 1 - \cos\hat{\psi} = \frac{u_0^2}{2} > \frac{u_0^2}{4} + 1 - \cos\hat{\psi},$$

which completes the proof.

We now repeat the entire procedure, starting at the maximum point instead of at u_0. We obtain the qualitative picture indicated in Theorem 4.3, of a curve oscillating about the r-axis with successively decreasing extrema (see Fig. 4.1). We have further:

Lemma 4.7. *Denoting by m a minimum point and by M a maximum point, there holds between any extremum and the following zero*

$$\left.\begin{aligned} -\frac{u}{2} &< \frac{r}{r^2 - r_m^2}\sin\psi < -\frac{u_m}{2} \\ -\frac{u_M}{2} &< \frac{r}{r^2 - r_M^2}\sin\psi < -\frac{u}{2} \end{aligned}\right. \tag{4.19a, b}$$

while between any zero and the following extremum there holds

$$\left.\begin{aligned} \frac{u_m}{2} &< \frac{r}{r_m^2 - r^2}\sin\psi < \frac{u}{2} \\ \frac{u}{2} &< \frac{r}{r_M^2 - r^2}\sin\psi < \frac{u_M}{2}. \end{aligned}\right. \tag{4.19c, d}$$

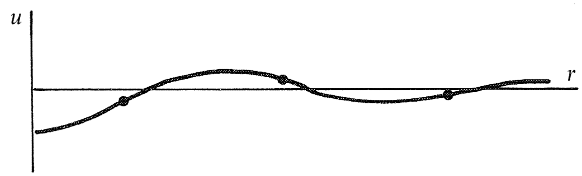


Figure 4.1. The case $u_0 \geq -2$; inflections.

Proof. All inequalities follow easily from (4.9) by an integration from the extremal point, since $\sin\psi=0$ at an extremum and $u(r)$ is monotonic between any extremum and adjacent zero.

Lemma 4.8. *On the interval up to and including the first maximum of* $u(r;u_0)$, *there holds* $(d/dr)k_l<0$; *on the interval up to and including the first inflection of* $u(r;u_0)$, *there holds* $(d/dr)k_m<0$.

Proof. We have always

$$\begin{aligned}\frac{d}{dr}k_l&=\frac{d}{dr}\frac{\sin\psi}{r}=\frac{(\sin\psi)_r}{r}-\frac{\sin\psi}{r^2}\\&=-\frac{1}{r}\left(u+2\frac{\sin\psi}{r}\right)\end{aligned}\tag{4.20}$$

by (4.12). On the interval preceding the first zero (4.19a) yields

$$\frac{d}{dr}\frac{\sin\psi}{r}<-\frac{1}{r}(u-u)=0.$$

Between the first zero and the first maximum we find from (4.19d)

$$\frac{d}{dr}\frac{\sin\psi}{r}<-\frac{u}{r}\left(1+\frac{r_M^2}{r^2}-1\right)=-\frac{r_M^2}{r^2}u<0.$$

Finally we note from (4.12) and (4.19a)

$$\begin{aligned}\frac{dk_m}{dr}&=-\frac{du}{dr}-\frac{1}{r}(\sin\psi)_r+\frac{\sin\psi}{r^2}\\&=-\frac{du}{dr}+\frac{1}{r}\left(u+2\frac{\sin\psi}{r}\right)\\&<-\frac{du}{dr}+\frac{1}{r}(u-u_0)\\&=-\frac{1}{r}\int_0^r\rho u_{\rho\rho}\,d\rho<0.\end{aligned}\tag{4.21}$$

It follows in particular that the inflection is isolated; a similar discussion applies to all later inflections.

To complete the proof of Theorem 4.3, we note that, in view of the above estimates, the function $v(r;u_0)\equiv u(r;u_0)u_0^{-1}$ satisfies the Bessel equation for $\mathsf{J}_0(r)$ with an (analytic) inhomogeneous term that tends uniformly to zero with u_0. The result then follows from the continuous dependence properties of the inhomogeneous Bessel equation.

4.6. Appearance of Vertical Points

From (4.10b) and (4.12) we find

Lemma 4.9. *For any u_0, the solution $u(r; u_0)$ can be continued at least until the value $r = -2/u_0$, with $\sin\psi < -\frac{1}{2}u_0 r$. There are no inflections in the region $ru < -1$.*

We intend to show that if $|u_0|$ is large, the trajectory cannot be extended indefinitely as a solution of (4.9).

Lemma 4.10. *If $u_0 < 0$, the lower circular arc*

$$v(r) = u_0 - \frac{2}{u_0} - \sqrt{\left(\frac{4}{u_0^2}\right) - r^2} \tag{4.22}$$

meets the hyperbola $rv = -1$ if and only if $u_0 \le -\sqrt{2}$. On the segment of the arc between the u-axis and the hyperbola there holds $v < u_0 - (2 - \sqrt{2}/u_0) < (\sqrt{2}/2) u_0$.

Proof. The indicated circular arc is inscribed in a square, of which a half side length is the horizontal segement joining $(0, u_0)$ to the hyperbola $ru = -2$ (see Fig. 4.2). If $u_0 = -\sqrt{2}$, then the arc is tangent to $ru = -1$ at $(1, -1)$ (see figure). In the general case it is tangent to the hyperbola $r[u - (u_0 - (2/u_0))] = -2/u_0^2$, obtained by similarity transformation from the symmetric configuration. On this latter hyperbola we have

$$ru + 1 = -\frac{2}{u_0^2} + r\left(u_0 - \frac{2}{u_0}\right) + 1.$$

If $u_0 < -\sqrt{2}$, then $u_0 - (2/u_0) < 0$, and thus $ru + 1 < 0$ when $ru_0 < -1$. That is, interior to the indicated square the hyperbola lies below $ru = -1$, hence the arc (4.22) must cross $ru = -1$ to achieve the required tangency.

If $u_0 > -\sqrt{2}$, then $u_0 - (2/u_0) > 0$, so that $ru_0 < -1$ implies $ru > -1$, from which we conclude that the arc (4.22) does not meet $ru = -1$.

When the arc does cross $ru = -1$, the first crossing point lies below the point of symmetry determined by $r = u_0 - (2/u_0) - v$. Thus, on the initial segment determined by (4.22) there holds $u_0 - (2/u_0) - v > -\sqrt{2}/u_0$. The proof is completed by the observation that in the case considered, $u_0 < 2/u_0$.

We now introduce the comparison lemma referred to in §4.4.

Lemma 4.11. *Let $F(r, t)$ satisfy $F_t > 0$, all $r > 0$. Let $v^{(1)}(r)$, $v^{(2)}(r)$ be functions defined in $a \le r \le b$ and such that $[F(r, v_r^{(1)})]_r \ge [F(r, v_r^{(2)})]_r$ on that in-*

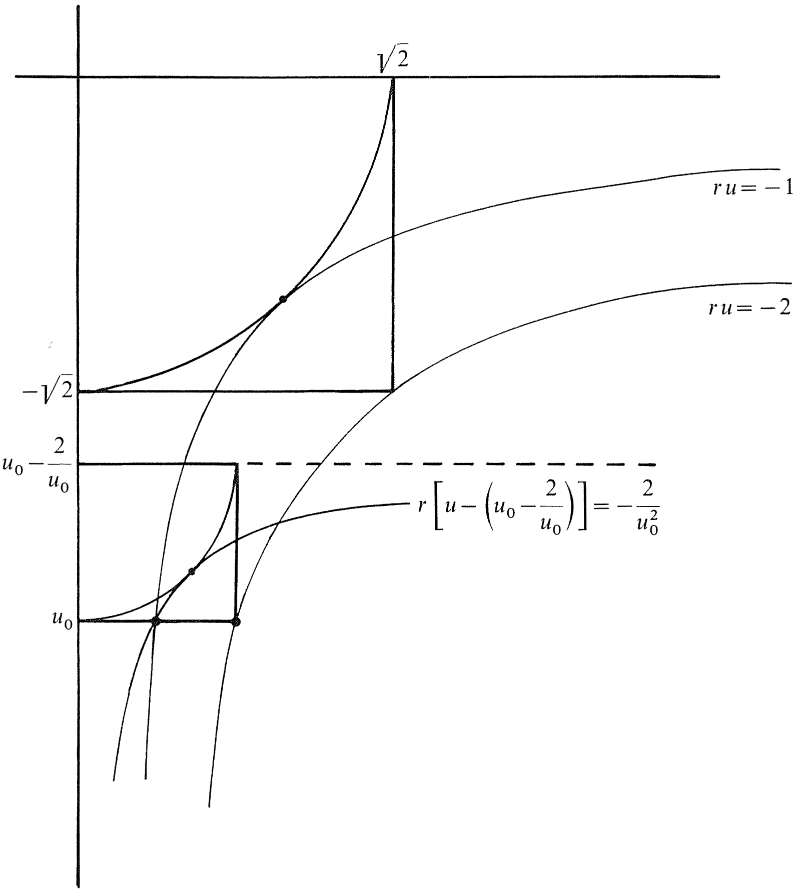


Figure 4.2. Construction for proof of Lemma 4.10.

terval. Suppose $v_r^{(1)}(a) \geq v_r^{(2)}(a)$. *Then* $v_r^{(1)}(b) \geq v_r^{(2)}(b)$, $v^{(1)}(b) - v^{(2)}(b) \geq v^{(1)}(a) - v^{(2)}(a)$, *and either equality holds if and only if* $v^{(1)}(r) \equiv v^{(2)}(r) + \text{const.}$ *on* $a \leq r \leq b$.

The proof follows immediately on an integration. We shall apply the lemma principally in two particular cases: $F = r(t/\sqrt{1+t^2})$, for which $(1/r)[F(r, u_r)]_r$ is twice the mean curvature of the rotation surface defined by $u(r)$, and $F = -(1/\sqrt{1+t^2})$, in which case $[F(r, u_r)]_r$ is the meridional curvature defined by $u(r)$.

In the present case we make the former choice for F and choose as a comparison surface the lower hemisphere (4.22), which has the (constant) mean curvature $-u_0/2$.

Lemma 4.12. *If* $u_0 < -\sqrt{2}$, *the solution* $u(r; u_0)$ *continues into the region* $ru < -1$, *but does not enter the region* $ru < -2$. *There holds* $k_m > 0$, $dk_m/dr < 0$ *until the solution curve either again crosses the hyperbola* $ru = -1$ *or a vertical point appears. Throughout this arc, there holds* $k_m < -u/2 < k_l < -u_0/2$.

Proof. Since $u > u_0$ on the initial trajectory (Lemma 4.1), there holds $(r \sin\psi)_r = -u < -u_0$. Using Lemma 4.11 to compare u with the function $v(r)$ defined by (4.22) and letting φ denote the inclination angle of $v(r)$, we find $\sin\psi < \sin\varphi$, $u < v$ on the interval $0 < r < -2/u_0$ in which v is defined. Thus, since $v(r)$ enters the region $ru < -1$ (Lemma 4.10), so does $u(r)$. We have

$$k_m = (\sin\psi)_r = -u - \frac{\sin\psi}{r} = -u - k_l$$

$$> -v + \frac{u_0}{2} > -v + \frac{u_0}{\sqrt{2}} > 0$$

by (4.10) and Lemmas 4.11 and 4.12. Thus there is no inflection on the interval, and we may apply Lemma 4.8 to find $(d/dr)k_m < 0$. The assertion $ru < -2$ follows from (4.10a), and the final assertion follows from (4.10) and Lemma 4.8.

Lemma 4.13. *If* $u_0 \le -2\sqrt{2}$ *then a vertical slope appears at a point* (r_1, u_1) *satisfying*

$$-\frac{2}{u_0} < r_1 < -\frac{1}{2}u_0\left(1 - \sqrt{1 - \frac{8}{u_0^2}}\right) \tag{4.23a, b}$$

$$u_0 - \frac{2}{u_0} < u_1 < u_0 - \frac{1}{2}u_0\left(1 - \sqrt{1 - \frac{8}{u_0^2}}\right). \tag{4.24a, b}$$

Proof. Since $\sin\psi > 0$ on the initial segment we may adopt u as independent variable. Integrating (4.12) in u and using (4.10b), we find

$$1 - \cos\psi > \frac{-1}{2}u(u - u_0);$$

thus, if $u_0 \le -2\sqrt{2}$ a vertical slope appears at a value $u_1 < \frac{1}{2}u_0[1 + \sqrt{1 - 8u_0^{-2}}]$.

To obtain a bound in the other direction we compare the solution curve with the circle $v(r)$ at corresponding values of u. Since $k_m = -(\cos\psi)_u < -\frac{1}{2}u_0 = -(\cos\varphi)_u$ by Lemma 4.12, we see that at each fixed u there holds $\cos\psi > \cos\varphi$. It follows that $u(r; u_0)$ can be continued at least to the height at which the comparison circle $v(r)$ becomes vertical. That is, $u_1 > u_0 - 2/u_0$. A further consequence is that $r_1 > -2/u_0$.

Finally we compare $u(r)$ with the vertical section $w(r)$ of the sphere of constant mean curvature $\beta^{-1}=-\frac{1}{4}u_0(1+\sqrt{1-8u_0^{-2}})$, with center at $(0,u_0+\beta)$, so that $u(0)=w(0)$. The curvature is chosen to correspond to the height $-2\beta^{-1}>u_1$, and so $u>w$, $u_r>w_r$ until the height u_1 is reached. The corresponding radius r_1 must be less than the radius of the sphere $w(r)$, thus $r_1<\beta$. The lemma is proved.

We note the vertical point of the comparison circle w occurs on the intersection of that circle with the hyperbola $ru=-2$. The conclusion of the lemma is illustrated in Fig. 4.3. The assertion can be rephrased in terms of the respective coordinates (ρ_1,v_1), (r_1,u_1), (σ_1,w_1) of the three vertical points. The relations (4.23) and (4.24) become

$$\begin{gathered}\rho_1<r_1<\sigma_1\\ v_1<u_1<w_1.\end{gathered}\tag{4.25}$$

We have shown that if $u_0\geq -2$ no vertical can appear, while if $u_0\leq -2\sqrt{2}$ a vertical must appear.

Lemma 4.14. *Let u_{0c} be the largest value of u_0 at which a vertical occurs. Then if $u_0=u_{0c}$ there is exactly one vertical point (r_c,u_c); it appears on the hyperbola $ru=-1$, and is an inflection for the solution curve. This curve can be continued beyond (r_c,u_c) uniquely as a solution of* (4.9), *and has the oscillatory behavior described in Theorem* 4.3. *It does not meet the hyperbola again until it has developed a positive maximum; in fact, its slope exceeds that of the hyperbola until the r-axis is crossed.*

Proof. Let (r_c,u_c) be a vertical point for the (limiting) solution curve, as $u_0\searrow u_{0c}$. Then $\psi_u=0$ at (r_c,u_c). By (4.12), $r_cu_c=-1$, as asserted. From (4.12)

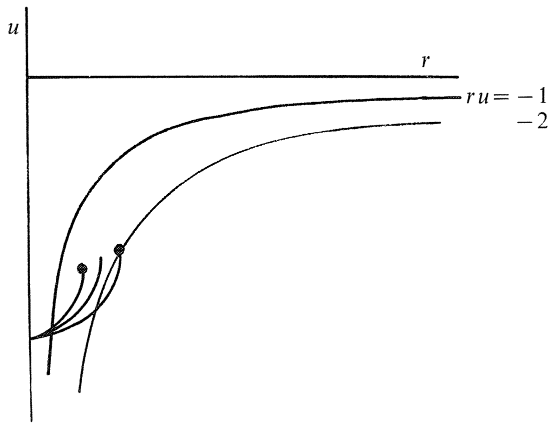


Figure 4.3. Initial comparison surfaces, $u_0\leq -2\sqrt{2}$.

we find at (r_c, u_c)

$$\frac{d}{du}k_m = -1 - \frac{\cos\psi}{r}\psi_u + \frac{\sin\psi}{r^2}r_u = -1,$$

from which follows that the curve can be continued locally beyond r_c as a solution of (4.9). In view of Lemmas 4.3 and 4.4, it will suffice to show that the curve does not develop another vertical or return to the hyperbola before crossing the r-axis.

We have from (4.9)

$$r\sin\psi - r_c = -\int_{r_c}^{r} \rho u \, d\rho < r - r_c \tag{4.26}$$

in the region $ru > -1$, so that $\sin\psi < 1$ in this region; thus verticals within the region are excluded. If the curve should return to the hyperbola in the region $u<0$, there would be an intermediate value $\hat{r}$ at which the slopes would be equal, so that

$$\sin\hat{\psi} = \frac{1}{\sqrt{1+\hat{r}^4}}.$$

But from (4.26) we see that if $u<0$ in the interval $r_c < r < \hat{r}$, then

$$\hat{r}\sin\psi = \frac{\hat{r}}{\sqrt{1+\hat{r}^4}} > r_c,$$

and thus $r_c < \frac{1}{2}\sqrt{2}$. On the other hand, $r_c > -2/u_{0c}$ by (4.10b), while by Lemma 4.13, $u_{0c} > -2\sqrt{2}$. Therefore $r_c > \frac{1}{2}\sqrt{2}$. This contradiction completes the proof of the lemma.

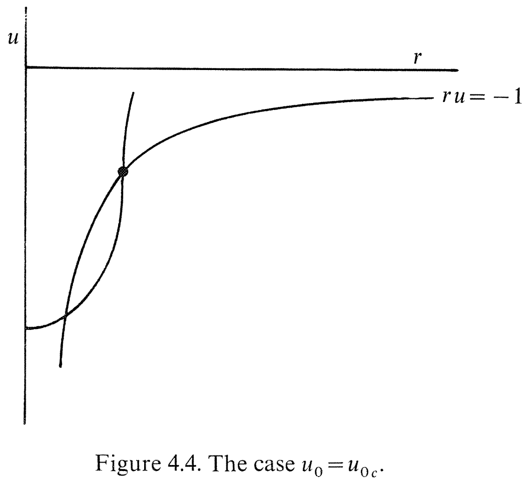


Figure 4.4. The case $u_0 = u_{0c}$.

We collect the results of this section in a general statement:

Theorem 4.4. *If* $u_0 \leq -\sqrt{2}$, *the solution curve must enter the region* $ru < -1$. *It does not enter the region* $ru \leq -2$. *It remains convex until it either leaves the region or a vertical occurs. There holds* $k_m < -\frac{1}{2}u < k_l$, *with* k_m, k_l *both decreasing,* $k_m + k_l = -u$. *There is a critical value* u_{0c}, $-2 > u_{0c} > -2\sqrt{2}$, *for which a (unique) vertical first appears, at a point* (r_c, u_c) *with* $r_c u_c = -1$; *see Fig.* 4.4. *There holds* $dk_m/du = -1$ *at* (r_c, u_c), *and the curve extends as a solution of* (4.9) *with the oscillatory behavior described by Theorem* 4.3. *If* $u_0 \leq -2\sqrt{2}$ *the location of the first vertical can be estimated by* (4.23) *and* (4.24).

4.7. Behavior for Large $|u_0|$

It is clear that every solution curve can be continued locally past the initial vertical point as a solution of the parametric system (4.8). We intend to show that the continuation can be effected globally, to a complete curve without limit sets or double points. Our basic tool will be Lemma 4.11; for comparison surfaces we have recourse to a discovery of Delaunay [41]:

Let an ellipse of major axis $2a$ *and distance* $2c$ *between focal points roll rigidly on an axis without slipping. Let* $\mathscr{C}$ *be the curve swept out by one of the focal points. Then the surface generated by rotating* $\mathscr{C}$ *about the axis has constant mean curvature* $H = (2a)^{-1}$.

We note that $\mathscr{C}$ is periodic with half-period τ satisfying $2a < \tau < \pi a$, and that each half-period can be represented in the interval $a - c \leq r \leq a + c$ by a single valued function $v(r)$ for which the equation

$$\frac{1}{r}(r \sin\psi)_r = \frac{1}{a} \tag{4.27}$$

holds, and for which $\sin\psi = 1$ at the two end points (see Fig. 4.5).

We indicate the use of the roulades by showing that if $|u_0|$ is sufficiently large, the curve $u(r; u_0)$ can be continued as a solution of the parametric system (4.8) until a second vertical appears at (r_2, u_2) with $r_2 u_2 > -1$.

We observe first by (4.12), (4.10b), and (4.24b) that

$$k_m = (\sin\psi)_r > -u_1 + \tfrac{1}{2}u_0 > -\tfrac{1}{2}u_0\sqrt{1 - 8u_0^{-2}} > 0$$

when $u_0 < -2\sqrt{2}$, and thus the curve turns back toward the u-axis and can be described (locally) as a solution of $(r\sin\psi)_r = -ru$. We compare it

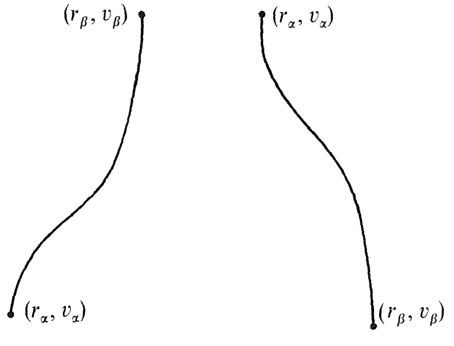


Figure 4.5. Delaunay arcs. (a) Outgoing. (b) Returning.

with a roulade $v(r)$ whose mean curvature is $-\frac{1}{2}u_1$ and for which $a+c = r_1$, which is positioned so that the vertical points coincide (see Fig. 4.6). Since $v_r(r_1) = -\infty$, Lemma 4.11 yields $u_r < v_r$, $u(r) > v(r)$ as long as the continuation of both u and v as single-valued functions is possible.

The curve $v(r)$ can be continued toward the u-axis only until the point $(a-c, u_1+\tau)$ with $a-c = -(2/u_1) - r_1 > 0$. At this point the slope is again infinite. It follows there is a value $r_2 > -2/u_1 - r_1$ beyond which this branch of the solution curve cannot be continued as a single valued function.

From the geometrical interpretation of τ as the half circumference of an ellipse with major axis $2a = -2/u_1$ and focal length $c = r_1 - a$, one finds the expression for the height of the roulade

$$\tau = 2a\,\mathsf{E}(k), \qquad k = \frac{c}{a}, \tag{4.28}$$

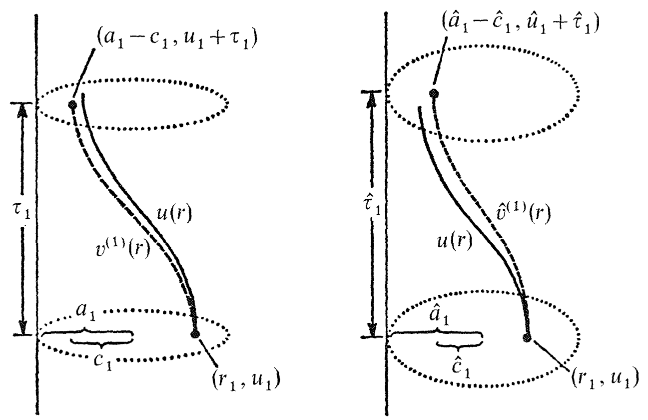


Figure 4.6. Comparison with roulades.

where $\mathsf{E}(k)$ is the complete elliptic integral of the second kind. For large $|u_0|$ we will have

$$\tau = -\frac{2}{u_1} - \frac{16}{3}\frac{\ln|u_1|}{|u_1|^3} + O\left(\frac{1}{|u_1|^3}\right). \tag{4.29}$$

Let us estimate r_2 from above. To do so, we compare $u(r)$ with a roulade $\hat{v}(r)$, which is determined by the conditions

$$\begin{aligned} \hat{a} &= -\frac{1}{u_1+\hat{\tau}} \\ \hat{a}+\hat{c} &= r_1 \\ \hat{\tau} &= \int_0^\pi (a^2-c^2\cos^2\varphi)^{1/2}\,d\varphi. \end{aligned} \tag{4.30}$$

A formal estimate shows that such a roulade exists if $u_1 < -2\sqrt{\pi}$.

The conditions (4.30) are chosen so that the roulade can be placed with its lower vertical point at (r_1, u_1) (see Fig. 4.6) and so that in that configuration its mean curvature will be exactly the one determined from the right side of (4.9) by setting u equal to the height at the upper vertical. Applying Lemma 4.11 again, we obtain $u_r > \hat{v}_r$, $u(r) < \hat{v}(r)$ for all $r < r_1$ for which $u(r) < u_1 + \hat{\tau}$. This condition clearly holds for r near r_1; since $\hat{v}(r) < u_1 + \hat{\tau}$, we conclude that it holds on the entire interval $\hat{a}+\hat{c} < r < r_1$, thus

$$0 > v_r(r) > u_r(r) > \hat{v}_r(r) > -\infty$$

on this interval, and hence the solution can be continued to the left of r_1, at least until the value

$$r_2 < -\frac{2}{u_1+\hat{\tau}} - r_1 = \beta_2. \tag{4.31}$$

For large $|u_0|$ we find

$$\hat{\tau} = -\frac{2}{u_1} - \frac{40}{3}\frac{\ln|u_1|}{|u_1|^3} + O\left(\frac{1}{|u_1|^3}\right) \tag{4.32}$$

and thus

$$r_2 < -\frac{2}{u_1} - r_1 - \frac{4}{u_1^3} + O\left(\frac{\ln|u_1|}{|u_1|^5}\right). \tag{4.33}$$

We now proceed, essentially, as in the proof of Lemma 4.13. Denoting by φ the inclination angle of v, we note that

$$\sin\psi > \sin\varphi = -\frac{1}{2}ru_1 + \frac{r_1}{r}\left(1+\frac{1}{2}r_1u_1\right);$$

thus from the preceding estimates and from (4.23) and (4.24) we find for $r<\beta_2$ that

$$\frac{\sin\psi}{r} > -\frac{3}{50}u_1^3 + O\left(\frac{\ln|u_1|}{|u_1|^5}\right).$$

Using this estimate, we integrate (4.12) in u from $u(\beta_2)$; since $\cos\psi<0$ until a vertical is reached, and since

$$\cos\psi(\beta_2) > \cos\varphi(\beta_2) = -\frac{1}{5}\sqrt{21} + O\left(\frac{\ln|u_1|}{|u_1|^2}\right),$$

we are led to a contradiction unless the curve becomes vertical before u has increased by an amount $-16u_1^{-3}$. That is, a vertical must appear at a value

$$u_2 < u_1 + \hat{\tau}_1 - 16u_1^{-3}.$$

The solution curve then turns back from the axis at (r_2, u_2) and initiates a further branch.

It is now clear how to proceed, and the successive comparisons yield a curve that continues with simple projection onto the u-axis until the r-axis is crossed. The following properties are easy corollaries of the method.

Lemma 4.15. *The successive horizontal distances of the inner vertical points from the u-axis, and of the outer verticals from the hyperbola* $ru = -2$, *increase monotonically. For sufficiently large* $|u|$, *there is exactly one inflection between any two successive vertical points.*

We prove also:

Lemma 4.16. *In the initial region* $u<0$, *the entire curve is bounded* (*strictly*) *between the u-axis and the hyperbola* $ru=-2$ (see Fig. 4.7).

Proof. We note from (4.12) that at any vertical point there holds

$$(\sin\psi)_r = -\frac{ru+1}{r}, \tag{4.34}$$

and thus each such point continues to an outgoing arc or returning arc according as $ru > -1$ or $ru < -1$. We integrate (4.9) on an outgoing arc starting from (r_α, u_α), α even $(r_0=0)$ to obtain

$$r\sin\psi = r_\alpha - \int_{r_\alpha}^{r} \rho u\, d\rho, \tag{4.35}$$

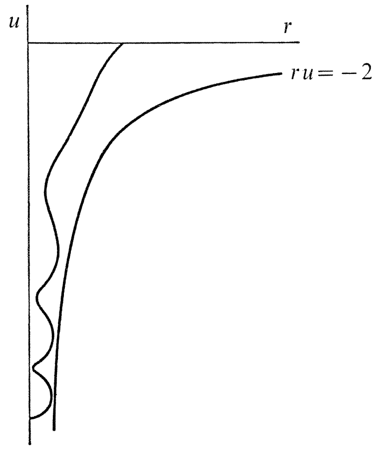


Figure 4.7. The initial region $u<0$.

which shows first that $u_r(r)>0$ on any such arc along which $u<0$. Integrating by parts in (4.35), we find

$$r\sin\psi - r_\alpha = \tfrac{1}{2}(u_\alpha r_\alpha^2 - ur^2) + \tfrac{1}{2}\int_{r_\alpha}^{r} \rho^2 u_\rho \, d\rho$$
$$> -\tfrac{1}{2}r_\alpha - \tfrac{1}{2}r(ur)$$

from which

$$ur > \frac{r_\alpha}{r} - 2\sin\psi > -2.$$

On an arc returning from (r_β, u_β) we similarly obtain

$$r_\beta - r\sin\psi = \tfrac{1}{2}(ur^2 - u_\beta r_\beta^2) + \tfrac{1}{2}\int_{r}^{r_\beta} \rho^2 u_\rho \, d\rho,$$

and since $u_\beta r_\beta > -2$ there follows

$$\sin\psi > -\frac{1}{2}ur - \frac{1}{2}\frac{1}{r}\int_{r}^{r_\beta} \rho^2 u_\rho \, d\rho,$$

from which we easily conclude $u_r<0$ in the region $u<0$. There follows immediately $r>r_\alpha>0$ along such an arc.

4.8. Global Behavior

The discussion thus far has shown that the solution curve can be continued upward without self-intersections until it reaches the r-axis. We wish to show that after crossing the r-axis it continues outward to infinity without crossing any previous portion of the arc. Our first step is to show that it crosses the r-axis with a positive slope.

Lemma 4.17. *Let $r=a_1$ be the first point at which the solution curve meets the r-axis. Then $0<u_r(a_1)<\infty$.*

Proof. Suppose $u_r(a_1)<0$, or equivalently, $\cos\psi_1<0$. The curve could then be continued backward into the negative u half plane until a first vertical (r_α, u_α) (see Fig. 4.8). By Lemma 4.16, $r_\alpha u_\alpha > -2$. We integrate (4.12) with respect to u, from u_α to 0, obtaining

$$\int_{u_\alpha}^{0} \frac{\sin\psi}{r}\,du = \cos\psi_1 + \frac{1}{2}u_\alpha^2. \tag{4.36}$$

To evaluate the left side of (4.36), we integrate (4.9) in r between r and r_α:

$$\begin{aligned} r_\alpha - r\sin\psi &= -\int_r^{r_\alpha} \rho u\, d\rho < \tfrac{1}{2}(r^2 - r_\alpha^2)u_\alpha \\ &< \tfrac{1}{2}r^2 u_\alpha + r_\alpha \end{aligned}$$

since $r_\alpha u_\alpha > -2$. Thus

$$\frac{\sin\psi}{r} > -\frac{1}{2}u_\alpha \tag{4.37}$$

on the entire arc. Placing (4.37) into (4.36), we find $\cos\psi_1 > 0$, which contradicts the assumption.

It remains to exclude the case $\cos\psi_1 = 0$. Were that to occur, we would find from (4.12) that $(\cos\psi)_u > 0$ at the point; thus if $\cos\psi_1 = 0$ there would again be a backward branch from a_1 into the negative u half plane, and we obtain a contradiction as before.

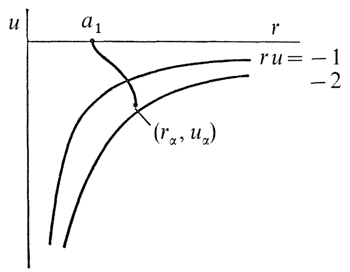


Figure 4.8. Proof of Lemma 4.17.

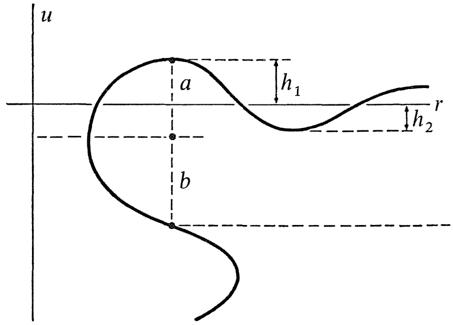


Figure 4.9. Proof of Lemma 4.18.

The proofs of Lemmas 4.4 to 4.6 now follow exactly as before, and we see that the solution curve continues from the point $(a_1, 0)$ as indicated in Fig. 4.1. We show that the curve does not intersect the initial branch in the region $u < 0$.

Lemma 4.18. *Let u_α, u_β be the heights at two successive points on the solution curve, such that $r_\alpha = r_\beta$ with an intervening vertical at (r_γ, u_γ). If $r_\gamma < r_\alpha$, then $\sin\psi_\beta < \sin\psi_\alpha$; if $r_\gamma > r_\alpha$, then $\sin\psi_\beta > \sin\psi_\alpha$* (see Fig. 4.9).

Proof. Suppose $r_\gamma < r_\alpha$. From (4.9) we find

$$r_\beta \sin\psi_\beta - r_\gamma = -\int_{r_\gamma}^{r_\beta} \rho u \, d\rho$$

$$r_\alpha \sin\psi_\alpha - r_\gamma = -\int_{r_\gamma}^{r_\alpha} \rho u \, d\rho$$

and thus, since $r_\alpha = r_\beta$,

$$r_\alpha(\sin\psi_\beta - \sin\psi_\alpha) = \int_{r_\gamma}^{r_\alpha} \rho(u^- - u^+) \, d\rho < 0;$$

here u^- and u^+ denote values on the lower and upper branches. The case $r_\gamma > r_\alpha$ follows similarly.

Corollary 4.18. *The r-coordinate, both of the inner and of the outer vertical points, increases monotonically as the curve is traversed from the initial point.*

Consider now the coordinate r_M of the first (positive) maximum. By the Corollary 4.18, either r_M exceeds the r-coordinate of any vertical point, or else there is a vertical segment $r = r_M$ joining two points separated by a single vertical, as indicated in Fig. 4.9.

From Lemma 4.18 follows that $a < b$ in Fig. 4.9, and thus $h_1 < b$. But $h_j < h_1$, any $j > 1$, by Lemma 4.4. Hence $h_j < b$, all $j > 1$, and thus intersections are excluded.

Combining the above results we obtain

Theorem 4.5. *The solution of the parametric system* (4.8) *defined by the datum* u_0 *can be continued indefinitely without limit sets or double points. It has the properties indicated in Lemmas* 4.15–4.17, *and has the form indicated in Figs.* 4.10–4.12.

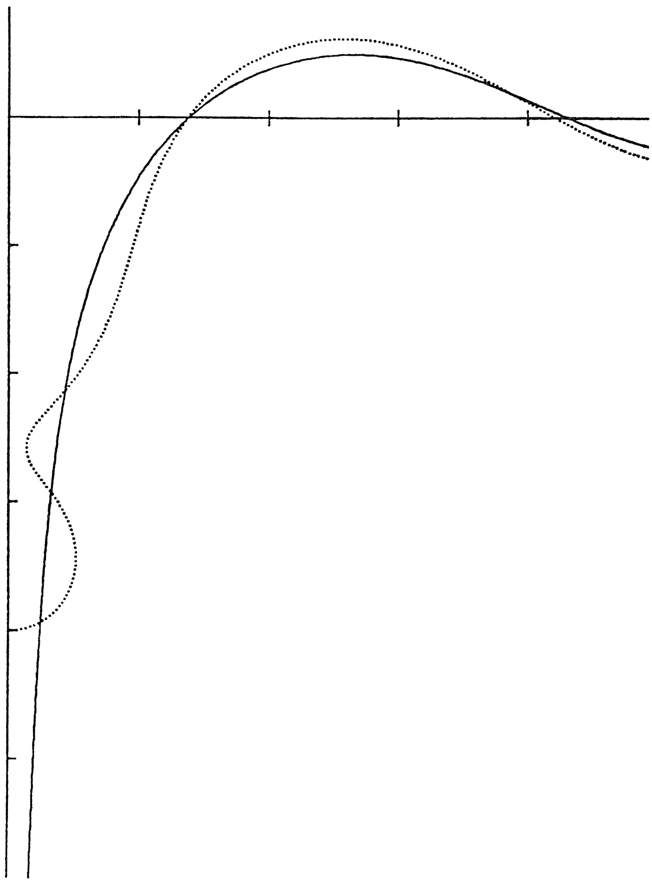


Figure 4.10. $u_0 = -4$; singular solution.

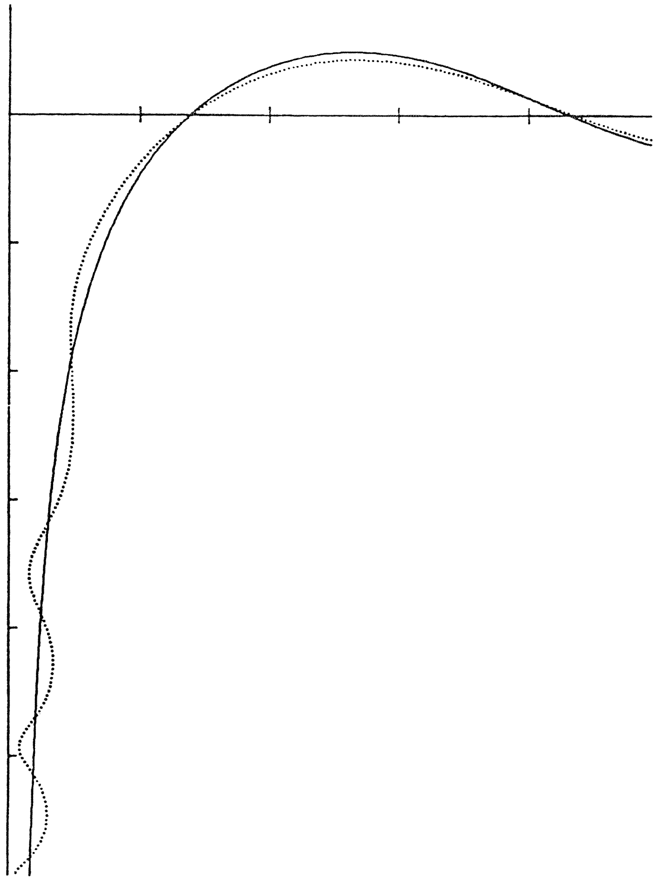


Figure 4.11. $u_0 = -8$; singular solution.

The figures show the results of computer calculations by Concus, corresponding to the values $u_0 = -4, \; -8, \; -16$, and also the singular solution, cf. §4.13.

4.9. Maximum Vertical Diameter

Regardless of u_0, there is a universal upper bound for the diameter of a drop at a vertical point.

Theorem 4.6. *Let $\delta \approx 2.47341$ be the unique positive root of the equation*

$$r^3 - 3^{3/2} r - 3^{3/4} = 0. \tag{4.38}$$

Then 2δ exceeds the diameter of any drop at a vertical point.

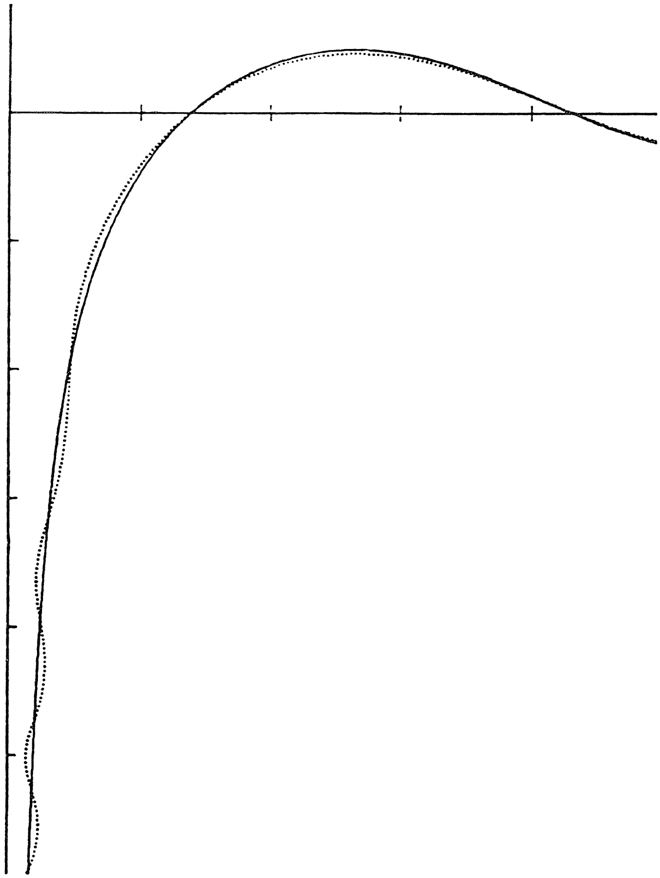


Figure 4.12. $u_0 = -16$; singular solution.

We base the proof on a lemma, which also has an independent interest.

Lemma 4.19. *Let $u(r)$ represent a solution curve passing through (a, u_a) with $-1 \le a u_a \le 0$, and such that*

$$a \sin\psi_a \ge \tfrac{1}{2} a. \tag{4.39}$$

Suppose $u(r) < 0$ in $a \le r < R$. Then $\sin\psi > 0$ on this arc segment. If the curve meets the hyperbola $ru = -1$ in a point (c, u_c) with $a < c < R$, then $c < 3^{1/4}$, and $\sin\psi_c > 1/2$.

Proof. We integrate (4.9) between α and r, obtaining after an integration by parts

$$r \sin\psi - \alpha \sin\psi_\alpha = \tfrac{1}{2}(\alpha^2 u_\alpha - r^2 u(r)) + \tfrac{1}{2}\int_\alpha^r \rho^2 u_\rho \, d\rho \tag{4.40}$$

from which, if $\alpha=a$,

$$r\sin\psi \geq -\tfrac{1}{2}r^2 u(r)+\tfrac{1}{2}\int_a^r \rho^2 u_\rho\, d\rho \tag{4.41}$$

by (4.39). Since $u(a)<0$, we conclude from (4.41) that $\sin\psi>0$ when r is sufficiently close to a. Thus, if $\sin\psi$ were to vanish at any points interior to $a\leq r<R$, there would be a minimum such value $r=r_\gamma>a$ at which that occurs. But (4.41) would then imply

$$0=r_\gamma \sin\psi_\gamma>\tfrac{1}{2}\int_a^{r_\gamma} \rho^2 u_\rho\, d\rho>0,$$

a contradiction. Thus, $\sin\psi>0$ on $a<r\leq c$, and hence $u_\rho>0$ on that interval. Setting $r=c$ in (4.41) yields

$$\sin\psi_c> -\tfrac{1}{2}c\, u_c=\tfrac{1}{2}, \tag{4.42}$$

as asserted.

Finally, we note that at $r=c$ the inclination of the solution curve cannot exceed that of the hyperbola. Thus, $\sin\psi_c\leq(1+c^4)^{-1/2}$, so that $c^4<3$ follows from (4.42).

We proceed to prove Theorem 4.6. We first observe from (4.12) that at any vertical point there holds

$$(\cos\psi)_u=\frac{1}{r}(ur+1) \tag{4.43}$$

which is positive if $ur>-1$. It follows that for any such vertical there must be a (next) vertical below it, at which $ur\leq-1$, and at which the diameter would clearly be larger. Thus, if we label the starting point (r_0,u_0), $r_0=0$, and the successive verticals by (r_j,u_j), $j\geq1$, then the maximum diameter will be attained at a point (r_{2j+1},u_{2j+1}), $-1\geq r_{2j+1}u_{2j+1}>-2$ (cf. Lemma 4.16). At the preceding point (r_{2j},u_{2j}) there holds either $r_{2j}=0$ (if $j=0$) or else $\sin\psi_{2j}=1$. In either event, (4.39) holds with $a=r_{2j}$. Also, $-1\leq r_{2j}u_{2j}\leq0$, and thus the curve crosses the hyperbola $ru=-1$ at a point (c,u_c), $r_{2j}<c<r_{2j+1}$. Finally, we observe that interior to the region $ru<-1$ the curve has no inflections (Lemma 4.12). Setting $\alpha=c$, $r=r_{2j+1}$ in (4.40) and applying Lemma 4.19, we find

$$r_{2j+1}^3-3^{3/2}r_{2j+1}-3^{3/4}<0. \tag{4.44}$$

The (single) positive solution of (4.38) exceeds any solution of (4.43), thus that solution provides the desired bound.

We note the following corollary of the method of proof:

Corollary 4.6. *At each intersection point* (r_c, u_c) *of the solution curve with the hyperbola* $ru = -1$, *such that* (r_c, u_c) *is connected, either to the initial point* $(0, u_0)$ *or to a vertical* $r_j u_j$ *with* $r_j u_j \geq -1$, *by an arc of the curve on which* $u < 0$ *and on which no other verticals appear, there hold* $\sin\psi_c > 1/2$, $c < 3^{1/4}$.

In particular, these inequalities hold at the first point of contact of $u(r; u_0)$ with the hyperbola. Corresponding to the largest value u_{0m} for which contact occurs before the r-axis is crossed, there hold $\sin\psi_c = 1/2$, $c = 3^{1/4}$.

4.10. Maximum Diameter

There is also (cf. §4.9) a universal upper bound for the coordinate $r = r_M$ of the first (and highest) maximum point.

Theorem 4.7. *Let R be the coordinate of the first crossing point of the solution curve of* (4.8) *with the r-axis, and let δ be as in Theorem* 4.6. *Let* $\mu \approx 2.888$ *be the unique root* $r > \delta$ *of*

$$\left(1 - \frac{\delta^2}{2}\right) \ln\frac{r}{\delta} + \frac{r^2 - \delta^2}{4} - \frac{1}{2}\left(\delta \ln\frac{r}{\delta}\right)^2 - \frac{1}{\delta^2} = 0. \tag{4.45}$$

Then $R < \mu$.

Proof. If no vertical points appear, the result is contained in Lemma 4.3(b). We may thus assume there is a (last) vertical (r_v, u_v), after which the solution continues as a graph $u(r) < 0$ satisfying (4.9), to the crossing point $(R, 0)$. By Lemma 4.17, $R > r_v$. From (4.9) we find on the interval $r_v < r < R$

$$\begin{aligned} r\sin\psi - r_v &= -\int_{r_v}^{r} \rho u \, d\rho \\ &= \tfrac{1}{2}(r_v^2 u_v - r^2 u) + \tfrac{1}{2}\int_{r_v}^{r} \rho^2 u'(\rho)\, d\rho \\ &> -\tfrac{1}{2} r_v + \tfrac{1}{2} r^2 u \end{aligned} \tag{4.46}$$

since (by 4.16) $\sin\psi > 0$ on the segment, and $r_v u_v > -1$. There follows $ru < 2\sin\psi$ and hence, from Lemma 4.3a, $F(R; a) < 0$ (as in the proof of Lemma 4.3(b)) for any a in the interval. Again a unique $\hat{r} > a$ is determined by $F(r; a) = 0$, and $\hat{r}$ is increasing in a when $a > a_0$ as determined by (4.17).

If $R < a_0$, there is nothing to prove. If $r_v < a_0 < R$, we set $a = a_0$ and determine $R < R_0$ as in the proof of Lemma 4.3(b).

If $r_v \geq a_0$, we recall that $r_v u_v > -1$, $\sin\psi_v = 1$; thus (4.15) can now be written

$$G(R;a) \equiv \left(1 - \frac{a^2}{2}\right)\ln\frac{R}{a} + \frac{R^2 - a^2}{4} + \frac{1}{2}\left(a \ln\frac{R}{a}\right)^2 - \frac{1}{a^2} < 0, \tag{4.47}$$

and (4.17) takes the form

$$\left(1 - \frac{a^2}{2}\right)\frac{\sqrt{2-a^2}}{a^2} + \frac{a^2}{4}\left[\exp\left[\frac{2}{a^2}\sqrt{2-a^2}\right] - 1\right] - \frac{2}{a^2} + \frac{1}{2} = 0 \tag{4.48}$$

which has the unique positive solution $b_0 \approx 1.157 < a_0$. Thus, in the equation $G(R;r_v) = 0$ we will have R increasing in r_v. We have however $r_v < \delta$ (Theorem 4.6), and it follows that R is bounded by the (unique) solution $r > \delta$ of (4.45), as was to be shown.

Theorem 4.8. *Let (r_M, u_M) be the coordinates of the first maximum point. Let R be as above, and let*

$$\lambda = \frac{1}{\sqrt{R}}\sqrt{R + \sqrt{2}}, \qquad \tau = \frac{1}{\lambda^2 - 1}. \tag{4.49}$$

Then

$$r_M^2 < \frac{4 + \tau}{(1+\tau)\ln\left(1 + \frac{1}{\tau}\right) - 1} \tag{4.50}$$

and hence, by Theorem 4.7, $r_M < 5.333$.

Proof. We consider the segment of the solution curve joining $(R,0)$ to (r_M, u_M). On that segment we have $u > 0$, $\sin\psi > 0$. Setting $v = \sin\psi$, we find from (4.12)

$$v_r = -u - \frac{v}{r} < -\frac{v}{r} \tag{4.51}$$

and thus

$$r v < R v_R \tag{4.52}$$

on the segment.

By Lemma 4.6,

$$u < u_M < \sqrt{2}\sqrt{1 - \cos\psi_R} = \sqrt{2}\frac{\sin\psi_R}{\sqrt{1 + \cos\psi_R}} < \sqrt{2}\, v_R. \tag{4.53}$$

Thus, by (4.9)

$$(r v)_r = -r u > -r u_M > -\sqrt{2}\, v_R r, \tag{4.54}$$

from which

$$v(r) > \frac{R v_R}{r}\left(1 + \frac{\sqrt{2}}{2} R\right) - \frac{\sqrt{2}}{2} v_R r. \tag{4.55}$$

Thus, $v(r)$ remains positive at least until the value λR, with λ determined by (4.49). We now observe $du/dr > v(r)$ and integrate (4.55) from R to r. We obtain

$$u(r) > R v_R \left\{ \left(1 + \frac{1}{2}\sqrt{2}R\right) \ln \frac{r}{R} - \frac{\sqrt{2}}{4} R \left(\frac{r^2}{R^2} - 1\right) \right\}. \tag{4.56}$$

Once the height $u(\lambda R)$ is attained, we find from (4.9)

$$(r v)_r = -r u < -r u(\lambda R) \tag{4.57}$$

and thus, for $\lambda R < r \le r_M$,

$$r v < \lambda R v_R - u(\lambda R) \frac{r^2 - (\lambda R)^2}{2}. \tag{4.58}$$

It follows that the value r_M, at which $v = \sin\psi = 0$, must satisfy

$$r_M^2 < \frac{2\lambda R v_{\lambda R}}{u(\lambda R)} + (\lambda R)^2. \tag{4.59}$$

We estimate $u(\lambda R)$ by (4.56), and we estimate $\lambda R v_{\lambda R}$ by placing (4.56) into (4.9) and integrating from R to λR. Using (4.49), we obtain the stated inequality (4.50).

We note that asymptotically as $R \to 0$, there holds from (4.49) and (4.50)

$$r_M^2 < \frac{4}{\ln(1/R)} + O\left(\frac{1}{\ln(1/R)}\right),$$

which tends to zero. In view of Theorem 4.7, it follows that Theorem 4.8 provides a universal (finite) upper bound for r_M, for all possible pendent drops. This bound has a physical significance, as any drop that extends beyond the coordinate r_M of the first maximum would have to pass through the supporting plane, which is physically unrealistic. Thus, (4.45), (4.49), and (4.50) *together provide a bound for the physical diameter of any pendent drop.*

4.11. Maximum Volume

If we restrict attention to (physical) drops as above, we may observe first that any drop can be replaced by one of larger or equal volume by extending it to the coordinate r_M. But the volume of that drop is easily

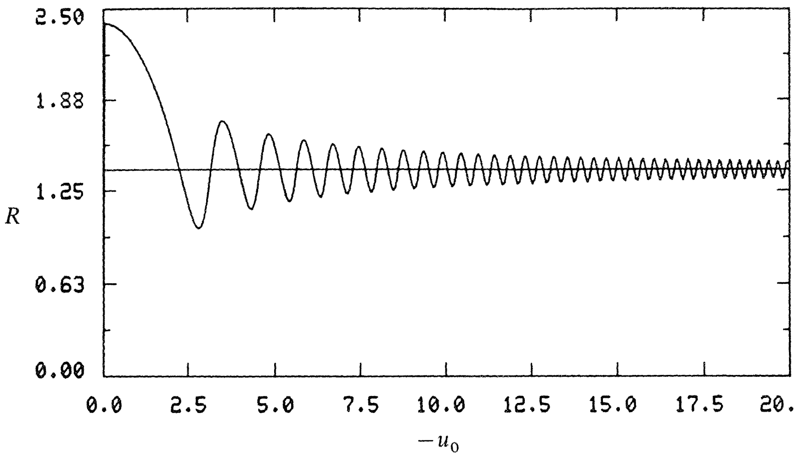


Figure 4.13. Initial null-point for drop profile and singular solution.

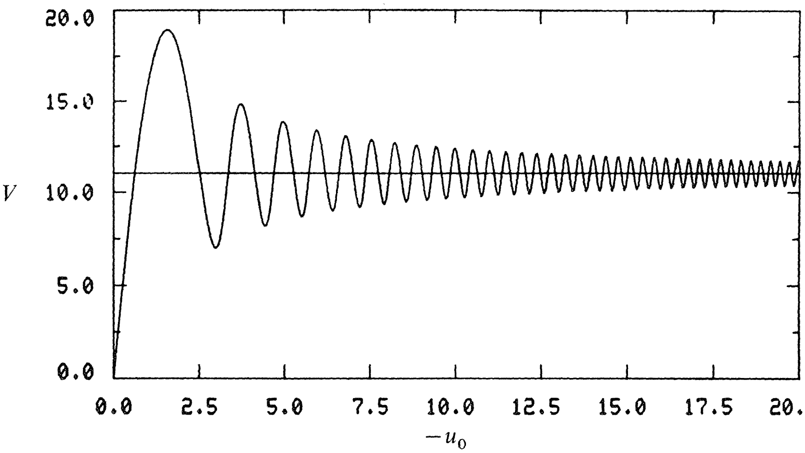


Figure 4.14. Maximal volume for pendent drop and singular drop.

calculated. It is, in terms of arc length s measured from the tip,

$$\begin{aligned} V_M &= \pi r_M^2 u_M - 2\pi \int_0^{s_M} \rho u \frac{d\rho}{ds} ds \\ &= \pi r_M^2 u_M + 2\pi \int_0^{s_M} \frac{d}{ds}(\rho \sin\psi)\, ds \\ &= \pi r_M^2 u_M \end{aligned} \tag{4.60}$$

by the governing equations (4.8). From Lemma 4.6 and the remarks at the end of the preceding section, we have immediately, in view of Theorem 4.8,

Theorem 4.9. *The quantity*

$$V^* = \pi\sqrt{2}(r_M)^2 < 126.4 \tag{4.61}$$

exceeds the volume of any (physical) pendent drop.

Figures 4.13 and 4.14 show results of computer calculations by *P.* Concus of R and V, with increasing $|u_0|$. See the remarks in Note 6 at the end of the chapter.

4.12. Asymptotic Properties

Since the solution exists globally for every u_0, it is of interest to examine its asymptotic behavior as $u_0 \to -\infty$. This behavior exhibits a number of striking properties, which are examined in some depth in [34]. The underlying idea is simply to iterate the procedure outlined in §4.7, with the hope of obtaining estimates sufficiently accurate that they will continue to have meaning in a fixed compact range, even as $u_0 \to -\infty$. Unfortunately that procedure, in which the detailed estimates are obtained by placing into the integrals that occur the upper and lower heights of comparison roulades, yields information that is not sufficiently precise, and with repeated iteration control over the solution is lost. A degree of success was however attained by placing into the integrals that appear not just the bounds on the roulades but the analytic pointwise expressions for the roulades. The estimates are thus made to depend on painstaking estimation of integrals of incomplete elliptic integrals. The details are tedious and complicated and will not be reproduced here. The results do seem, however, rather striking and we therefore present them in outline. For a full presentation we refer the reader to [34].

We set $r_0 = 0$, denote the successive vertical points by (r_j, u_j), and write

$$c_j = |r_j + 1/u_j|, \qquad k_j = -c_j u_j, \qquad \delta c_j = c_{j+1} - c_j.$$

Set

$$q(k) = \mathsf{E}(k) - \frac{2}{3k^2}[(1+k^2)\,\mathsf{E}(k) - (1-k^2)\,\mathsf{K}(k)],$$

where K and E are complete elliptic integrals of the first and second kind. We then find

$$\delta c_j = -\frac{2k_j q(k_j)}{u_j^3} + A_j \frac{1}{\sqrt{k_j}\,u_j^4}, \qquad j \text{ even},$$

with an analogous (more complicated) expression for j odd. Here $|A_j| < A < \infty$, uniformly for all sufficiently large $|u_j|$, in any range $0 < k_0 \leq k < 1$.

Let $u_{m_1}^2$ be the (unique) solution of the equation

$$-\frac{4\pi}{u^2}\ln\left(\frac{4\pi}{u^2}\right) = \frac{1}{6A}$$

and set

$$k^{(1)} = \max\left\{k\colon -2kq(k) \leq -2A\left(1-k+\frac{4\pi}{u_{m_1}^2}\right)\ln\left(1-k+\frac{4\pi}{u_{m_1}^2}\right)\right\}.$$

For sufficiently large $|u_j|$ we may then write

$$\delta c_j = -P_j c_j^3$$

with

$$\min_{k \geq k^{(1)}} \frac{|q(k)|}{k^2} < P_j < \max_{k \geq k^{(1)}} \frac{3|q(k)|}{k^2}. \tag{4.62}$$

We then obtain, for any fixed N and $|u_0|$ sufficiently large,

$$2NP \sim c_N^{-2} - u_0^2$$

for some P in the range (4.44). The symbol $\sim$ denotes an asymptotic relation. We find correspondingly

$$u_N^2 \sim u_0^2 - 2\Lambda N, \qquad 2 \leq \Lambda \leq \pi.$$

Theorem 4.10. *Given any $k^{(0)}$ with $k^{(1)} < k^{(0)} < 1$, there exists $\eta(k^{(0)}) > 0$ such that the solution curve, starting at $(0, u_0)$, "separates" from the axis $r = 0$ and the hyperbola $ru = -2$, after an interval $u_N - u_0 = [1 - (1-\eta)^{1/2}]\,|u_0|$, in the sense that for all $u > u_N$ the curve lies between the hyperbolas $ru = -1 \pm k^{(0)}$. Above u_N, the curve begins to contract toward $ru = -1$, and at the heights $|u_j| < |u_{N^{(1)}}| = (1-\eta)^{1/2}|u_0|$ there holds $k < k^{(0)}$. The curve can be continued through successive verticals to a height $|u_{N^{(2)}}| \sim A|u_0|^{7/9}$, for suitably large A, at which level it has contracted toward $ru = -1$ in the ratio*

$$\frac{c_{N^{(1)}}}{c_0} \sim \left(\frac{\Lambda}{\Lambda + P}\right)^{1/2}$$

For any α, $(23/9) < \alpha < 3$, the curve can be continued further through successive verticals and is confined to a strip of sensibly constant width until a height $|u_{N^{(3)}}| \sim |u_0|^{(2\alpha+1)/9}$. For smaller values of $|u|$ (relative to u_0) vertical points presumably cease to appear; however, the curve lies within a strip about $ru = -1$, of width determined by $k = A|u|^{-2/7}$, for sufficiently large

A (*independent of* u_0). *Specifically, there exists A such that for any fixed* (*sufficiently large*) $\hat{u}$, *there holds, for* $(\hat{r}, \hat{u})$ *on the solution curve,*

$$|\hat{r} - 1/\hat{u}| < A\,|\hat{u}|^{-9/7}$$

uniformly in u_0, *as* $u_0 \to -\infty$.

The four different stages of global asymptotic behavior are sketched in Fig. 4.15.

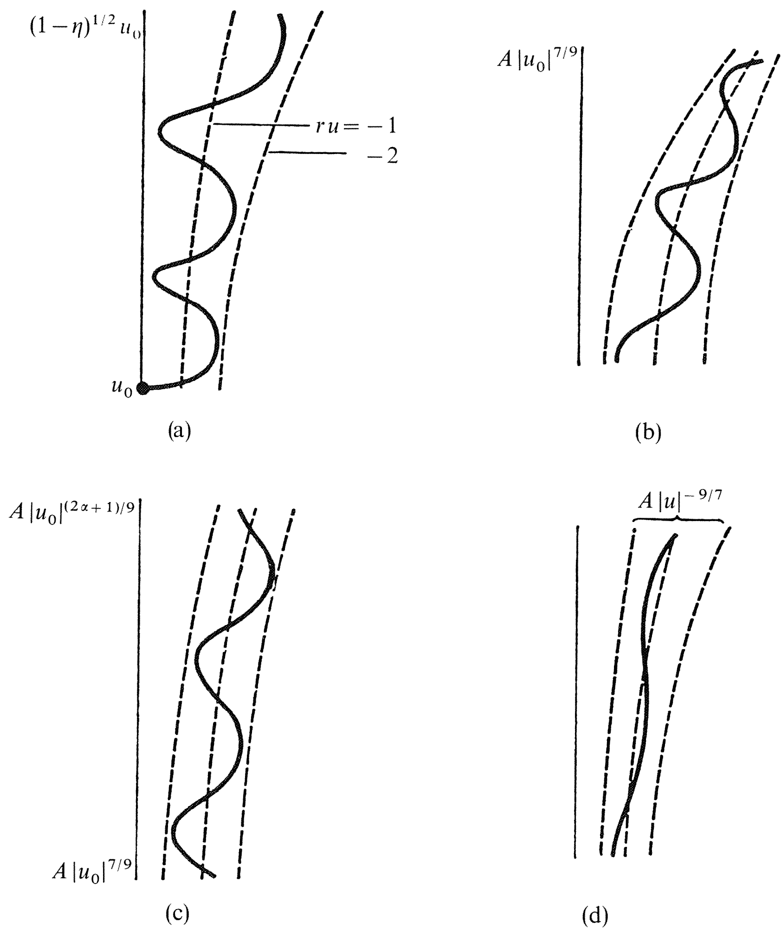


Figure 4.15. Asymptotic behavior for large $|u_0|$ at four levels (scales differ). (a) Initial separation from axis. (b) Contraction toward hyperbola. (c) Confinement to strip of constant width. (d) Behavior far from $|u_0|$.

4.13. The Singular Solution

If $\kappa \geq 0$ in the underlying equation (3.6), it has been shown [50] that every isolated singularity of a solution is removable. The theorem holds without growth or symmetry conditions. If $\kappa < 0$, the behavior of solutions can be quite different, although there are some conceptual analogies. The following result is due to Concus and Finn [32].

Theorem 4.11. *There exists a solution $U(r)$ of (4.9) with a nonremovable isolated singularity at the origin. The solution admits the (divergent) asymptotic development*

$$U(r) \sim -\frac{1}{r} + \frac{5}{2}r^3 - \frac{567}{8}r^7 + \frac{123149}{16}r^{11} - \frac{212466731}{128}r^{15} + \cdots. \tag{4.63}$$

The proof given in [32] is technically complicated and we shall not present it here in detail; we mention only that it proceeds through a representation of the nonsingular part of $U(r)$ by a Fresnel kernel. The resulting integral operator is then shown to contract a suitable ball in a Banach space into itself.

The solution $U(r)$ was observed independently in a computational study by Huh [103].

Conjecture 1. *$U(r)$ is—up to inessential transformations—the unique solution of (4.9) with a nonremovable isolated singularity.*

The following results [33], while they fall short of a complete uniqueness proof, do show that no other growth at a singular point is possible.

Theorem 4.12. *Let $u(r)$ be a solution of (4.9) in an interval $0<r<R$. Then either $u(r)$ can be defined at $r=0$ so as to satisfy $\operatorname{div} \mathrm{T}\, u = -u$ in the entire open ball $|x|<R$, or else for any two constants $\lambda_0 > \pi + \sqrt{2}$, $\lambda_1 > \sqrt{2}$, there holds*

$$-\lambda_0 r < U(r) + |u(r)| < \lambda_1 r \tag{4.64}$$

for all sufficiently small r.

Theorem 4.13. *Let $u(r)$ be a solution of (4.9) in $0<r<R$. Then either $u(r)$ can be extended to a solution of $\operatorname{div} \mathrm{T} u = -u$ in the open ball $|x|<R$, or for any $\gamma > \frac{1}{2}(\pi + \sqrt{2})^2$ there holds*

$$\sin\psi(r) > 1 - \gamma r^4 \tag{4.65}$$

for all sufficiently small r.

Again the proofs are technical and we do not present them here.

Although Theorem 4.11 ensures existence only in a sufficiently small neighborhood of $r=0$, computer calculations suggest that the solution can be continued indefinitely. Thus:

Conjecture 2. *$U(r)$ can be extended to a solution of* (4.9) *throughout the interval* $0<r<\infty$.

We observe next from Theorems 4.12 and 4.13 that any singular solution must have at $r=0$ exactly the growth order of the hyperbola $ru=-1$, which figured prominently throughout the discussion of the solutions of (4.8) for large $|u_0|$. In fact, it is shown in [34] that for any sequence $u_0^j \to \infty$, a subsequence converges, uniformly in compacta, to a solution of (4.8) without boundary points, limit sets or double points. The limit solution is asymptotic to the hyperbola as $r \to 0$ and projects simply onto the r-axis for large r. We are led to:

Conjecture 3. *The global solutions of* (4.8) *corresponding to initial datum u_0 converge as $u_0 \to -\infty$, uniformly with all derivatives in any compact set, to the* (*unique*) *singular solution* $U(r)$.

In Figs. 4.10–4.12 are shown the results of computer calculations that appear to support the conjectures.

There is a more striking form of Theorem 4.12, which emphasizes the constraint imposed on the solution by the nonlinearity in the equation, and which also does not require rotational symmetry [54]:

Theorem 4.14. *Let $u(x)$ be a solution of* $\operatorname{div} \mathsf{T} u = -u$ *in the punctured disk* $0<r<R$. *Suppose there exists r_0, $0<r_0<R$, such that on $|x|=r_0$ there holds*

$$u(x) \geq -\frac{1}{r_0} + \frac{\pi}{2} r_0. \tag{4.66}$$

Then for all x in $0<|x|\leq r_0$ there holds

$$u(x) \geq -\frac{1}{r_0} + \frac{\pi}{2} r_0 - \sqrt{4r_0^2 - r^2} + \sqrt{3}\, r_0. \tag{4.67}$$

If (4.66) *holds with the reverse inequality, then so does* (4.67).

Thus, if particular bounds above or below are known on a single circumference, then corresponding bounds must hold throughout the punctured disk. In particular, any solution whose growth rate at the singular point is significantly less than that of $U(r)$ must be bounded at the point.

We have also [54]:

Theorem 4.15. *Let*

$$\mu = \underset{0<\varepsilon<1}{\text{l.u.b.}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \frac{\alpha}{\sqrt{1-\alpha^2}}\, d\sigma$$

where

$$\alpha = \frac{1-\varepsilon+(1-\sigma)^2}{2(1+\sigma)}.$$

Let $u(x)$ *satisfy* $\operatorname{div} Tu = -u$ *in* $0<r<R$. *Then there is no* r_0, $0<r_0<\frac{1}{2}R$, *for which*

$$|u(x)| \geq \frac{1}{r_0} + \mu r_0$$

in $0<|x|\leq r_0$.

Thus, there is no solution with uniform growth rate exceeding that of $U(r)$.

Both results are proved by comparison with suitable roulades.

L. Simon has shown that there exist solutions of $\operatorname{div} Tu = -u$ that are bounded at isolated singular points.

4.14. Isolated Character of Global Solutions

It is possible to continue the solution $u(r; u_0)$ not only upward from $(0; u_0)$ as we have done, but also downward, simply by choosing the initial value of ψ to be π rather than zero. In that case the curve bends initially downward rather than upward (see Fig. 4.16(a)); if we join the

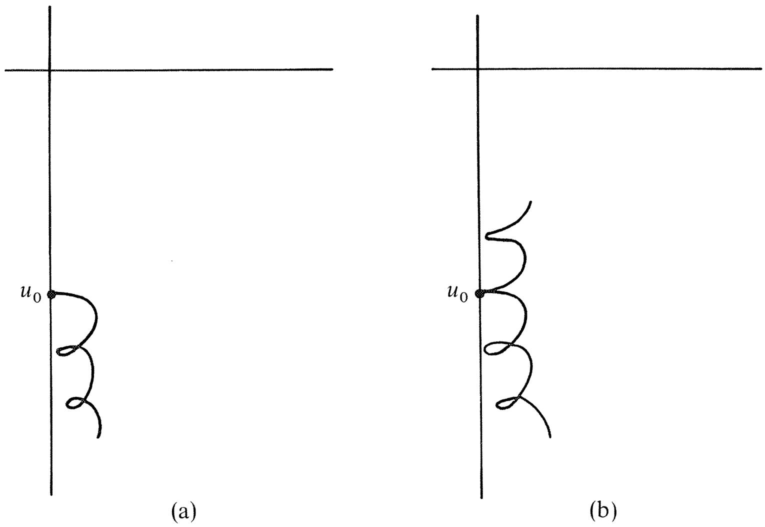


Figure 4.16. Continuation in extended sense (not to scale).

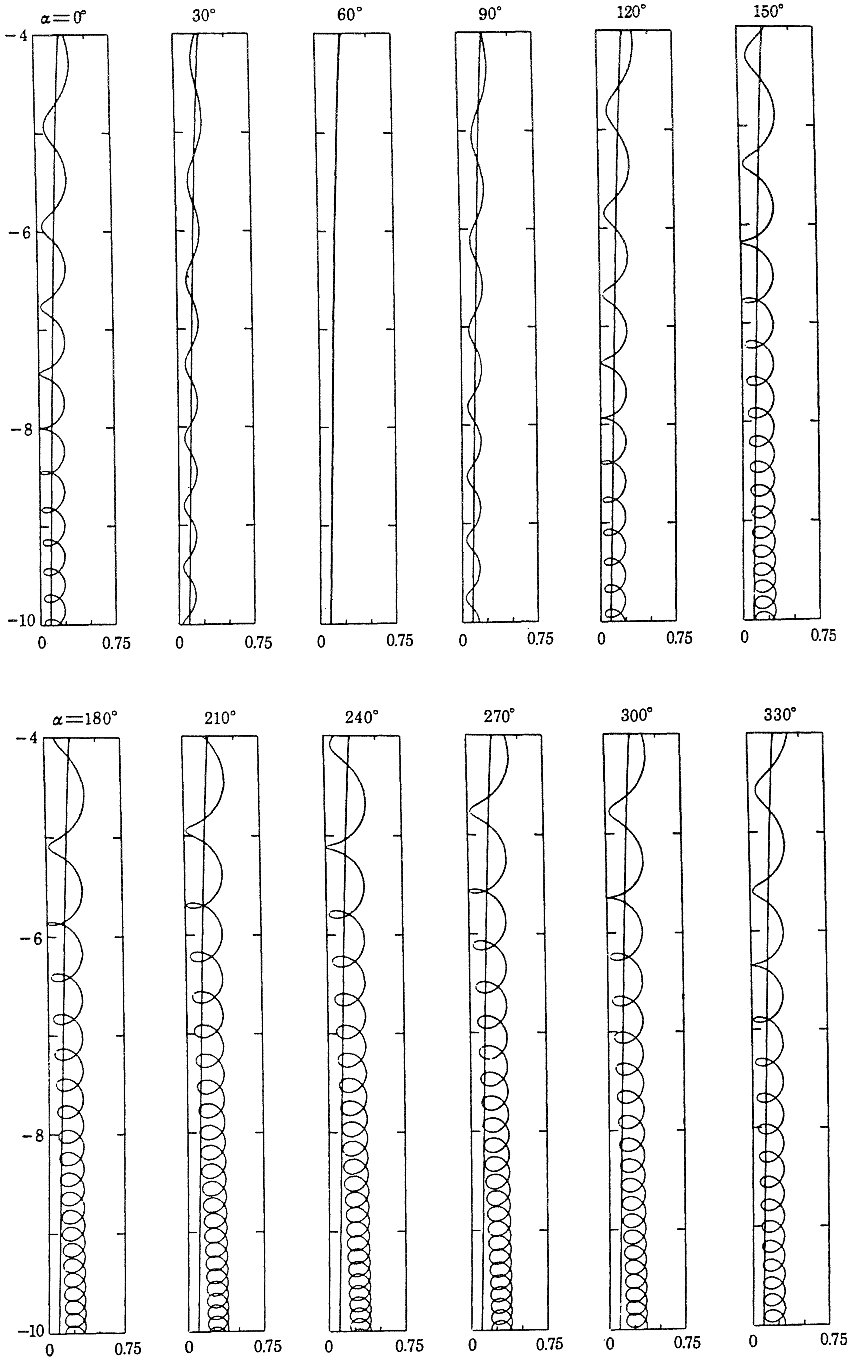


Figure 4.17. Singular solution and other solutions of (4.8).

two branches together, we obtain what can be considered as a single solution arc, in the extended sense that it passes through a singular point at $(0, u_0)$, at which ψ has a jump discontinuity. We justify the procedure by looking at a situation in which $|u_0|$ is large; at one step upward ψ changes abruptly near the u-axis, from $\psi \sim \pi$ on the upper "branch" to $\psi \sim 0$ on the lower one (see Fig. 4.16(b)).

The behavior of the global configuration is considered in [34] with the aid of computer calculations; the case $u_0 = -8$ is illustrated as the first curve in Fig. 4.17. Above the height u_0 the configuration looks roughly like a succession of roulades of ellipses, while below that height the appearance resembles roulades of hyperbolas, with the characteristic double points of those curves.

The solution curves of (4.8) in Fig. 4.17 were obtained by choosing as initial point p the intersection of $u(r; -8)$ with the singular solution, and then choosing initial angles in increments of $\pi/6$, measured counterclockwise from the arc of $u(r; -8)$ emanating from p in the direction of increasing u. One sees that in several of these curves a similar transition in behavior occurs, even though no contact is made with the u-axis (and thus no singular point appears). Thus it seems natural to consider the solutions we have constructed (with their extensions) as isolated limiting configurations of the manifold of all solutions of (4.8). If we adjoin these "ideal elements" and agree to consider them as global solutions, we are led to:

Conjecture 4. *The singular solution $U(r)$ is the only (nontrivial) solution of* (4.8) *that can be extended indefinitely without double points.*

4.15. Stability

There is a considerable literature on stability of pendent drops. From a theoretical point of view, the references [120, 146, 147, 18, 130, 26, 187] indicate some of the major contributions. The deepest and most definitive results are those of Wente [187]. Of the three problems studied by Wente, we restrict attention here to the drop of prescribed volume pendent from a horizontal plate.

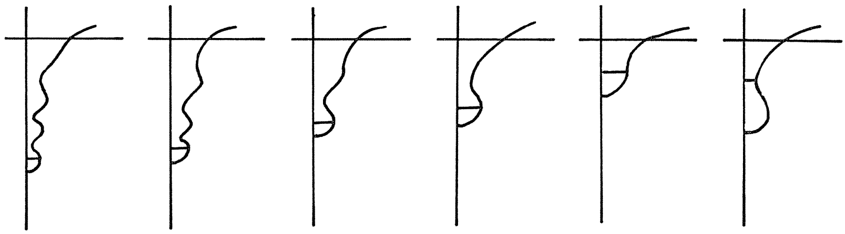
Wente adopts—as do his predecessors—a criterion of local static stability, according to which a configuration in static equilibrium is called stable if the second variation of energy is positive for all smooth variations of the surface that leave the volume unvaried. We indicate here some of his results. We denote by γ the angle of contact with the wall, measured within the fluid, and we remark that according to another theorem of Wente, the drop is symmetric about a vertical axis; thus the above results on existence and shape of the drops apply.

Theorem 4.16. a) *Suppose* $0<\gamma<\pi$. *Then for any sufficiently small volume* V, *there exist stable drops that are convex and resemble spherical caps. These drops are generated by profile curves whose tip is at* u_0, *where* $|u_0|$ *is large. As* V *increases, so does* u_0, *until an inflection point is reached. This drop is stable. With further increase of* V, u_0 *decreases; the drop profile contains an inflection but continues in some interval to remain stable. Instability occurs prior to appearance of a second inflection.*

b) *If* $\gamma=0$ *all profile curves contain an inflection. The curves are generated by starting with the solution* $u\equiv 0$, *then letting* u_0 *decrease and considering the portion of the curve up to the first maximum. The limit of stability is reached prior to the appearance of a vertical point, i.e., before* $u_0=-2\sqrt{2}$ (*cf. Lemma* 4.13).

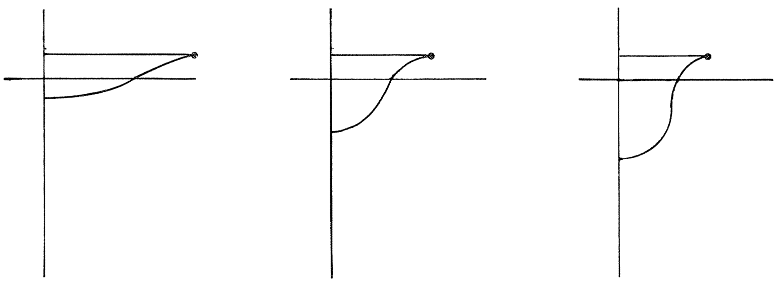
c) *For any angle of contact, the drop height increases monotonically with volume throughout the range of stability.*

These various steps in drop formation are illustrated in Figs. 4.18 and 4.19. Stable configurations are obtained from a small part of the drop



(a) (b) (c) (d) (e) (f)

Figure 4.18. Drop formation with increasing volume, $\gamma=\pi/2$. The line segments indicate the plane of support. Configurations (a)-(e) are always stable; (f) will be stable if the increase of volume from (e) is sufficiently small.



(a) b) (c)

Figure 4.19. Drop formation with increasing volume, $\gamma=0$. Configurations (a) and (b) may be stable; (c) is unstable, since vertical appears.

near the tip if $\gamma>0$, or from a portion of the drop without vertical points and extending to the first maximum, if $\gamma=0$.

It should be noted the theorem does not assert that all stable drops are obtained in the indicated way; in particular the uniqueness of a stable drop with given γ and V has not been proved. These properties are however reduced by Wente to a monotonicity property of a mapping, which is supported by computer calculations.

Wente also establishes stability criteria for two other configurations, in which the fluid drop hangs from a fixed circular aperture, under conditions either of constant pressure (the nearly empty medicine dropper) or of constant volume (the filled medicine dropper). In the latter case he shows that stable configurations can occur in which both a bulge and a neck appear, and he gives explicit geometric criteria for instability.

Notes to Chapter 4

1. §4.2. As was already mentioned, the local existence was first proved by Lohnstein [119]. An alternative proof, due to Concus and Finn [34], is based on the Schauder fixed point theorem and has the advantage that it makes no use of the analyticity properties of the equation; the method is thus better suited for characterizing other equations that share analogous properties. It also seems likely that the Picard iterative procedure, as developed by Johnson and Perko [105] for the case $\kappa>0$, could be used also in the present case. Brulois [21] observed that when $\kappa>0$, the Lohnstein-Wente proof can be simplified greatly by using the one-dimensional solution as a majorant.

2. Remark following (4.9). The case $u_0>0$ can be interpreted physically as describing the free surface of a bubble in equilibrium at the top of a tank filled with liquid.

3. Theorem 4.3. J.B. Serrin has proved that the successive extremal heights tend to zero (oral communication); cf. Note 11.

4. Lemma 4.6. The result $u_M<\sqrt{2}$ was first proved by Wente [187], using another method.

5. Theorem 4.5. The pendent drop solution does not provide a minimum for the potential energy (e.g., the energy can always be reduced by lowering most of the mass to a sufficiently small height and then connecting it to the plane by a thin tube containing the remaining mass). However, Giusti [87] and, independently, Gonzales, Massari, and Tamanini [92], obtained partial results by introducing a "floor" as obstacle and considering only configurations that lie on or above the floor. If the gravitational field is sufficiently small, they are then able to prove existence of a minimizing configuration that is in contact with the ceiling but not with the floor. Earlier results in this direction had already been obtained by Wente [185], who proved existence, for sufficiently small gravitational field, of a drop of liquid hanging from a prescribed aperture of general shape. As already indicated in connection with Chapter 3, an existence result obtained in this way yields simultaneously a stability property of the configuration.

6. §§4.4–4.11. All estimates are in nondimensional form. To obtain the corresponding dimensional results for a given physical configuration we need only multiply each spatial coordinate that appears in any relation by $\sqrt{\kappa}$. Thus, for a water drop hanging from a glass plate in vacuo in the earth's gravitational field, for which situation one has $\gamma \approx 0$ and $\kappa \approx 29$, we find $\delta \approx 0.46$ cm, $\mu \approx 0.54$ cm, $r_M < 0.99$ cm, and $V^* \approx 0.81$ cm^3. These values, although they are within the range of reality, are certainly not sharp. The results do provide, however, general qualitative information that could not have been obtained from methods that were previously employed.

7. §4.11. Lohnstein [120] was apparently the first to suggest that there should be a finite upper bound for the volumes of all physical pendent drops, although it seems likely he had in mind only those drops up to a (not clearly defined) limit of stability. Lohnstein supported his view by a procedure that was in part numerical. Again from a numerical point of view, the question was considered in full generality by Hida and Nakanishi [100], who made computer calculations of the volumes of drops, as function of the parameter u_0. They found an oscillating pattern that seemed to approach a finite limit as $u_0 \to -\infty$. Later calculations by Concus with more sophisticated computers, of V and also of R (Figs. 4.13 and 4.14), suggest that either these quantities converge very slowly to the corresponding values for the singular solution $U(r)$, or else they oscillate about those values. The computer results provide support for the conjectures of §4.13, at least in the sense that they do not conflict with the conjectures.

The first attempt to obtain a global description of pendent drops seems due to Bashforth and Adams [8], who used the problem to test a new numerical procedure. Lohnstein was aware of this work but overlooked the section on pendent drops. In a later note [120] he remarks "... entnahm ich kürzlich das genannte Werk der hiesigen Königlichen Bibliothek, und da fand ich dann die erwähnte Tabelle, deren rechtzeitige Benutzung mir viel Arbeit gespart hätte".

Kelvin [179] proposed a geometric integration procedure; he was the first to discover global solutions with repeated bulges. His result in a specific case is reproduced in Fig. 4.20.

Modern computing methods have facilitated the numerical integration

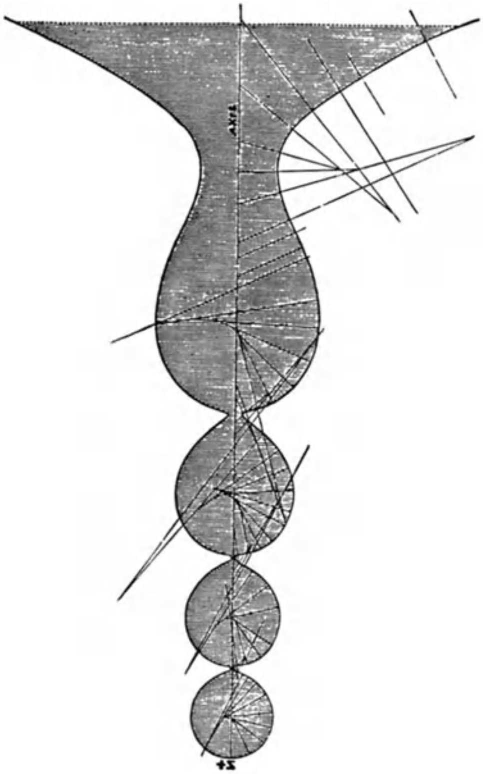


Figure 4.20. Pendent drop (Thomson, 1886).

of (4.8), and many particular cases in agreement with Kelvin's discovery have since been worked out, see, e.g., [103, 100, 144, 96, 34]. Hida and Nakanishi [100] were apparently the first to find configurations with a large number of bulges. Although such solutions are globally unstable, individual portions of them can appear as stable configurations that would be physically observed, cf. Theorem 4.16.

8. The bound (4.61) for the volume has an interesting heuristic consequence. Imagine a symmetric sessile drop of volume exceeding V^*; the existence of such a drop as a formal solution of the equations follows from Theorem 3.2. We now rotate the plane Π of support about an axis through the diameter of the wetted disk D, supposing that the adhesion forces suffice to keep the wetted surface unchanged. It is shown in [162, 67] (see also Chapter 8, Theorem 8.3) that the contact angle distribution cannot remain constant when Π is tilted; nevertheless it seems reasonable to suppose that, at least for small angles of tilt, a formal solution of the basic equations continues to exist. If Π could be rotated through an angle π, we would obtain a pendent drop configuration, which by

Wente's theorem [186] must be symmetric and governed by the system (4.8). But by Theorem 4.9, no such solution exists. Thus, at some angle φ_0 of tilt, the formal solution must either penetrate Π or develop an instability of a sort that prevents its further continuation. The latter behavior would differ from what was encountered in this chapter, in which the formal solution in its dependence on u_0 does not seem cognizant of any instability. Some evidence of such noncontinuability was in fact encountered by Milinazzo (see the Appendix in [72], also the discussion in §8.15).

9. §4.15. In his discussion of stability, Wente restricts himself to variations directed normal to the surface. Since such a variation of the (physical) surface will in general not maintain contact with the supporting plane, he extends the original surface through the plane and makes an appropriate variation of the extension. The cases $\gamma=0, \pi$ require separate discussion. The procedure is satisfactory for the particular problem considered, in the sense that every neighboring surface that can be achieved by a variation of the type considered in Chapter 1 can be achieved also (when $\gamma \neq 0, \pi$) by such a normal variation; however, in a general situation, e.g., if Π is replaced by a curved surface, the procedure would seem to require more differentiability hypotheses than should be needed. A formula for the second variation corresponding to the variations introduced in Chapter 1 does not seem to appear in the literature.

The Wente procedure would not be satisfactory from the point of view of establishing the equilibrium equations from physical principles (see Chapter 1) as it does not correspond to the principle of virtual work, which requires that physical particles be mapped into physical particles.

10. A further class of symmetric capillary surfaces with striking properties was investigated by Vogel [182]; see also his later papers [183, 184].

11. *Added in proof*: M.F. Bidaut-Veron has proved **Conjecture 2** (§4.13) and has established a uniqueness result that overlaps Theorems 4.12 and 4.13. Her proof yields also that the successive extremal heights tend to zero (cf. Note 3).

Chapter 5

Asymmetric Case; Comparison Principles and Applications

5.1. The General Comparison Principle

We wish to estimate the behavior of capillary surfaces in tubes of general section. We approach the problem in the spirit of the preceding chapter, and attempt to gain information by comparison of a given solution with an explicitly known "near solution" of the equation. Since symmetry will no longer be assumed, we need a more sophisticated tool than Lemma 4.11, and we have recourse to a form of the maximum principle for elliptic equations. The statement we give is formally analogous to the familiar maximum principle for the Laplace operator; however there is an important new feature, which reflects itself in some striking distinctions, between the behavior of capillary surfaces and that of surfaces determined by the solutions of linear elliptic equations.

The material of this chapter will for the most part be restricted to the case $\kappa > 0$; for purposes of later reference some of the results will be formulated to include also the case $\kappa = 0$. We restrict attention to graphs over a base domain Ω; as in §1.9, we are led to an equation

$$\operatorname{div} \mathsf{T}u = \kappa u, \qquad \mathsf{T}u = \frac{1}{\sqrt{1+|\nabla u|^2}} \nabla u \tag{5.1}$$

in Ω, under the boundary condition

$$\nu \cdot \mathsf{T}u = \cos\gamma \tag{5.2}$$

on $\Sigma = \partial\Omega$. We write $\mathsf{N}u \equiv \operatorname{div} \mathsf{T}u - \kappa u$.

Theorem 5.1 (Concus and Finn [30]). *Suppose $\kappa \geq 0$, $\mathsf{N}u \geq \mathsf{N}v$ in Ω. We suppose Σ admits a decomposition $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_0$, such that*

$$\begin{aligned} v &\geq u && \text{on } \Sigma_\alpha \\ \nu \cdot \mathsf{T}v &\geq \nu \cdot \mathsf{T}u && \text{on } \Sigma_\beta \end{aligned}$$

and Σ_0 can be covered, for any $\varepsilon>0$, by a countable number of disks B_{δ_i} of radius δ_i, such that $\Sigma\delta_i<\varepsilon$. It is assumed that $\Sigma_\beta\in C^{(1)}$, however no regularity hypotheses are needed on Σ_α or Σ_0. We conclude:

i) *if $\kappa>0$ or if $\Sigma_\alpha\neq\emptyset$, then $v\geq u$ in Ω; equality holds at any point if and only if $v\equiv u$.*
ii) *if $\kappa=0$, $\Sigma_\alpha=\emptyset$, then $v(x)\equiv u(x)+$const. in Ω.*

N.B. If $\Sigma_0=\emptyset$ the result is (essentially) the classical maximum principle. But if Σ_0 contains even a single point, the statement would be false for linear equations unless growth hypotheses at Σ_0 are introduced. The result as stated thus depends essentially on the particular nonlinearity in the equation. We note the hypothesis on Σ_0 is equivalent to the requirement that Σ_0 have one-dimensional Hausdorff measure zero.

Proof of Theorem 5.1. Suppose there were a subset of Ω in which $u>v$. Then, for ε small enough, the set

$$\Omega_0^M(\varepsilon)=\{x\in\Omega: 0<u-v<M\}\cap\{\Omega\setminus\cup B_{\delta_i}\}$$

would be non-null for some $M>0$. Define Ω^-, Ω^+ analogously, corresponding to $u-v<0$ and $u-v>M$, set $\Lambda=\Omega\cap\partial\{\bigcup B_{\delta_i}\}$ and define the function $w(x)$ in Ω by

$$w=\begin{cases}0 & \text{in } \Omega^-\\ u-v & \text{in } \Omega_0^M\\ M & \text{in } \Omega^+.\end{cases}$$

Set $\Omega_\varepsilon=\Omega\setminus\bigcup B_{\delta_i}$. Then (see Fig. 5.1)

$$\begin{aligned}0\leq\int_{\Omega_\varepsilon} w(\mathsf{N}u-\mathsf{N}v)\,dx=&-\int_{\Omega_\varepsilon}\nabla w\cdot(\mathsf{T}u-\mathsf{T}v)\,dx\\ &-\kappa\int_{\Omega_\varepsilon} w(u-v)\,dx+\int_\Lambda w(\mathsf{T}u-\mathsf{T}v)\cdot\nu\,ds\\ &+M\int_{\Sigma^+}(\mathsf{T}u-\mathsf{T}v)\cdot\nu\,ds+\int_{\Sigma_0^M} w(\mathsf{T}u-\mathsf{T}v)\cdot\nu\,ds\end{aligned}\tag{5.3}$$

or, assigning symbols to the integrals on the right in order of appearance,

$$0\leq -Q-W+\mathscr{I}_\Lambda+\mathscr{I}^++\mathscr{I}_0^M.\tag{5.4}$$

We have clearly $W\geq0$, equality holding only if $\kappa=0$. Since $\Sigma^+\subset\Sigma_\beta$ we find $\mathscr{I}^+\leq0$. Also, all points of Σ_0^M at which $w\neq0$ are in Σ_β; hence $\mathscr{I}_0^M\leq0$. With regard to Q, we may write

$$Q=\int_{\Omega_0^M}\nabla(u-v)\cdot(\mathsf{T}u-\mathsf{T}v)\,dx.$$

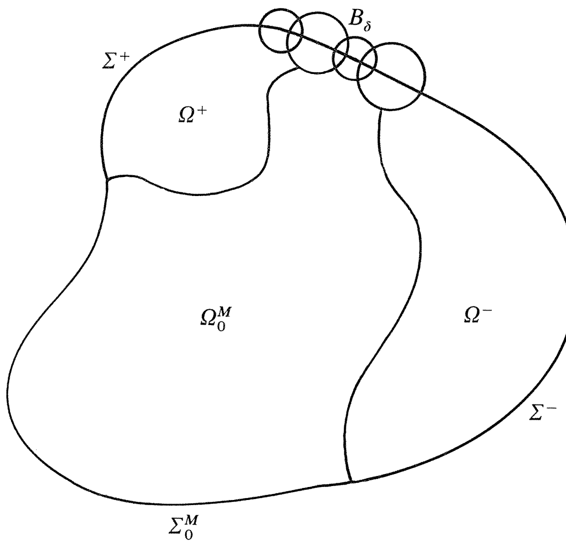


Figure 5.1. Construction for proof of Theorem 5.1.

We write $\nabla u = p$, $\nabla v = q$, $\mathsf{T}u = \mathsf{A}(p)$. Consider the function

$$\mathscr{F}(\tau) = (p-q)\cdot[\mathsf{A}(q+\tau(p-q)) - \mathsf{A}(q)], \qquad 0 \le \tau \le 1.$$

We have $\mathscr{F}(0)=0$, $\mathscr{F}(1) = \nabla(u-v)\cdot(\mathsf{T}u - \mathsf{T}v)$. Since

$$\mathscr{F}'(\tau) = \frac{|p-q|^2}{[1+|q+\tau(p-q)|^2]^{3/2}} \ge 0$$

we conclude at once that $Q \ge 0$, and equality holds only if $\nabla u \equiv \nabla v$ in Ω_0^M.

Finally we observe that $|\mathsf{T}f| < 1$ for arbitrary f with finite first derivatives; it follows that $|\mathscr{I}_\Lambda| < 4\pi M\varepsilon$.

Letting $\varepsilon \to 0$, we find from (5.4)

$$\lim_{\varepsilon\to 0}(Q+W) \le 0$$

from which, since both integrands are nonnegative, $Q = W = 0$ for any ε.

If $\kappa > 0$, we conclude at once that the construction of Ω_0^M is not possible for any ε, M; that is, $v \ge u$ in Ω. If $\kappa = 0$ the procedure tells us that if there is a subset on which $v < u$ then $v \equiv u + \text{const.}$ in Ω_0^M. It follows that $\lim_{\varepsilon\to 0}\Omega_0^M = \Omega$, so that $v \equiv u + \text{const.}$ in Ω, and either $\Sigma_\alpha = \emptyset$ or $v \equiv u$ in Ω. However, if $\kappa = 0$ and $\Sigma_\alpha = \emptyset$, then an arbitrary constant can be added to v without changing the hypotheses. Thus, in every such case there holds $v \equiv u + \text{const.}$, as was to be shown.

It remains to show that in every case for which $v \geq u$ in Ω, equality at any point implies equality throughout Ω. To do so, we use the following result, due to Hopf [101]:

Lemma 5.1. *Let $w(x)$ satisfy an inequality*

$$a(x,y)\,w_{xx} + 2b(x,y)\,w_{xy} + c(x,y)\,w_{yy} + d(x,y)\,w_x + e(x,y)\,w_y - \kappa w \geq 0 \tag{5.5}$$

in an open disk D, with $\kappa \geq 0$, $ac - b^2 > \delta > 0$. Suppose $w < 0$ in D, $w(p) = 0$ at some $p \in \partial D$ at which w is continuous. Then the normal derivative $\partial w/\partial v]_p > 0$. (If w is not differentiable at p, the lower derivate is positive.)

In our case, the inequality $\mathsf{N}u \geq \mathsf{N}v$ can be put into the form (5.5) by setting $w = u - v$ and using the mean value theorem. Suppose $w \leq 0$ in Ω, $w \not\equiv 0$, and suppose there is a non-null set $P \subset \Omega$ on which $w = 0$. There would then be a point $q \in \Omega \backslash P$, whose distance to $\partial\Omega$ exceeds its distance to P. The open disk D about q, of maximal radius such that $D \subset \Omega \backslash P$, would then have positive distance from $\partial\Omega$ and would contain on its boundary a point $p \in P$. Applying Lemma 5.1 at p, we find $\partial w/\partial v > 0$ at p, which would imply the existence of points in Ω at which $w > 0$. This contradiction completes the proof of Theorem 5.1.

Corollary 5.1. *If $\kappa > 0$, the solution of the capillary problem* (5.1), (5.2) *for a domain Ω, with smooth boundary up to an exceptional set Σ_0, is unique. If $\kappa = 0$, the solution is unique up to an additive constant.*

Proof. If $\mathsf{N}u = 0$, $\mathsf{N}v = 0$, then $\mathsf{N}u = \mathsf{N}v$. By (5.2), $v \cdot \mathsf{T}u = v \cdot \mathsf{T}v$ on $\Sigma_\beta = \Sigma \backslash \Sigma_0$, and $\Sigma_\alpha = \emptyset$. By Theorem 5.1, if $\kappa > 0$ then $v \geq u$ and also $u \geq v$ in Ω, so that $u \equiv v$. If $\kappa = 0$, we apply the theorem to obtain $u \equiv v + \text{const.}$ in Ω.

5.2. Applications

Theorem 5.2 (Concus and Finn [31]). *Let u define a capillary surface over Ω, so that* (5.1) *holds with $\kappa > 0$. Let B_δ be a disk of radius δ, with $B_\delta \subset \Omega$, as in Fig.* 5.2. *Then*

$$u < \frac{2}{\kappa\delta} + \delta \tag{5.6}$$

in B_δ.

The theorem asserts that there is no way in which the boundary of Ω or the contact angle could be changed outside B_δ, so as to raise liquid above the bound (5.6) in B_δ.

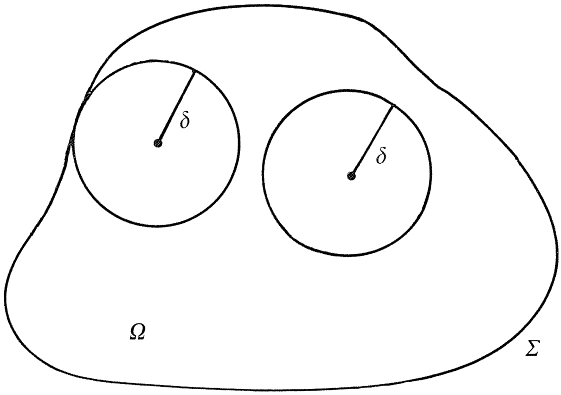


Figure 5.2. Comparison disks for Theorem 5.2.

Proof. Choose δ', $0<\delta'<\delta$, let $B_{\delta'}$ be a disk of radius δ', concentric to B_δ, and let v' denote a lower hemisphere over $B_{\delta'}$, whose lowest point has the height $v'_0=2/\kappa\delta'$. Then $B_{\delta'}\Subset\Omega$, and

$$\operatorname{div}\mathsf{T}v'=\frac{2}{\delta'}=\kappa v'_0\leq\kappa v' \quad \text{in } B_{\delta'}$$

$$v\cdot\mathsf{T}u<1, \qquad v\cdot\mathsf{T}v'=1 \quad \text{on } \Sigma_{\delta'}=\partial B_{\delta'}.$$

We apply Theorem 5.1, with $\Sigma_\beta=\Sigma_{\delta'}$, $\Sigma_\alpha=\Sigma_0=\emptyset$. Thus

$$u<v'<\frac{2}{\kappa\delta'}+\delta'$$

in $B_{\delta'}$. Letting $\delta'\to\delta$, the result follows.

Corollary. *If every point of Ω lies interior to a disk $B_\delta\subset\Omega$, then* (5.6) *holds throughout Ω.*

Theorem 5.3. *Let Σ have a continuous normal except perhaps on a set Σ_0 as above. Let u define a capillary surface S over Ω, which meets the bounding cylinder Z over $\Sigma\backslash\Sigma_0$ in the angle γ, so that $v\cdot\mathsf{T}u=\cos\gamma$ on $\Sigma\backslash\Sigma_0$. Let B_δ be a disk of radius δ such that $\Omega\cap B_\delta\neq\emptyset$ and such that a lower hemisphere v over B_δ meets Z (if at all) in angles not exceeding γ; that is, $v\cdot\mathsf{T}v\geq\cos\gamma$ on $(\Sigma\backslash\Sigma_0)\cap B_\delta$ (see Fig.* 5.3). *Then* (5.6) *holds in $\Omega\cap B_\delta$.*

Proof. We wish to apply Theorem 5.1 in the domain $\Omega\cap B_\delta$. As set Σ_β we may choose the set $\Sigma_\delta\cap\Omega$, together with the set $(\Sigma\backslash\Sigma_0)\cap B_\delta$. The remaining set Σ_0^* on the boundary of $\Omega\cap B_\delta$ consists of $\Sigma_0\cap\Omega$ and the set of endpoints of maximal open intervals of $\Sigma_\delta\cap B_\delta$. The set of such intervals

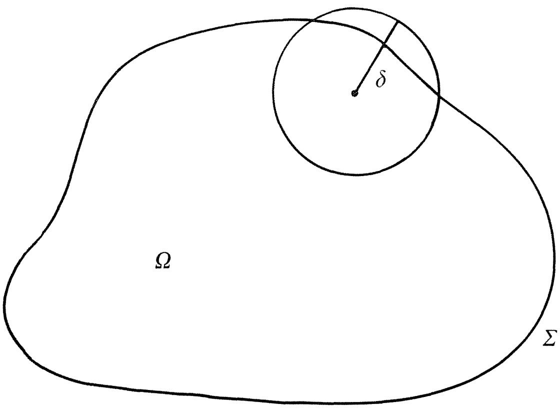


Figure 5.3. Comparison disk for Theorem 5.3.

is countable, hence so is the associated set of endpoints, hence this set has zero Hausdorff measure; thus Theorem 5.1 applies and the result follows.

Theorem 5.4 ([31]). *Suppose Ω has a corner with opening angle 2α, suppose $\mathsf{N}u=0$ in Ω, $\nu\cdot\mathsf{T}u=\cos\gamma$ on $\Sigma^*\setminus V$, where Σ^* is the subset of Σ cut out by B_δ as indicated in Fig. 5.4. Suppose in addition that $\alpha+\gamma\geq\pi/2$. Then* (5.6) *holds throughout $\Omega^*=\Omega\cap B_\delta$.*

Proof. A lower hemisphere v over B_δ makes a constant angle γ_0 with a vertical cylinder over Σ^* (except at the isolated point V where the angle

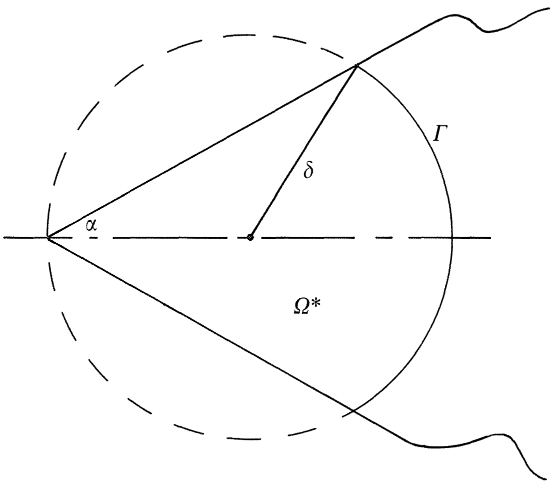


Figure 5.4. Comparison disk for Theorem 5.4.

is not defined), and $\gamma_0 = (\pi/2) - \alpha$ (see figure). The result is thus a corollary of Theorem 5.3 above.

N.B. This result holds for *every* solution u in Ω that satisfies $\nu \cdot Tu = \cos\gamma$ on $\Sigma^* \setminus V$. No growth condition at V is needed, nor is any bound imposed for u on Γ. The reasoning however fails if $\alpha + \gamma < \pi/2$. In fact, in this case the solutions behave quite differently.

Theorem 5.5 ([31]). *In the corner configuration above, suppose $\alpha + \gamma < \pi/2$. Let r, θ be polar coordinates centered at V, set $k = \sin\alpha/\cos\gamma$. There exists a constant A, not depending on the particular solution considered, such that*

$$\left| u - \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{k\kappa r} \right| < A \tag{5.7}$$

in Ω^*.

Proof. We seek a "near solution" in the form of a function whose level curves are circular arcs that meet Σ in the angle γ, see Fig. 5.5. Any such function has the form

$$f\left(\frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{r}\right), \tag{5.8}$$

and any function $f(x)$ with $f'(x) \neq 0$ yields the required property. Choosing $f(x) \equiv x$, and setting

$$v = \frac{1}{k\kappa} f \tag{5.9}$$

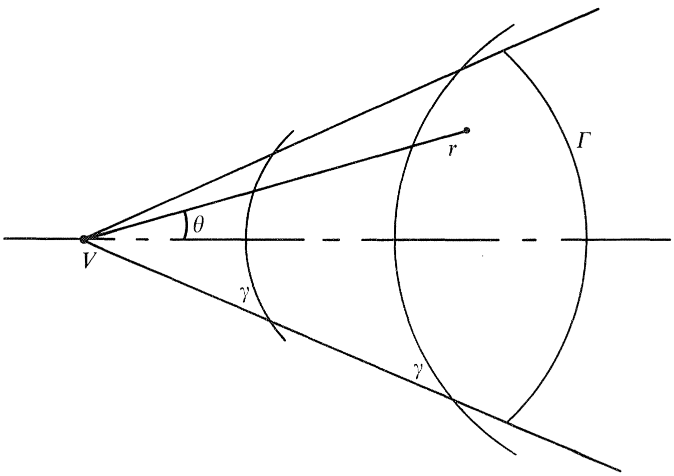


Figure 5.5. Coordinate system and Ansatz circles.

with the above indicated argument, we find after some calculation

$$\operatorname{div} \mathsf{T} v = \kappa v + \eta, \qquad |\eta| = O(r^3) \tag{5.10}$$

in Ω, and

$$\nu \cdot \mathsf{T} v = \cos\gamma - \zeta, \qquad \zeta = O(r^4),\ \zeta > 0 \tag{5.11}$$

on $\Sigma^* \setminus V$.

The arc Γ can be covered by a finite number of balls $B_{\delta_i} \subset \Omega$, hence by Theorem 5.2 above there is a constant M such that for any solution u in Ω, $|u| < M$ on Γ. Setting $w = v - A'$, we have

$$\operatorname{div} \mathsf{T} w = \kappa w + (\eta + \kappa A') \tag{5.12}$$

in Ω^*. We choose A' so that the two conditions

$$\left.\begin{aligned} w < u &\quad \text{on } \Gamma \\ \eta + \kappa A' > 0 &\quad \text{in } \Omega^* \end{aligned}\right. \tag{5.13}$$

are both satisfied.

Since $\nu \cdot \mathsf{T} v < \cos\gamma$ on $\Sigma^* \setminus V$, we may apply Theorem 5.1 (Σ_α becomes now the arc Γ) to obtain

$$u - v > -A' \tag{5.14}$$

in Ω^*.

If $\gamma \neq 0$ and we replace γ in the argument of v by any $\hat{\gamma} < \gamma$, we will have

$$\nu \cdot \mathsf{T} \hat{v} > \cos\gamma \tag{5.15}$$

on $\Sigma^* \setminus V$ for all sufficiently small δ. Having chosen δ we may choose A'' so that, now setting $w = \hat{v} + A''$, we will have

$$\left.\begin{aligned} w > u &\quad \text{on } \Gamma \\ \eta - \kappa A'' < 0 &\quad \text{in } \Omega^*. \end{aligned}\right. \tag{5.16}$$

Theorem 5.1 now yields

$$u - \hat{v} < A'' \tag{5.17}$$

in Ω^*.

An estimate of this form holds for any $\hat{\gamma} < \gamma$; we wish to obtain a corresponding result for γ. To do so, choose $r_0 < \min\{1, \delta\}$ and sufficiently small that $\hat{\gamma}$ can be chosen to satisfy $\cos\hat{\gamma} > \cos\gamma + C r_0^4$ for some constant C. We choose $\hat{\gamma}$ in the range

$$\cos\gamma + C r_0^4 < \cos\hat{\gamma} < \cos\gamma + 2 C r_0^4 \tag{5.18}$$

and observe from the explicit form of $v(\cdot;\gamma)$ that there is a constant C_0 such that (5.18) implies

$$\hat{v} \equiv v(\cdot;\hat{\gamma}) < v(\cdot;\gamma) + C_0 \frac{r_0^4}{r} \tag{5.19}$$

uniformly in $r \le r_0$, and with C_0 independent of r_0 in the range considered. In particular, in the interval $r_0^2 \le r \le r_0$ there holds

$$v(\cdot;\hat{\gamma}) < v(\cdot;\gamma) + C_0 r_0^2 \tag{5.20}$$

and hence by (5.17)

$$u < v(\cdot;\hat{\gamma}) + A'' < v(\cdot;\gamma) + A'' + C_0 r_0^2. \tag{5.21}$$

Using again (5.18), we may choose $\hat{\gamma}_1$, $\hat{\gamma} \le \hat{\gamma}_1 \le \gamma$, such that for $r \le r_0^2$ there holds

$$v(\cdot;\hat{\gamma}_1) < v(\cdot;\gamma) + C_0 \frac{r_0^8}{r} \tag{5.22}$$

so that in the range $r_0^4 \le r \le r_0^2$ there holds

$$v(\cdot;\hat{\gamma}_1) < v(\cdot;\gamma) + C_0 r_0^4. \tag{5.23}$$

We note $(\partial/\partial(\cos\gamma))v > 0$; thus $v(\cdot;\hat{\gamma}_1) > v(\cdot;\gamma)$ and it follows that on the arc $r = r_0^2$

$$u - v(\cdot;\hat{\gamma}_1) < A'' + C_0 r_0^2. \tag{5.24}$$

We now apply the proof of (5.17), in the domain $r < r_0^2$ and with $\hat{\gamma}$ replaced by $\hat{\gamma}_1$, to obtain

$$\begin{aligned} u &< v(\cdot;\hat{\gamma}_1) + A'' + C_0 r_0^2 \\ &< v(\cdot;\gamma) + A'' + C_0(r_0^2 + r_0^4) \end{aligned} \tag{5.25}$$

in the range $r_0^4 \le r \le r_0^2$. Iteration of the procedure, with r_0^2 replaced successively by r_0^4, r_0^8, ..., yields the estimate, for all $r \le r_0$,

$$u < v + A'' + C_0 \frac{r_0^2}{1 + r_0^2} \tag{5.26}$$

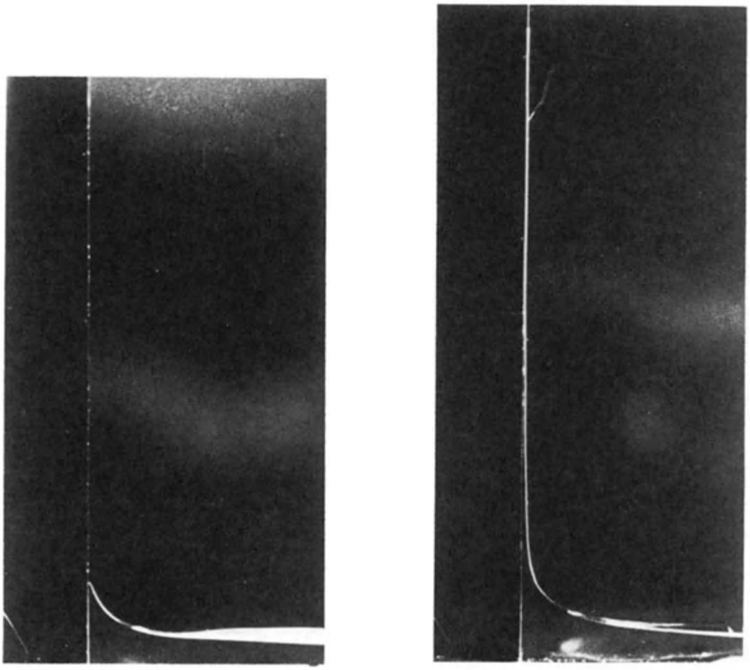
which completes the proof of (5.7) in the case $\gamma > 0$.

Finally, suppose $\gamma = 0$. In this case the boundary condition holds exactly for v, that is, we may set $\zeta \equiv 0$ in (5.11). Thus, there is no need to introduce $\hat{\gamma}$ and the original reasoning yields the desired estimate.

Corollary 5.5. *Capillary surfaces in a domain with corner depend discontinuously on the boundary data. That is, throughout the closed interval* $\pi/2 \geq \gamma \geq \pi/2 - \alpha$ *every solution* u *satisfies* $u < (2/\kappa\delta) + \delta$ *at all points of* Ω^*, *whereas for any* $\gamma < \pi/2 - \alpha$ *there holds* $u \to \infty$ *for any approach to* V *in the corner.*

The discontinuous dependence can be verified experimentally with the help of two acrylic plastic plates and a level surface. Figure 5.6 shows an experiment performed by Tim Coburn in the Medical School at Stanford University. A small amount of water was placed in a wedge opening formed by the plates, and the opening was then varied slightly about the critical angle. In Fig. 5.6(a) the opening is just above critical and the maximal rise height of the water is seen to be slightly under the bound given by (5.6). In Fig. 5.6(b) the opening has been closed about 2°. The observed rise height is more than ten times the bound predicted by (5.6).

The existence of solutions in domains with corners was shown by Emmer [46] for the case $\alpha + \gamma > \pi/2$, and later by Finn and Gerhardt [68] without restriction on angle (cf. §7.10).



(a) (b)

Figure 5.6. Discontinuous dependence on boundary angle. (a) $\alpha \geq$ critical. (b) $\alpha <$ critical.

Theorem 5.4 yields boundedness at a corner when $\alpha+\gamma\geq\pi/2$, but provides no further information on local regularity of the solution surfaces. This question was taken up concomitantly and independently from different points of view by Korevaar [107] and by Simon [166]. Simon showed that if $\alpha+\gamma>\pi/2$ and $\alpha<\pi/2$, then $u(x)$ is of class $C^{(1)}$ up to V. Later Tam [173] obtained the deeper result:

Theorem 5.6. *If* $\alpha+\gamma\geq\pi/2$ *and* $\alpha<\pi/2$, *then the normal vector to* S *is continuous up to* V.

The result of Korevaar indicated above clarifies the role of the hypothesis $\alpha<\pi/2$ and shows that Theorem 5.6 is definitive. Korevaar proved:

Theorem 5.7. *If* $\alpha>\pi/2$ *at a vertex* V, *then there exist solutions of* (5.1) *and* (5.2), *defined in a neighborhood of* V *in* Ω, *that are discontinuous at* V.

It should be noted that in view of Theorem 5.3, only bounded discontinuities can occur.

To simplify notation, we present the reasoning only in the special case $\alpha=3\pi/4$. Given the vertex V with angle 2α, we extend it to a bounded domain Ω as indicated in Fig. 5.7. Here $\varepsilon>0$ is to be determined, Ω is to be smooth except for V and its reflection V'. According to the results of Emmer and of Finn and Gerhardt referred to above, for any γ in the

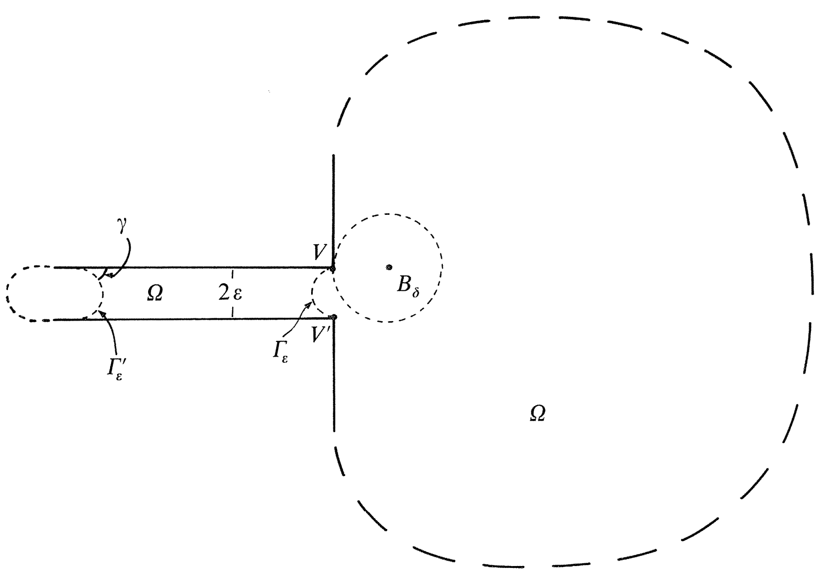


Figure 5.7. Geometry of base domain for Theorem 5.7.

range $0 \leq \gamma < \pi/2$ there is a unique solution of (5.1), (5.2) in Ω. We consider that solution in its dependence on ε, all other parameters being fixed.

We first observe that for any fixed sufficiently small $\delta > 0$ there holds by Theorem 5.2

$$u < \frac{2}{\kappa \delta} + \delta \tag{5.27}$$

in B_δ (see Fig. 5.7), uniformly as $\varepsilon \to 0$. Now consider a circular arc Γ_ε (of radius $\varepsilon \sec \gamma$) making equal angles γ with the parallel segments, as indicated in Fig. 5.7. We extend the arc to half the underside of a torus, whose principal radii are R, $\varepsilon \sec \gamma$ (see Fig. 5.8), and which meets the "strip" region again in a reflection Γ_ε' of Γ_ε. This torus has mean curvature satisfying

$$H \geq \frac{1}{2}\left(\frac{1}{\varepsilon} - \frac{1}{R-\varepsilon}\right); \tag{5.28}$$

thus, if we denote its defining function by v_ε, we will have

$$\operatorname{div} \mathrm{T} v_\varepsilon = 2H \geq \left(\frac{1}{\varepsilon} - \frac{1}{R-\varepsilon}\right). \tag{5.29}$$

Let us move the torus vertically, until its maximum height $\hat{v}$ satisfies

$$\hat{v} = \frac{1}{\kappa}\left(\frac{1}{\varepsilon} - \frac{1}{R-\varepsilon}\right). \tag{5.30}$$

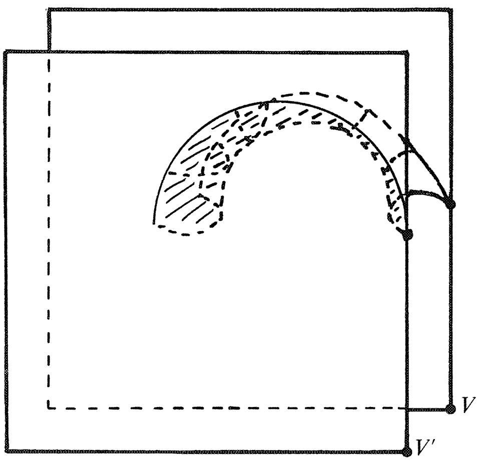


Figure 5.8. Comparison torus for Theorem 5.7.

We will then have

$$\operatorname{div} \mathsf{T} v_\varepsilon \geq \kappa \hat{v} \geq \kappa v_\varepsilon \tag{5.31}$$

throughout the region of definition of v_ε.

The torus meets the bounding walls over the parallel lines in the constant angle γ, thus

$$\nu \cdot \mathsf{T} v_\varepsilon = \cos\gamma \tag{5.32}$$

on these lines. On Γ_ε and Γ_ε' there holds

$$\nu \cdot \mathsf{T} v_\varepsilon = -1. \tag{5.33}$$

If we choose the four intersection points of Γ_ε, Γ_ε' with the boundary as the singular set Σ_0, then the conditions of Theorem 5.1 (with u and v interchanged) will be satisfied over the domain $\Omega_\varepsilon \subset \Omega$ covered by v_ε; we conclude $u > v_\varepsilon$ in Ω_ε. But $v_\varepsilon \geq \hat{v} - R$ at all points of definition, and thus the solution u satisfies

$$u > \frac{1}{\kappa}\left(\frac{1}{\varepsilon} - \frac{1}{R-\varepsilon}\right) - R \tag{5.34}$$

throughout Ω_ε. Comparing the inequalities (5.27) and (5.34), we see that for small enough ε, u must be discontinuous at V, V'.

5.3. Domain Dependence

We now consider the following question: does a "small" capillary tube always raise liquid to a higher level than does a "big" one? To express the idea precisely, consider domains Ω^i, Ω^0, with $\Omega^i \subset \Omega^0$, $\Omega^i \neq \Omega^0$, and solutions u^i in Ω^i, u^0 in Ω^0, with the same constant data γ on Σ^i, Σ^0. The question then becomes: for $x \in \Omega^i$, is $u^i(x) > u^0(x)$? A related question is: does u^i raise a larger volume of liquid than does u^0 over the section Ω^i?

We consider first some special situations.

Theorem 5.8 ([55]). *If Ω^0 is a disk, then always $u^i > u^0$ in Ω^i.*

Proof. Let ν^0 denote, at any point in Ω^0, a radially directed unit vector. By the symmetry of u^0, we have

$$\nu^i \cdot \mathsf{T} u^0]_{\Sigma^i} \leq \nu^0 \cdot \mathsf{T} u^0]_{\Sigma^i}$$

where ν^i is the unit exterior normal on Σ^i. By the convexity of the surface u_0, we find

$$\nu^0 \cdot \mathsf{T} u^0]_{\Sigma^i} \leq \nu^0 \cdot \mathsf{T} u^0]_{\Sigma^0} = \cos\gamma,$$

equality holding only at points of Σ^0.

Therefore

$$\nu^i \cdot \mathsf{T}u^i]_{\Sigma^i} = \cos\gamma \geq \nu^i \cdot \mathsf{T}u^0]_{\Sigma^i},$$

hence by Theorem 5.1 we have $u^i > u^0$ in Σ^i.

Theorem 5.9. *If* $\gamma = 0$ *then always* $u^i > u^0$.

Proof.

$$\nu \cdot \mathsf{T}u^0]_{\Sigma^i} < 1 = \nu \cdot \mathsf{T}u^i]_{\Sigma^i}.$$

Theorem 5.10 (Siegel [163]). *If* Ω^i *is a disk that can contact* Σ^0 *at any* $p \in \Sigma^0$ *from inside* Ω^0, *then* $u^i > u^0$ *in* Ω^i.

Proof. We shall assume that the derivatives of u^0 up to the second order are bounded in Ω^0. The proof will thus not apply when $\gamma = 0$; however, that case is covered by Theorem 5.9 above. The corresponding bounds for the derivatives of u^i in Ω^i follow from the material of Chapter 2, since u^i is a symmetric solution in a disk.

Carrying out the differentiation in the equation $\operatorname{div} Tu = \kappa u$, we find an equation

$$a(p,q)\,u_{xx} + 2b(p,q)\,u_{xy} + c(p,q)\,u_{yy} = \kappa u \tag{5.35}$$

for each of u^i, u^0, with $p = u_x$, $q = u_y$, and $ac - b^2 > 0$. Subtraction of the equation for u^i from the one for u^0 leads to

$$a^0 w_{xx} + 2b^0 w_{xy} + c^0 w_{yy} + (a^0 - a^i)\,u^i_{xx} + \cdots = \kappa w \tag{5.36}$$

with $w = u^0 - u^i$, $a^0 = a(p^0, q^0)$, etc. Applying the mean value theorem to expressions of the form $a^0 - a^i$, we are led to an equation

$$a(x,y)\,w_{xx} + 2b(x,y)\,w_{xy} + c(x,y)\,w_{yy} + d(x,y)\,w_x + e(x,y)\,w_y - \kappa w = 0 \tag{5.37}$$

for w in Ω^i; here $ac - b^2 > \delta > 0$, $|a| + |b| + |c| + |d| + |e| < M < \infty$, independent of the position of Ω^i in Ω^0.

Suppose the theorem were false; then there would exist $\Omega^i \subset \Omega^0$, and solutions u^i, u^0 with $0 < \gamma < \pi/2$, such that $u^0 > u^i$ at some point of Ω^i. By Theorem 5.1 there exists $p^i \in \Sigma^i$ at which $u^0 > u^i$. Applying again Theorem 5.1 to the difference between u^0 and the solution $v^0 \equiv 0$, we find there exists $p^0 \in \Sigma^0$, with $u^0(p^0) > u^0(p^i) > u^i(p^i)$. We may in fact choose p^0 so that $u^0(p^0) = \max_{\Sigma^0} u^0$.

Now move Σ^i rigidly so that it contacts Σ^0 at p^0 from within Ω^0. The solution u^i is invariant relative to points of Ω^i, and is symmetric in Ω^i. Thus,

$$u^i]_{\Sigma^i} < u^0(p^0).$$

We may now decrease the radius of Σ^i, while maintaining contact at p^0. Since (see Chapter 2) $u^i]_{\Sigma^i} \to \infty$ as the radius $\to 0$, there will be a critical radius at which $u^i]_{\Sigma^i} = u^0(p_0)$. Since $u^0(p) < u^0(p_0)$, for all $p \in \Omega^0$, we have $w(p) \le 0$, for all $p \in \Sigma^i$, while $w(p) < 0$ in Ω^i, $w(p^0) = 0$ (we have used Theorem 5.1 twice).

We now have recourse to the Hopf Lemma 5.1. We may assume for simplicity that v has the direction of the positive x-axis. Then, since u^0 achieves a maximum at p^0 relative to Σ^0, $\partial u^0/\partial v = u_x^0$ and $u_y^0 = 0$. Similarly, $\partial u^i/\partial v = u_x^i$, $u_y^i = 0$. Thus, by Lemma 5.1

$$\begin{aligned} \cos\gamma = v \cdot \mathrm{T}u^0]_{p^0} &= \frac{u_x^0}{\sqrt{1+(u_x^0)^2}} > \frac{u_x^i}{\sqrt{1+(u_x^i)^2}} \\ &= v \cdot \mathrm{T}u^i]_{p^0} = \cos\gamma. \end{aligned}$$

This contradiction establishes the result.

Theorem 5.11. *Suppose the length of Σ^i exceeds that of Σ^0. Then u^i raises the larger volume over its section, that is,*

$$\int_{\Omega^i} (u^i - u^0)\,dx > 0.$$

Proof. We write for simplicity Σ^i, Σ^0 for the respective lengths. We have

$$\begin{aligned} \kappa \int_{\Omega^i} (u^i - u^0)\,dx &= \kappa \int_{\Omega^i} u^i\,dx - \kappa \int_{\Omega^0} u^0\,dx + \kappa \int_{\Omega^0 \setminus \Omega^i} u^0\,dx \\ &= (\Sigma^i - \Sigma^0)\cos\gamma + \kappa \int_{\Omega^0 \setminus \Omega^i} u^0\,dx. \end{aligned}$$

Applying Theorem 5.1 to the two solutions u^0, $v \equiv 0$ in Ω^0, we find $u^0 > 0$ in Ω^0, and the result follows.

5.4. A Counterexample

In all the preceding cases, either the "smaller" capillary tube yielded a pointwise higher solution, or it at least raised a larger volume of fluid over its section. We now show by example that the reverse situation can also occur (Finn [55]). The underlying idea is to construct a domain Ω^0 whose boundary Σ^0 contains a point p of large curvature surrounded by large segments of small curvature, and to show that such a domain raises more fluid near p than does an inner domain Ω^i whose boundary curvature changes are less rapid.

Definition. A subdomain $\mathcal{N} \subset \Omega$ is said to satisfy an internal sphere condition with constants δ, γ in Ω, if for all $x \in \mathcal{N}$, there holds $x \in B_\delta$ such that the lower hemisphere v over B_δ meets the vertical walls over $\Sigma = \partial\Omega$, from within the component of $B_\delta \cap \Omega$ containing x, in angles not exceeding γ, or does not meet the walls (see Fig. 5.9).

We write $\mathcal{N} \in ISC(\delta; \gamma; \Omega)$. The definition is to be understood in the sense that the conditions hold up to a set of Hausdorff measure zero on Σ, on which no conditions need be imposed.

Theorem 5.12. *Let $\mathcal{N} \in ISC(\delta; \gamma; \Omega)$, let $u(x)$ denote a capillary surface in Ω with boundary angle γ, and $0 \le \gamma \le \pi/2$. Then*

$$u(x) < \frac{2}{\kappa\delta} + \delta$$

in $\mathcal{N}$.

Proof. The proof is as in Theorem 5.3 above.

Theorem 5.13. *Let Ω^0 be a domain for which there is a point $p \in \Sigma^0$ with the properties:*

i) *There is a neighborhood $\mathcal{N}_p$ of p in Ω^0 such that $\mathcal{N}_p \in ISC(\delta; \gamma; \Omega^0)$ for some $\delta > 0$, $0 < \gamma < \pi/2$.*

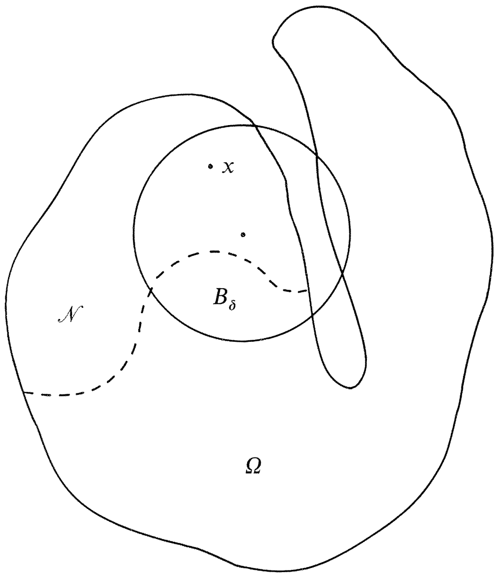


Figure 5.9. Internal sphere condition.

ii) *The curvature* H_p *of* Σ^0 *at* p *(considered as positive when the curvature vector points into* Ω^0*), satisfies* $H_p \cos\gamma > (2/\delta) + \kappa\delta$.

Then there exists $\Omega^i \subset \Omega^0$, *such that the corresponding solution* u^i *in* Ω^i *satisfies*

$$\int_{\Omega^i} (u^0 - u^i)\, dx > 0.$$

Note that the theorem fails when $\gamma = 0$, cf. Theorem 5.9.

Proof of Theorem 5.13. We choose coordinates so that Σ^0 can be represented near p in the form

$$y = \frac{a}{2} x^2 + o(x^2), \qquad a = H_p$$

$$y' = ax + o(x).$$

For $\alpha < a$ and $\varepsilon > 0$ (to be chosen) we introduce in Ω a curve Γ defined by

$$y = \frac{\alpha}{2} x^2 + \frac{\varepsilon}{2}.$$

For ε sufficiently small, $\Gamma \cap \Sigma^0$ consists of two points determined by $x = \pm\sqrt{\varepsilon/\lambda} + o(\varepsilon^{1/2})$, and $\Gamma \subset \mathcal{N}_p$; here $\lambda = a - \alpha$.

We compute, for the difference in lengths,

$$\Sigma^* - \Gamma = \frac{a+\alpha}{3\sqrt{\lambda}} \varepsilon^{3/2} + o(\varepsilon^{3/2}) \tag{5.38}$$

while for the area $\mathscr{A}$ bounded between Γ and Σ^*, we find

$$\mathscr{A} = \frac{2}{3\sqrt{\lambda}} \varepsilon^{3/2} + o(\varepsilon^{3/2}). \tag{5.39}$$

Since $\mathcal{N}_p \in ISC(\delta; \gamma; \Omega)$ we have

$$u < \frac{2}{\kappa\delta} + \delta$$

in $\mathcal{N}_p$, hence also in $\mathscr{A}$. Thus, letting Ω^i be the part of Ω above Γ, we have

$$
\begin{aligned}
\kappa\int_{\Omega^i}(u^0-u^i)\,dx &= \kappa\int_{\Omega^0}u^0\,dx-\kappa\int_{\Omega^i}u^i\,dx-\kappa\int_{\mathscr{A}}u^0\,dx\\
&=(\Sigma^*-\Gamma)\cos\gamma-\kappa\int_{\mathscr{A}}u^0\,dx\\
&>(\Sigma^*-\Gamma)\cos\gamma-\left(\frac{2}{\delta}+\kappa\delta\right)\mathscr{A}\\
&>\frac{1}{3\sqrt{\lambda}}\varepsilon^{3/2}[a+\alpha-2H_p+\sigma]\cos\gamma+o(\varepsilon^{3/2})
\end{aligned}
\tag{5.40}
$$

for some $\sigma>0$, by the above estimates and the hypothesis (ii). We now choose $\alpha<a=H_p$ sufficiently large that $a+\alpha-2H+\sigma>0$. Having done so, we may choose ε sufficiently small that $\kappa\int_{\Omega^i}(u^0-u^i)\,dx>0$, which was to be shown.

If the theorem is to be of any use, we must show that it is not vacuous, that is, we must show that there exist Ω^0, $p\in\Sigma^0$ for which the hypotheses (i) and (ii) are satisfied. We give an explicit construction, by starting with B_δ, then constructing $\mathscr{N}_p$ so that the *single* B_δ will work for all $x\in\mathscr{N}_p$, then finally extending $\mathscr{N}_p$ to Ω^0.

Consider a ball B_δ and corresponding lower hemisphere v. If $0<\gamma<\pi/2$, the set of all vertical planes cutting v in the (constant) angle γ, envelop a vertical circular cylinder. The traces on the plane of B_δ are a concentric circle Σ_γ and the set of its tangent lines. Through any point q in the annulus between Σ_γ and Σ_δ pass exactly two such tangent lines. *Any vertical cylinder passing through q, in a direction which when extended does not meet Σ_γ, will meet v over q in an angle less than γ* (see Fig. 5.10). This determines at q a permissible range of directions for the curve Σ^0

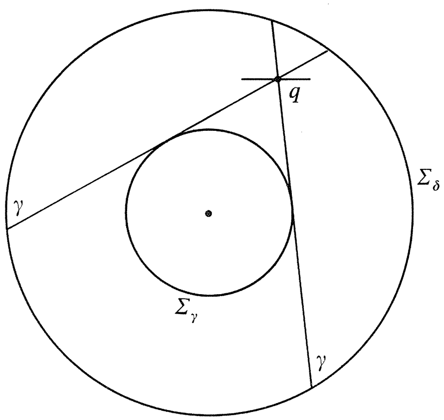


Figure 5.10. Permissible direction.

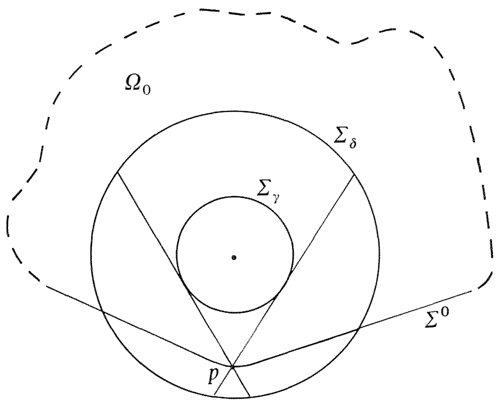


Figure 5.11. Construction of Ω_0.

that is to be constructed. Further, *if a straight line L has a permissible direction at q, then L has a permissible direction at all points of* $L \cap B_\delta$.

Given γ in the range $0<\gamma<\pi/2$, we now choose $p \in B_\delta$ exterior to Σ_γ, and a direction at p within the permissible range. Most simply, we choose a direction orthogonal to the segment joining p to the center of B_δ. We may satisfy the condition (ii) by taking a circular arc Σ^0 through p with the given direction, of curvature $H_p > (1/\cos\gamma)((2/\delta)+\kappa\delta)$, with the curvature vector directed toward the center of B_δ. We consider a sufficiently small segment of Σ^0, so that the tangent directions will still lie within the possible range, and then extend Σ^0 with straight segments until the boundary Σ_δ is crossed (see Fig. 5.11). Σ^0 can then be completed to a closed curve (bounding a domain Ω^0) in any way at all, provided only that the curve does not again enter B_δ. We may then choose $\mathcal{N}_p = B_\delta \cap \Omega^0$, and the construction is complete. For all $x \in \mathcal{N}_p$, there holds $x \in B_\delta$, and the lower hemisphere v over B_δ meets the vertical cylinder over Σ^0 in angles not exceeding γ, as follows from the method of construction.

The construction fails when $\gamma=0$, as it must (see Theorem 5.9).

5.5. Convexity

The symmetric surfaces studied in Chapters 2 and 3 are all convex (even absolutely monotonic, see Brulois [21]); we may ask whether there are natural conditions on the section Ω of a capillary tube, under which the convexity property will be retained.

Theorem 5.14 (Korevaar [109]). *Let Ω be a bounded domain of class $C^{(1)}$ and strictly convex, in the sense that any segment joining points of Σ lies,* *except for its end points, in Ω. Let $u(x)$ be a solution of* (5.1) *in Ω, differentiable up to Σ, with $\nu \cdot \mathrm{T}u = 1$ on Σ. Then $u(x)$ describes a strictly convex surface over Ω.*

The result is obtained as special case of a general procedure that has also found applications in other directions (see, e.g., [22, 106, 110, 112]). Let $x_1, x_3 \in \Omega$, let $0 \leq \lambda \leq 1$, and set

$$x_2 = \lambda x_3 + (1-\lambda) x_1.$$

The "concavity function" $\mathfrak{K}$, defined by

$$\mathfrak{K}(x_1, x_3, \lambda) \equiv u(x_2) - \lambda u(x_3) - (1-\lambda) u(x_1),$$

will be ≤ 0 for all x_1, x_3 in Ω and $\lambda \in [0,1]$ if and only if $u(x)$ determines a convex surface over Ω.

Lemma 5.2. *Let $u \in C(\bar{\Omega})$ and suppose $u(x)$ is not convex. Then $\mathfrak{K}(x_1, x_3, \lambda)$ attains its positive supremum for points x_1, $x_3 \in \bar{\Omega}$, such that $x_1 \neq x_3$, $0 < \lambda < 1$.*

Proof. Let $(x_1^k, x_2^k, \lambda^k)$ be a maximizing sequence; we may assume $(x_1^k, x_2^k, \lambda^k) \to (x_1, x_2, \lambda)$, and hence $\mathfrak{K}(x_1^k, x_2^k, \lambda^k) \to \mathfrak{K}(x_1, x_2, \lambda)$. We have $x_1 \neq x_3$, $0 < \lambda < 1$, since $\mathfrak{K} > 0$.

We now write the equation (5.1) in the form (5.35)

$$a^{ij} u_{ij} = \kappa u \tag{5.41}$$

and observe that a^{ij} is a function (only) of Du = gradient of u, satisfying $a^{ij} = a^{ji}$, $[a^{ij}] > 0$.

Lemma 5.3. *Let $u \in C^{(2)}(\Omega)$ satisfy* (5.41) *in Ω. Then $\mathfrak{K}(x_1, x_3, \lambda)$ cannot have a positive local maximum at any set of three interior points.*

Proof. From $\nabla_{x_1} \mathfrak{K} = 0$ we find

$$0 = (1-\lambda) Du(x_2) - (1-\lambda) Du(x_1);$$

similarly from $\nabla_{x_3} \mathfrak{K} = 0$ we obtain

$$0 = \lambda Du(x_2) - \lambda Du(x_3).$$

Therefore, since $0 < \lambda < 1$,

$$Du(x_1) = Du(x_2) = Du(x_3). \tag{5.42}$$

Now let us make a rigid translation of all three points by a vector v. Setting

$$F(v) = \mathfrak{K}(x_1 + v, x_3 + v, \lambda)$$

we find, for the derivative in direction v,

$$D_v F(0) = 0 \tag{5.43}$$

and for the Hessian

$$[D^2 F(0)] \leq 0 \tag{5.44}$$

since F has a local maximum at 0. Since $a^{ij} = a^{ji}$ and $[a^{ij}] > 0$, we obtain

$$a^{ij} F_{ij} \leq 0, \tag{5.45}$$

and choosing a^{ij} to be the value corresponding to the (equal) gradients (5.42), we may write (5.45) in the form

$$a^{ij} u_{ij}(x_2) - \lambda a^{ij} u_{ij}(x_3) - (1-\lambda) a^{ij} u_{ij}(x_1) \leq 0. \tag{5.46}$$

In view of the equation (5.41), we may write

$$u(x_2) \leq \lambda u(x_3) + (1-\lambda) u(x_1),$$

which contradicts the hypothesis that $\mathfrak{K}$ has a positive maximum at (x_1, x_3, λ).

We return to the proof of the theorem. Under the hypotheses, if u is not convex, then by Lemma 5.2 a positive maximum is achieved by $\mathfrak{K}$, while by Lemma 5.3 and the strict convexity of Ω, at least one of the points x_1 or x_3 must lie on Σ. Suppose $x_1 \in \Sigma$ and let s denote arc length along the segment $\overline{x_1 x_3}$. Since $\nu \cdot \mathrm{T} u = 1$ on Σ, we find $\partial u / \partial s]_{x_1} = \infty$, with u increasing as x_1 is approached. Thus $\mathfrak{K}$ could be increased by moving x_1 a small distance into Ω along $\overline{x_1 x_3}$ with x_2, x_3 held fixed. This contradiction establishes the result.

The condition $\nu \cdot \mathrm{T} u = 1$ ($\gamma = 0$) of Theorem 5.14 is essential. Korevaar showed this by using the special properties of the "wedge solution" of Theorem 5.5.

Notes to Chapter 5

1. Theorem 5.1. Our proof has suppressed some details regarding existence and integrability of ∇w. The theorem holds in fact under much weaker hypotheses than we have indicated, cf. Emmer [46], Finn and Gerhardt [68], and Chapter 7, Theorem 7.7.

2. Theorems 5.4 and 5.5, Corollary 5.5. The results reappear in a less precise form in Mason and Morrow [122].

3. Theorems 5.4 and 5.5. It is a simple exercise to show that if a function $u(x, y)$ is of class $C^{(1)}$ up to the vertex V of a wedge domain Ω with opening angle 2α, and if $\nu \cdot \mathrm{T}u = \cos\gamma \equiv \text{const.} \neq 0$ on the sides, then $\alpha + \gamma \geq \pi/2$. Thus the criterion for discontinuous behavior appears again in terms of properties of general functions. Nevertheless, for any given γ it is easy to give examples of functions $u(x, y)$ that are continuous in the closure of Ω and $C^{(1)}$ except at V, and such that $\nu \cdot \mathrm{T}u = \cos\gamma$ on the sides. We therefore see that the two theorems express a subtler property, appropriate to the solutions of (5.1).

4. Theorems 5.4 and 5.5, Corollary 5.5. Recorded observations of the behavior of fluid in a vertical wedge can be traced at least to Brook Taylor [176], who examined water in a wedge formed by two glass plates with opening angle $2\alpha \approx 2.5°$, and identified the curves of contact on the plates as being "very near to the common hyperbola." Further experiments under varying conditions were performed by Hauksbee [97, 98, 99]. "Proofs" of hyperbolic form appear in Musschenbroek [142], in Ferguson and Vogel [49], and in Princen [150, p. 367ff.]. The methods in the former two papers lead to an asymptotically hyperbolic form for any opening angle less than π, a result that conflicts with Theorem 5.4. Princen develops his method only for the case $\gamma = 0$, stating that "the problem can be solved readily for any contact angle". Had he looked more closely at the case of general γ, he might have been led to the discontinuous dependence (Corollary 5.5). His procedure is in any event inexact, and even for $\gamma = 0$ the result falls short of Theorem 5.5.

5. Theorem 5.5. The result has been applied to problems of oil recovery from pore networks, see Campbell and Orr [23].

6. Corollary 5.5. The experiment of Coburn (essentially a kitchen sink experiment) establishes that the contact angle of water with acrylic plastic lies in the range between 78° and 81°. An experiment under controlled laboratory conditions could presumably yield still greater accuracy.

7. §5.3. The question considered here was raised informally by M. Miranda.

8. Theorem 5.8. The method has a much broader range of application and has been used to obtain local bounds from below under fairly general conditions (Finn [55]). If the methods of proof of Theorems 5.2 and 5.8 are combined, we obtain immediately that if Ω lies interior to a disk of radius R then the capillary surface u over Ω with boundary angle γ

satisfies

$$u > \left(\frac{2}{\kappa R} - \frac{R}{1+\sin\gamma}\right)\cos\gamma,$$

and thus $u \to \infty$ uniformly if κ (or equivalently the gravitational force) tends to zero. An alternative proof of this latter result was given by Emmer [47].

9. Theorem 5.10. Siegel [163] has given still another comparison result for the case in which Ω^i is a disk. Let B be the maximal concentric disk, in the sense that the solution in B is vertical on ∂B. Siegel proved that *if* $B \cap \Omega^0$ *is convex, then* $u^i > u^0$ *in* Ω^i. The proof is obtained from the monotonicity of $k_l = \sin\psi/r$ for the symmetric solution. Siegel used this result to prove the uniqueness of the capillary surface in an infinite wedge.

10. Theorem 5.13. A somewhat different result in this direction, for a situation in which the rise height in the corner is infinite, was given by Concus and Finn [35].

Chapter 6

Capillary Surfaces Without Gravity

6.1. General Remarks

On the earth's surface, the shape of the free surface in a capillary tube is determined by an interaction of surface and gravity forces. If a significant mass of fluid is present the gravity forces will predominate; thus the effect of rising liquid is observed only in tubes of small diameter.

If gravity is removed the effect becomes a consequence purely of surface forces, and geometrically similar free surfaces appear in geometrically similar containers. Thus the form of a capillary surface can change markedly, depending on whether or not an external force (gravity) field is present. In some situations the change can be quite striking, as it has a discontinuous character.

The problem cannot be considered without volume constraint as in Chapters 2 and 5, as it is then physically unrealistic and in general admits no solution. We may however close one end of the tube and attempt to cover the "bottom" Ω by a finite volume of liquid. It may not be *a priori* clear how much volume is needed for the purpose, however if the mathematical formalism leads to a graph over Ω that is bounded below, then a physical solution can be obtained by adding a suitable constant. We note again in this connection that in view of Miranda's Theorem 2.2 of [139] it is natural to seek solution surfaces as graphs over Ω, as no surface over Ω that is not a graph could minimize the energy.

The material of Chapters 2 and 5 could also have been presented from the above point of view. With the aid of Theorem 5.1, one sees easily that in a uniform (or null) gravitational field directed toward Ω through the fluid, changes of volume are effected simply by raising or lowering the surface uniformly over Ω.

In the absence of an external force field, equations (1.51) and (1.44) take the form

$$\operatorname{div} \mathsf{T} u = 2H \equiv \text{const.} \tag{6.1}$$

in Ω, while the boundary condition remains

$$\nu \cdot \mathsf{T} u = \cos \gamma \tag{6.2}$$

on $\Sigma = \partial \Omega$.

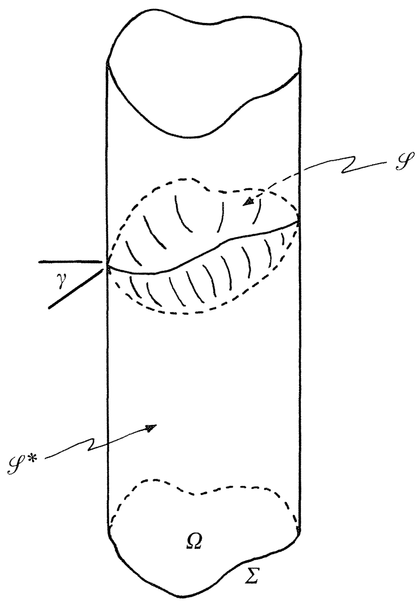


Figure 6.1. Capillary tube; general section, volume constrained case.

The constant H is not arbitrary: integrating (6.1) over Ω and using the divergence theorem, we obtain

$$2H\Omega = \oint_\Sigma \nu \cdot \mathsf{T}u\, ds = \Sigma \cos\gamma \tag{6.3}$$

by (6.2). Thus, H is known in advance and (6.1) becomes

$$\operatorname{div} \mathsf{T}u = \frac{\Sigma}{\Omega}\cos\gamma. \tag{6.4}$$

The problem thus becomes: *to find a surface of constant mean curvature determined by* (6.3), *which meets the cylindrical boundary* $\mathscr{S}^*$ *over* Σ *in the prescribed constant angle* γ; see Fig. 6.1.

6.2. A Necessary Condition

It was pointed out by Concus and Finn [30] that the problem (6.4), (6.2) in general admits no solution. Consider an arbitrary subdomain $\Omega^* \subset \Omega$ and let $\Gamma = \partial\Omega^* \cap \Omega$, $\Sigma^* = \partial\Omega^* \cap \Sigma$ (Fig. 6.2). If we now integrate (6.4) over Ω^* and use the boundary condition (6.2) on Σ^*, we find

$$\left(\frac{\Sigma}{\Omega}\Omega^* - \Sigma^*\right)\cos\gamma = \int_\Gamma \nu \cdot \mathsf{T}u\, ds. \tag{6.5}$$

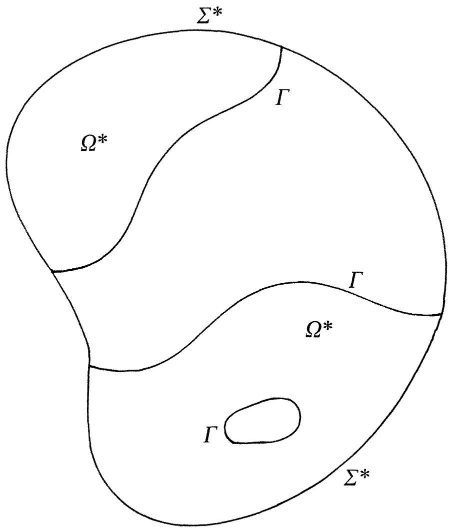


Figure 6.2. Subsets for the general necessary condition.

For any differentiable function f there holds

$$|\mathsf{T}f| = \frac{1}{\sqrt{1+|\nabla f|^2}} |\nabla f| < 1. \tag{6.6}$$

We write $H_\gamma = (\Sigma/\Omega)\cos\gamma$ and place this bound into (6.5); we find immediately [30].

Theorem 6.1. *A necessary condition for the existence of a solution of* (6.4), (6.2) *in Ω is that the functional*

$$\Phi[\Omega^*] \equiv \Gamma - \Sigma^* \cos\gamma + H_\gamma \Omega^* > 0 \tag{6.7}$$

for every Ω^* of the type considered, with $\Omega^* \neq \phi, \Omega$.

We note that the original equation does not appear in any way in (6.7); the question hinges on a geometrical property of the domain Ω.

Let us examine this result in the special case in which Ω contains a corner of opening angle 2α (Fig. 6.3). We find, for the indicated Γ,

$$\Phi[\Omega^*] = 2l(\sin\alpha - \cos\gamma) + O(l^2). \tag{6.8}$$

Letting $l \to 0$, we obtain the result [30]:

Theorem 6.2. *If Ω contains a corner with interior angle 2α, and if $\alpha + \gamma < \pi/2$, then the problem* (6.4), (6.2) *admits no solution.*

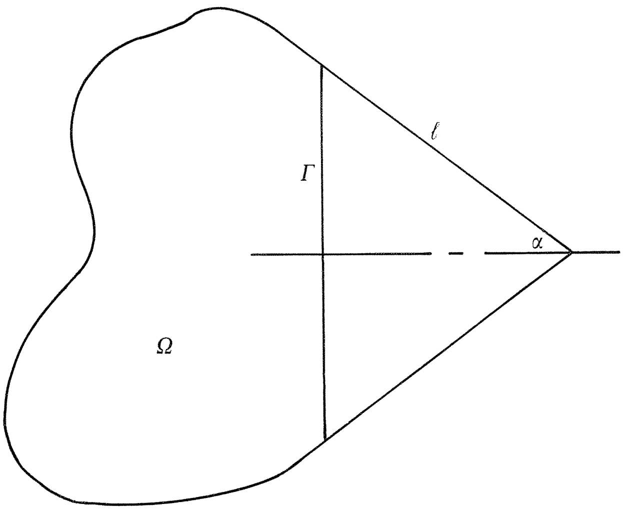


Figure 6.3. Nonexistence criterion.

The difficulty does not arise from the boundary discontinuity at the vertex V. One sees easily that the same contradiction can be obtained when the boundary is smoothed at V. Thus, we are faced with an elliptic boundary value problem that arises directly from physical considerations, which is in general not well posed, even in a smooth convex domain.

The condition we have found is remarkable in that it is sharp. Let Σ be a regular polygon and C the circumscribed circle (Fig. 6.4). A lower hemisphere whose equatorial circle lies over C provides an explicit solution of (6.4), (6.2) (up to the set Σ_0 of vertices, see Theorem 5.1) for the case $\alpha+\gamma=\pi/2$. The solution is analytic up to the polygonal walls and continuous in the closed region. Replacing the hemisphere by spherical caps of increasing radius, we find explicit solutions of the problem for any γ satisfying $\alpha+\gamma\geq\pi/2$. Thus, *there is a discontinuous dependence on boundary data.* As γ decreases from $\pi/2$ to $(\pi/2)-\alpha$, the solutions exist and remain equibounded and analytic in Ω. But if $\gamma<(\pi/2)-\alpha$, no solution exists.

Theorem 6.2 should be compared with Corollary 5.5, in which the identical geometrical condition appears and leads also to a discontinuous dependence. In a gravity field this behavior manifests itself in the solution becoming unbounded; when the gravity field is absent the solution simply disappears.

The behavior just described was verified experimentally by W. Masica at the NASA Lewis Laboratory in a "drop tower", which provided about five seconds of "free fall" without gravity; see Fig. 6.5. If $\alpha+\gamma\geq\pi/2$ the "spherical cap" solution is observed; if $\alpha+\gamma<\pi/2$ the fluid ascends into

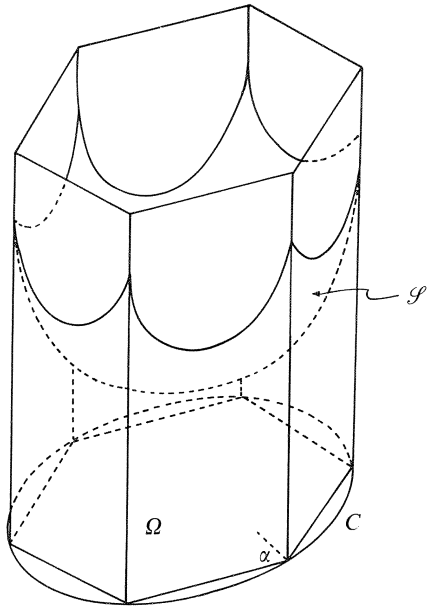
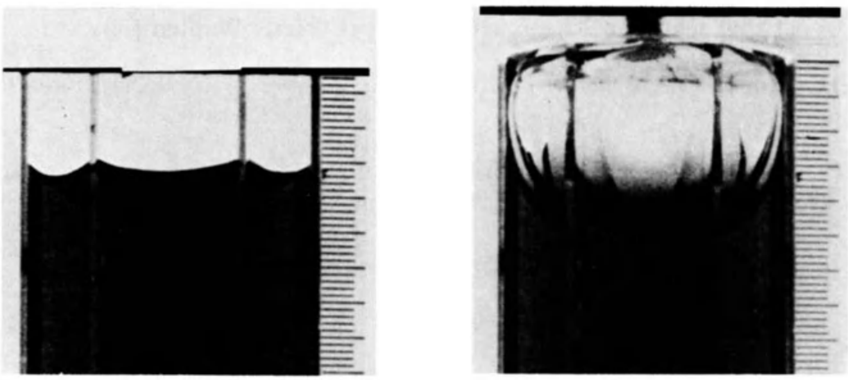


Figure 6.4. Spherical solution surface over regular hexagon.



(a) (b)

Figure 6.5. Discontinuous dependence. (a) $\alpha \geq$ critical. (b) $\alpha <$ critical.

the corners, to infinity or to the top of the container, whichever comes first. Thus, the discontinuous dependence is reflected in reality and is not an idiosyncrasy of the mathematical formalism.

We point out here the consequence of Theorem 5.1, that *whenever a solution exists, it is uniquely determined up to an additive constant.* In this result, any set of Hausdorff measure zero on the boundary can be neglected, thus the presence of the vertices in the above example, at which the data (6.2) cannot be prescribed, does not affect the uniqueness of the solution.

From Theorem 6.1 we obtain easily the following result (cf. [30], Corollary 3.3).

Theorem 6.3. *Capillary surfaces without gravity are always unstable with respect to boundary perturbations, in the sense that for any boundary Σ for which a solution of* (6.4), (6.2) *exists, there is an arbitrarily small perturbation $\Sigma \to \hat{\Sigma}$ such that $\hat{\Sigma}$ admits no solution. If $\gamma = 0$, the perturbation can be chosen to be arbitrarily small, both in Σ and in the unit normal vector to Σ.*

The situation in which there is an interval on Σ with curvature $H^\Sigma \geq H_\gamma$ has a special interest. With a view to a later application, we formulate it for the case in which γ is allowed to vary on Σ. The criterion (6.7) then becomes

$$\Phi[\Omega^*] \equiv \Gamma - \int_{\Sigma^*} \cos\gamma \, ds + H_\gamma \Omega^* > 0 \tag{6.9}$$

with

$$H_\gamma = \frac{1}{\Omega} \oint_\Sigma \cos\gamma \, ds. \tag{6.10}$$

Under this definition, Theorem 6.1 holds as before. We find [30]:

Theorem 6.4. *There cannot hold $\gamma = 0$ on any arc of Σ whose curvature H^Σ satisfies $H^\Sigma > H_\gamma$.*

Proof. Suppose there were such an arc, let p be a point on it, and choose coordinates as indicated in Fig. 6.6. We may then write near p,

$$\Sigma\colon\ y = \tfrac{1}{2} a x^2 + \cdots$$

with $a = H^\Sigma(p)$. Choose $\bar{a}$, $H_\gamma < \bar{a} < H^\Sigma(p)$, and introduce an arc Γ defined by

$$\Gamma\colon\ y = \varepsilon + \tfrac{1}{2} \bar{a} x^2 + \cdots.$$

If $\varepsilon > 0$ is small enough, Γ will cut off an arc Σ^* on Σ on which $H^{\Sigma^*} > \bar{a}$, and on which $\cos\gamma = 1$. A calculation yields

$$\Phi[\Omega^*] = \frac{3\sqrt{a - \bar{a}}}{(2\varepsilon)^{3/2}} \left[H_\gamma - \frac{a + \bar{a}}{2} + o(1) \right]$$

which is negative for small enough ε, thus yielding a contradiction to Theorem 6.1.

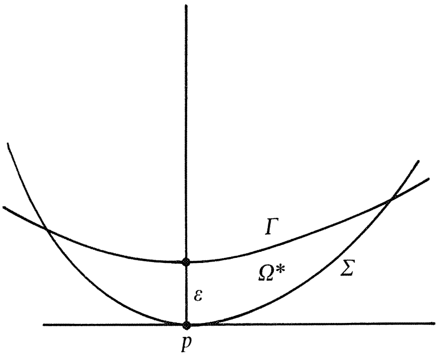


Figure 6.6. Construction for Theorem 6.4.

There is a special interest in the case in which the inequality between H_γ and H^Σ is not required to be strict [53].

Theorem 6.4(a). *Let $u(x)$ be bounded in a neighborhood of an arc $\Sigma^* \subset \Sigma$, for which $H^{\Sigma^*} \geq H_\gamma$. Then there cannot hold $\gamma = 0$ on Σ^*.*

The hypothesis of boundedness for $u(x)$ in this result is necessary, see Spruck [170]. With regard to both the Theorems 6.4 and 6.4a, we remark that it is not possible to replace the arc in question by an isolated point. For example, a lower unit hemisphere defines a solution of (6.1) with $H_\gamma \equiv (\Sigma/\Omega)\cos\gamma = 2$ in $|x| < 1$. The inequality $H^\Sigma > H_\gamma$ is satisfied on the arc Σ: $|x - \frac{3}{4}| = \frac{1}{4}$. The hemisphere, considered as a solution interior to Σ, defines a continuous γ on Σ, and $\gamma = 0$ at the contact point of Σ with $|x| = 1$.

More precise and inclusive formulations of Theorems 6.4 and 6.4(a) appear as Theorem 3 in [30] and as Theorems 1 and 2 in [53].

We have further:

Theorem 6.5. *Let an arc $\Sigma^* \subset \Sigma$ have curvature $H^{\Sigma^*} < H_\gamma$. Then $u(x)$ is bounded adjacent to Σ^*.*

The proof is obtained easily with the aid of the roulade of an ellipse, see §4.7. If $H^{\Sigma^*} < H_\gamma$, we can position an "outer" circumference of a roulade with mean curvature $H = H_\gamma/2$, so as to contact Σ within Ω at a given $p \in \Sigma^*$. A segment S cut off by a chord Γ (see Fig. 6.7) could then be found so that $S \subset \Omega$. Adding a constant to the roulade function $v(x)$, we can arrange to have $u(x) < v(x)$ on Γ. Using that $\nu \cdot Tv \equiv 1$ on the circumference, we obtain from Theorem 5.1 that $u < v$ throughout S, and thus $u(p) < \infty$.

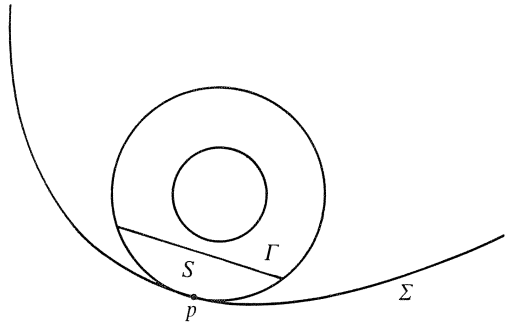


Figure 6.7. Construction for Theorem 6.5.

6.3. Sufficiency Conditions

The basic results that underlie all later work are due to E. Giusti. These results are based on a variational procedure and require for compactness purposes an interpretation of (6.7) in terms of "Caccioppoli" sets of finite perimeter. For discussion of these sets and their properties, see, e.g., [89, 126]. The *perimeter in* Ω of a set E is defined in terms of the characteristic function φ_E of E by the relation

$$P(E;\Omega)=\int_\Omega |D\varphi_E| = \sup\{\int_\Omega \varphi_E \operatorname{div} g\, dx\} \tag{6.11}$$

among all vector valued $g \in C_0^1(\Omega)$ with $|g|\le 1$. One verifies that if $\Omega^* \subset \Omega$ has smooth boundary $\Gamma \cup \Sigma^*$ (see §6.2), then $P(\Omega^*;\Omega)=\Gamma$. Correspondingly, $\Sigma^* = \int_\Sigma \varphi_{\Omega^*}^T ds$, $\varphi_{\Omega^*}^T$ being the trace on Σ of φ_{Ω^*}. When the context is clear, we shall omit the superscript T.

Given a function $f \in L^1(\Omega)$, we define its *variation over* Ω to be

$$V[f;\Omega]=\int_\Omega |Df| \tag{6.12}$$

as above. We introduce the class $BV(\Omega)$ of functions f, for which

$$\int_\Omega |f|\,dx + \int_\Omega |Df| < \infty. \tag{6.13}$$

Giusti proved, essentially, the following results:

Theorem 6.6 [85]. *If* $\gamma=0$, *and if* (6.7) *holds for every* $\Omega^* \neq \emptyset$, Ω *with finite perimeter, then there exists a solution of* (6.4), (6.2); *the solution is unique up to an additive constant.*

Theorem 6.7 [84]. *Suppose $\gamma>0$ and suppose* (6.7) *holds for every $\Omega^*\neq\emptyset$, Ω with finite perimeter. Suppose further that for some $\mu<1/\cos\gamma$ and Υ depending only on Ω, an inequality*

$$\int_\Sigma |f|\,ds\le\mu\int_\Omega|Df|+\Upsilon\int_\Omega|f|\,dx \tag{6.14}$$

holds for all $f\in BV(\Omega)$. Then there exists a solution of (6.4), (6.2). *The solution is bounded and unique up to an additive constant.*

In the above results, it is assumed that Ω is sufficiently smooth that an isoperimetric inequality (relative to Ω) of the form

$$\min\{E,\Omega\backslash E\}<C\Gamma^2 \tag{6.15}$$

holds for any $E\subset\Omega$. Such an estimate is valid, essentially, whenever a Poincaré inequality holds, see, e.g., [89]. The inequality (6.15) is to be understood in the sense

$$\min\{\int_\Omega\varphi_E\,dx,\int_\Omega(1-\varphi_E)\,dx\}<C(\int_\Omega|D\varphi_E|)^2$$

for some C depending only on Ω.

The inequality (6.14) appears first in Emmer [46], who proved it with $\mu=\sqrt{1+L^2}$ for any Lipschitz domain with Lipschitz constant L (see also [126, p. 203]). Thus the requirement (6.14) is satisfied in particular for any smooth domain. We will want to consider domains in which corners with inward opening angle 2α can appear. Emmer's inequality applies to such situations, and from Theorems 6.2 and 6.7 we see that the constant $\sqrt{1+L^2}$ cannot be improved in general. However, for domains with reentrant corners, for which $2\alpha>\pi$, the Emmer result would restrict the opening angle to $\alpha<(\pi/2)+\gamma$. It will turn out however that existence can hold without such restriction when $2\alpha>\pi$, and even inward cusps can be permitted. To this end we establish Emmer's result under somewhat weaker conditions than appear in the literature.

We consider Ω as a metric space with distance function $d(p,q)=$ infimum of Euclidean lengths of paths joining p to q in Ω, and we suppose its closure $\bar\Omega$ to be covered by a partition of unity with particular properties. Specifically, we suppose $\bar\Omega$ covered by a finite number N of open (in $\bar\Omega$) sets Ω_i, to each of which is associated a function $\varphi_i(x)\ge 0$ with $\varphi_i(x)\in C_0^\infty(\Omega_i)$ and $\sum_1^N\varphi_i(x)=1$. With regard to these sets, we assume:

i) there is an at most finite set of points $\{p_k\}=P\subset\Sigma$, and an associated $N_P\le N$, such that $\Sigma\backslash P\subset\bigcup_1^{N_P}\Sigma_j$, where $\Sigma_j=\Sigma\cap\Omega_j\neq\emptyset$ is an open connected set in the relative topology of Σ.

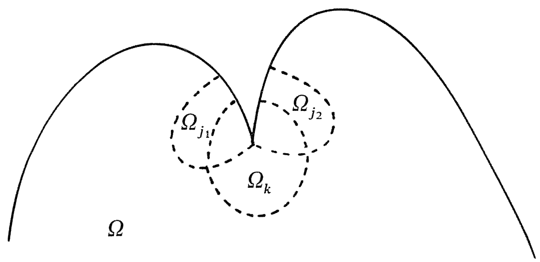


Figure 6.8. The sets Ω_j and Ω_k at p_k.

ii) to each $p_k \in P$ there is a unique $\Omega_k \ni p_k$, and $\Sigma_k = \Sigma \cap \Omega_k$ meets exactly two (adjacent) sets $\Sigma_j \subset \Sigma \setminus P$ (see Fig. 6.8).

iii) there exists $\tau > 0$ such that each $\Sigma_j \subset \Sigma \setminus P$ can be represented over some interval $a_j < x < b_j$ by a Lipschitz function $y = \psi_j(x)$ with Lipschitz constant L_j, and such that the set $\{(x,y): a_j < x < b_j, \ -\tau < y - \psi_j(x) < 0\}$ lies in Ω. Two such sets adjacent to a point $p_k \in P$ do not intersect (see Fig. 6.9(a)).

Lemma 6.1. *Under the above conditions, let β be a continuous function defined on Σ, let $\beta_j = \max_{\Sigma_j} |\beta|$. Let $\mathscr{A}_\delta \subset \Omega$ be the strip of width δ adjacent to Σ. Then for any $f \in BV(\Omega)$ there holds*

$$\left|\int_\Sigma \beta f \, ds\right| \le \mu \int_{\mathscr{A}_\delta} |Df| + \Upsilon(\Omega;\delta) \int_{\mathscr{A}_\delta} |f|; \tag{6.16}$$

here $\mu = \max \beta_j \sqrt{1 + L_j}$, taken over all j for which $\Omega_j \cap \mathscr{A}_\delta \cap \operatorname{supp} f \neq \emptyset$.

We note that the conditions of the lemma allow a finite number of reentrant corners without restriction on angle and without affecting the

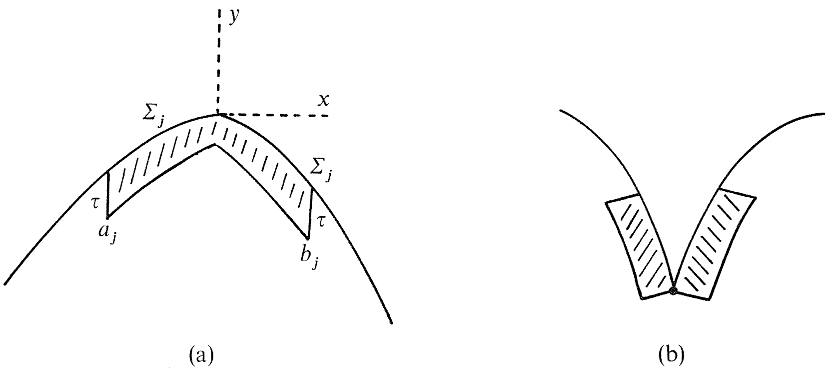


Figure 6.9. (a) One "Lipschitz" set. (b) Two "Lipschitz" sets.

choice of μ; even cusps and doubly covered portions of Σ can be admitted, see Fig. 6.9(b). An outward cusp is however excluded by the condition that the sets constructed under (iii) above should not intersect. The same condition imposes the restriction that P contain no vertex of an outward corner; each such vertex p_j of a corner with opening $2\alpha<\pi$ lies in a Σ_j, $j\leq N_P$, for which $L_j\geq\cot\alpha$.

Proof of Lemma 6.1. Set $f_j=f\varphi_j$. Choose δ small enough that the endpoints $(y=\psi_j-\tau)$ of each of the intervals constructed under (iii) above lie in $\Omega\setminus\mathscr{A}_\delta$. Let $\eta(x)\in C^\infty(\bar{\Omega})$, with $0\leq\eta\leq 1$, $\eta\equiv 1$ on Σ, $\eta\equiv 0$ in $\Omega\setminus\mathscr{A}_\delta$. Suppose first $j\leq N_P$. We have

$$f\varphi_j=\int_{-\delta}^{0}(\eta\varphi_j f)_y\,dy$$

so that

$$|f|\varphi_j\leq\int_{-\delta}^{0}|f_y|\varphi_j\,dy+\Upsilon_j(\Omega;\delta)\int_{-\delta}^{0}|f|\,dy.$$

Integrating with respect to x and recalling that $\operatorname{supp}\varphi_j\subset\Omega_j$, we get

$$\frac{1}{\sqrt{1+L_j^2}}\int_\Sigma|f|\varphi_j\,ds\leq\int_{\mathscr{A}_\delta}|Df|\varphi_j+\Upsilon_j\int_{\mathscr{A}_\delta}|f| \tag{6.17}$$

and thus

$$\left|\int_\Sigma\beta f\varphi_j\,ds\right|\leq\beta_j\sqrt{1+L_j^2}\int_{\mathscr{A}_\delta}|Df|\varphi_j+\beta_j\sqrt{1+L_j^2}\,\Upsilon_j\int_{\mathscr{A}_\delta}|f|. \tag{6.18}$$

If $j>N_P$, we consider separately the integrals corresponding to the intersections with the two adjacent Σ_j (see ii) above). Since the respective sets constructed under (iii) do not overlap, the evaluations are additive and we are led to the same result (6.18). Summing over all j for which $\Omega_j\cap\mathscr{A}_\delta\cap\operatorname{supp}f\neq\emptyset$, setting $\Upsilon=\Sigma\beta_j\sqrt{1+L_j^2}\,\Upsilon_j$, and noting that if (6.18) holds for any δ it holds for all larger δ, we obtain the stated result.

Thus we see that (6.14) can be justified by reasonable geometric considerations. In fact, some condition of the form (6.14) is necessary, as we see from the following observation:

Lemma 6.2. *Suppose there exists a solution $u(x)$ of* (6.4), (6.2) *in Ω, corresponding to $\gamma<\pi/2$. Then an inequality* (6.14) *holds, with $\mu=1/\cos\gamma$, for any $f\in BV(\Omega)$.*

Proof. An integration by parts yields

$$\int_\Omega f\operatorname{div}\mathrm{T}u\,dx=\int_\Sigma f\nu\cdot\mathrm{T}u-\int_\Omega Df\cdot\mathrm{T}u.$$

Since $\operatorname{div} Tu = (\Sigma/\Omega)\cos\gamma$, $\nu \cdot Tu]_\Sigma = \cos\gamma$, $|Tu| < 1$, we obtain the explicit estimate

$$\left|\int_\Sigma f\,ds\right| \le \frac{1}{\cos\gamma}\int_\Omega |Df| + \frac{\Sigma}{\Omega}\int_\Omega |f|\,dx \tag{6.19}$$

from which the statement follows.

Thus, subject to clarification of the case in which (6.14) holds with $\mu = 1/\cos\gamma$, an inequality of that form is seen to be both necessary and sufficient for existence. The limiting case $\mu = 1/\cos\gamma$ is delicate and is not completely settled; we return to it in §§7.8, 7.9.

6.4. Sufficiency Conditions II

The hypothesis of Theorems 6.6 and 6.7 that (6.7) hold for all $\Omega^* \neq \emptyset$, Ω is more difficult to interpret geometrically. As Giusti pointed out, it is not evidently useful in the form given, as in order to apply it one would have to examine every subdomain of Ω. One does obtain, however, from the theorems the following useful result:

Corollary 6.6, 6.7. *If a solution exists for $\gamma = \gamma_0$ and if $\gamma_0 \le \gamma_1 \le \pi/2$, then a solution exists for $\gamma = \gamma_1$.*

Several attempts have been made to obtain sufficiency criteria in terms of accessible parameters of Ω. Giusti and Weinberger [85] showed that *if Σ is convex and if the curvature k of Σ satisfies $0 < k < \Sigma/\Omega$, then a solution exists for $\gamma = 0$ (and hence, by the above corollary, for every γ).* Chen [24] showed that the convexity requirement cannot be discarded; in that respect Finn and Giusti [70] showed that even the condition $|k| < \Sigma/\Omega$ would in general not suffice. Chen [24] showed however that *whenever a disk of radius Ω/Σ can be rolled around Σ interior to Ω, then a solution exists for every γ.*

In seeking conditions for existence, Chen introduced the notions of "neck domain" and "tail domain". A subdomain $\Omega^* \subset \Omega$ cut off by Γ is called a *tail domain* if Γ is a circular arc of radius Ω/Σ that meets Σ with angle $\gamma = 0$ measured in Ω^*, and if there is no other such arc interior to Ω^* (Fig. 6.10). Chen showed that *if Ω contains a tail domain, then no solution exists for $\gamma = 0$.*

A neck domain is illustrated in Fig. 6.11. Chen showed that for arbitrarily small openings, solutions can exist; thus his sufficiency condition above is not necessary.

Finn [56] gave the following reformulation of Giusti's theorems:

A sufficient condition for existence of a solution is that there exist a vector field $w = (w^1, w^2)$ in $\bar{\Omega}$, with

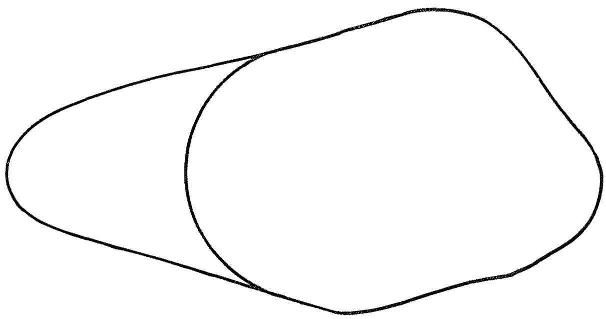


Figure 6.10. Tail domain.

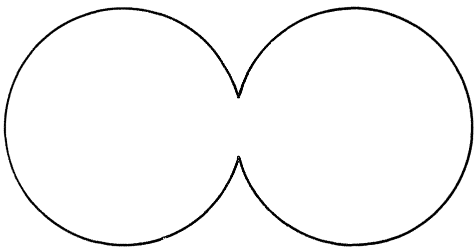


Figure 6.11. Neck domain.

$$\begin{aligned} \operatorname{div} w &= \frac{\Sigma}{\Omega} && \text{in } \Omega \\ \nu \cdot w &= 1 && \text{on } \Sigma \\ |w| &< \frac{1}{\cos\gamma} && \text{in } \bar{\Omega}. \end{aligned} \tag{6.20a, b, c}$$

To prove the result, we integrate (6.20a) over $\Omega^* \subset \Omega$ and apply the divergence theorem. The relation (6.14) follows as in Lemma 6.1.

We note that in contrast to the previously cited results, which apply only to the case $\gamma = 0$, this criterion applies for every γ.

For particular domains Ω, a field $w(x, y)$ can be constructed explicitly. For example, for the parallelogram of Fig. 6.12, with coordinate origin at the point of symmetry, the field $w(x, y)$ with

$$\begin{aligned} w^1 &= \frac{1}{a \sin 2\alpha}\left[x + \left(\frac{1}{b} - \frac{1}{a}\right) y \cot 2\alpha\right] \\ w^2 &= \frac{1}{b \sin 2\alpha} y \end{aligned} \tag{6.21}$$

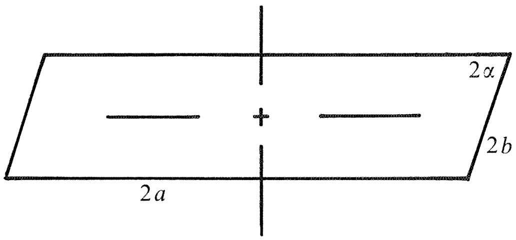


Figure 6.12. Parallelogram domain: solution exists if $\alpha+\gamma\geq\pi/2$.

has the indicated properties whenever $\alpha+\gamma>\pi/2$, and thus a solution exists in that case. We shall show in Theorems 7.10 and 7.11 that a solution exists also for $\alpha+\gamma=\pi/2$. Since by Theorem 6.2 there can be no solution when $\alpha+\gamma<\pi/2$, the result is sharp; that is, the identical criterion applies to a parallelogram as does for a regular polygon.

The result does not generalize to every polygon. Consider the trapezoidal configuration of Fig. 6.13. According to the above result, if $a=b$ (that is, for any rectangle) there is a solution whenever $\gamma>\pi/4$. Let us choose Γ, Ω^* as indicated, with Γ bisecting the altitude. After some manipulation the inequality (6.7) – for that choice of Γ – takes the form

$$\cos\gamma<\frac{2(b+a)^2}{|(b-a)-(b+a)\cos 2\alpha|}\,\frac{\cos 2\alpha}{b-a}. \tag{6.22}$$

Let a, b be prescribed, $a<b$. Let γ be *arbitrary* in $0\leq\gamma<\pi/2$. From (6.22) we see immediately that *there exists* $\varepsilon>0$ *such that whenever* $|(\pi/4)-\alpha|<\varepsilon$ *there is no solution to the problem.* Choosing γ in the range $\pi/4<\gamma<\pi/2$, a solution will exist in any rectangle, but for α sufficiently close to $\pi/4$, there will hold $\alpha+\gamma>\pi/2$ and the solution will nevertheless fail to exist in the trapezoid. In a sense, the closer the trapezoid is to rectangular shape, the more solutions are excluded from existence. Thus, the criterion for a rectangle (or parallelogram) does not apply to a trapezoid; a kind of unstable dependence on the data occurs, which differs from the angle discontinuity. *An arbitrarily small deviation from the rectangular configuration, throughout which the condition* $\alpha+\gamma>\pi/2$ *holds uniformly, can lead from existence to nonexistence of a solution.*

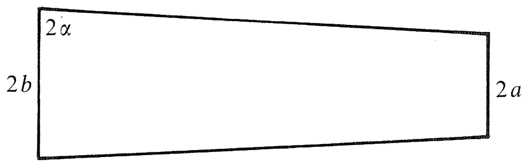


Figure 6.13. Trapezoidal domain: for fixed a,b and any $\gamma<\pi/2$, solution fails to exist when $|\pi/4-\alpha|<\varepsilon$.

6.5. A Subsidiary Extremal Problem

In seeking general conditions for existence of a solution, we may try to minimize $\Phi[\Omega^*]$. The procedure is facilitated if we observe a striking analogy with the original variational problem for $\mathscr{S}$. Recalling the definition $H_\gamma = (\Sigma/\Omega)\cos\gamma$, the energy expression (1.24) for the original problem becomes, up to the factor σ,

$$\mathscr{E}[\mathscr{S}] \equiv \mathscr{S} - \mathscr{S}^* \cos\gamma + H_\gamma V \tag{6.23}$$

where $\mathscr{S}^*$ is the wetted boundary area. If we compare (6.23) with

$$\Phi[\Omega^*] \equiv \Gamma - \Sigma^* \cos\gamma + H_\gamma \Omega^*, \tag{6.24}$$

we see that we are confronted with the same type of variational problem as the original one. The only differences are i) it is now in one lower dimension, ii) the multiplier H_γ is now no longer an unknown but is prescribed in advance, and iii) the container has now a general, rather than cylindrical, form, so that the problem must be studied parametrically. As indicated by our choice of notation (see Note 1), we focus attention on the set Ω^* enclosed by Γ and Σ^*, as determined by its characteristic function φ_{Ω^*}.

6.6. Minimizing Sequences

We consider a minimizing sequence $\{\Omega_j^*\}$ for the functional $\Phi[\Omega^*]$. For each member of the sequence there holds $\Omega_j^* \subset \Omega$, $\Sigma_j^* \subset \Sigma$. In view of the minimizing property, $\Phi[\Omega_j^*]$ is bounded above, hence Γ_j is bounded in the sequence; that is, the sets $\{\Omega_j^*\}$ have uniformly bounded perimeters in Ω. We conclude (cf. Theorem 1.19 in [89]) that *there is a subsequence of the* $\{\varphi_{\Omega_j^*}\}$ *that converges in* $L^{(1)}(\Omega)$ *to* φ_{Ω^0}, *and that*

$$\Gamma^0 = \int_\Omega |D\varphi_{\Omega^0}| \le \liminf \int_\Omega |D\varphi_{\Omega_j^*}|.$$

Lemma 6.3 (Lower Semicontinuity). *Let* Ω *satisfy the conditions of Lemma* 6.1, *and suppose that for any* $\varepsilon > 0$ *a partition can be found such that* $\max_k \sqrt{1+L_k^2} < (1+\varepsilon)/\cos\gamma$. *Let* $\{\Omega_j^*\}$ *be a sequence as above, and let* $\{\gamma_j\}$ *be a corresponding sequence of boundary angles, such that* $\gamma_j \to \gamma$. *Then*

$$\Phi[\Omega^0;\gamma] \le \liminf \Phi[\Omega_j^*;\gamma_j]. \tag{6.25}$$

Proof (Gerhardt [74]). We may write

$$\begin{aligned}\Phi[\Omega^0;\gamma]-\Phi[\Omega_j^*;\gamma_j]\le\Big(\int_\Omega|D\varphi_{\Omega^0}|-\int_\Omega|D\varphi_{\Omega_j^*}|\Big)+H_\gamma\int_\Omega|\varphi_{\Omega^0}-\varphi_{\Omega_j^*}|\,dx\\ +\cos\gamma\int_\Sigma|\varphi_{\Omega^0}-\varphi_{\Omega_j^*}|\,ds+\varepsilon_1^*\int_\Omega|\varphi_{\Omega_j^*}|\,dx\\ +\varepsilon_2^*\int_\Sigma|\varphi_{\Omega_j^*}|\,ds\end{aligned}$$

where ε_1^*, $\varepsilon_2^*\to 0$. To the term $\int_\Sigma|\varphi_{\Omega^0}-\varphi_{\Omega_j^*}|\,ds$ we apply (6.16) with $\beta\equiv\cos\gamma$; we obtain

$$\begin{aligned}\Phi[\Omega^0;\gamma]-\Phi[\Omega_j^*;\gamma_j]\le\Big(\int_{\Omega_\delta}|D\varphi_{\Omega^0}|-\int_{\Omega_\delta}|D\varphi_{\Omega_j^*}|\Big)\\ +H_\gamma\int_\Omega|\varphi_{\Omega^0}-\varphi_{\Omega_j^*}|\,dx+\Upsilon\int_{\mathscr{A}_\delta}|\varphi_{\Omega^0}-\varphi_{\Omega_j^*}|\,dx\\ +(2+\varepsilon)\int_{\mathscr{A}_\delta}|D\varphi_{\Omega^0}|+\varepsilon\int_{\mathscr{A}_\delta}|D\varphi_{\Omega_j^*}|+\varepsilon_1^*\Omega+\varepsilon_2^*\Sigma,\end{aligned}$$

where we have set $\Omega_\delta=\Omega\setminus\mathscr{A}_\delta$. We have already observed that the superior limit of the first three terms on the right is nonpositive. The fourth term can be made arbitrarily small by suitable choice of δ, following which the fifth term can be made small by choice of ε, since the $\{\Omega_j^*\}$ have bounded perimeters. Thus, $\limsup(\Phi[\Omega^0;\gamma]-\Phi[\Omega_j^*;\gamma_j])\le 0$, as was to be shown.

6.7. The Limit Configuration

It can happen that the perimeter Γ^0 of Ω^0 in Ω is the null set. If $\Gamma^0\neq\emptyset$, we apply a theorem of Massari [123], which yields that Γ^0 *consists of analytic arcs in* Ω. We wish to characterize the geometry of the arcs in relation to that of Ω.

Lemma 6.4. *The arcs of Γ^0 are circular, of radius $R_\gamma=H_\gamma^{-1}=\Omega/\Sigma\cos\gamma$, and Ω^0 lies on the side of Γ^0 opposite to that into which the curvature vector points.*

Proof. Since Γ^0 minimizes among all configurations, it must do so among neighboring curves, so that its first variation must vanish. Given a point of Γ^0, we adopt a coordinate system so that Γ^0 is a graph $u(x)$ in some interval $a\le x\le b$, with Ω^0 below the curve, and we make a variation with support in this interval. Neglecting terms in Φ that remain constant, the expression to be varied becomes

$$\int_a^b \sqrt{1+|\nabla u|^2}\,dx + H_\gamma \int_a^b u\,dx, \tag{6.26}$$

the Euler equation for which is

$$\frac{d}{dx}\frac{u_x}{\sqrt{1+u_x^2}} = H_\gamma, \tag{6.27}$$

and thus the curvature of Γ^0 is identically H_γ, as was to be shown.

Corresponding to $\Gamma^0 = \emptyset$ there are two possibilities: either $\Omega^0 = \emptyset$ or $\Omega^0 = \Omega$. In the former case one has immediately that the corresponding $\Phi^0 = 0$. In the latter case, $\Phi^0 = -\Sigma\cos\gamma + H_\gamma \Omega = 0$, since $H_\gamma = (\Sigma/\Omega)\cos\gamma$. We have proved:

Lemma 6.5. *The nonexistence of a minimizing set* $\neq \emptyset, \Omega$ *for* $\Phi[\Omega^*;\gamma]$ *in* Ω *is equivalent to* $\Phi[E;\gamma] > 0$ *for all admissible* $E \neq \emptyset, \Omega$.

In view of Theorem 6.7, we may rephrase this result as follows whenever the regularity condition (6.14) holds:

Theorem 6.8 (Nonexistence-Existence Principle). *The nonexistence of a minimizing solution* $\neq \emptyset$, Ω *in* Ω *to the subsidiary variational problem for* $\Phi[\Omega^*;\gamma]$ *is equivalent to the existence of a solution of the original variational problem* (6.4), (6.2) *for a capillary surface over* Ω. *The surface is determined, when it exists, as a solution of* (6.4), *with* (6.2) *holding in a generalized sense, see Chapter* 7.

We recall that (6.14) is not explicitly needed when $\gamma = 0$. If $\gamma > 0$ and Σ is piecewise smooth, it suffices to have $\alpha + \gamma > \pi/2$ at every corner of opening angle $2\alpha < \pi$, while if $2\alpha \geq \pi$, no condition is needed. If $\alpha + \gamma < \pi/2$, then according to Theorem 6.2 there is no solution to the capillary problem over Ω. The case $\alpha + \gamma = \pi/2$ has a special interest; it will be shown in the following chapter that a generalized solution then always exists when (6.7) holds, the associated surface energy may however be infinite in some cases.

Lemma 6.6. *Let* Ω *satisfy the condition* (6.14) *with* $\mu < 1/\cos\gamma$ *and also the isoperimetric condition* (6.15). *Then there exists* $\delta(\Omega;\gamma) > 0$ *such that* $\Phi[\Omega^*;\gamma]$ *and also the "adjoint" functional*

$$\Psi[\Omega^*;\gamma] \equiv \Gamma + \Sigma^* \cos\gamma - H_\gamma \Omega^* \tag{6.28}$$

are both positive whenever $\Omega^* < \delta$.

Proof. By (6.14) we have

$$\Phi[\Omega^*;\gamma] \ge (1-\mu\cos\gamma)\Gamma - C\Omega^*$$

for some fixed constant C. If $\Omega^* < \delta$ and δ is small enough, then (6.15) yields

$$\Phi[\Omega^*;\gamma] \ge (1-\mu\cos\gamma - C\sqrt{\delta})\Gamma > 0$$

for small enough δ. The corresponding result for Ψ requires only the use of (6.15).

From Lemma 6.6 we obtain

Lemma 6.7. *Under the conditions of Lemma* 6.6, *the number of components in a minimizing set* Ω^0 *is finite.*

We also have:

Lemma 6.8. *Under the conditions of Lemma* 6.6, *the number of circular arcs* Γ *on the boundary of a minimizing set* Ω^0 *is finite.*

Proof. It suffices to show that there exists $\delta > 0$ such that no arc $\hat{\Gamma}$ bounds with $\hat{\Sigma} \subset \Sigma$ a set $\hat{\Omega}$ of diameter $< \delta$. By Lemma 6.6, we may assume that $\hat{\Omega}$ contains no component of Ω^0. Removal of $\hat{\Gamma}$ thus leads to a change

$$\begin{aligned} \delta\Phi &= -\hat{\Gamma} - \hat{\Sigma}\cos\gamma + H_\gamma\hat{\Omega} \\ &< (-1 + CH_\gamma\sqrt{\hat{\Omega}})\hat{\Gamma} \end{aligned}$$

by the isoperimetric inequality (6.15), and thus $\delta\Phi < 0$ if δ is small enough.

Using Lemma 6.4, we find:

Lemma 6.9. *If two boundary arcs* Γ *of the minimizing set meet on* Σ *so as to form an angle* ω *within* Ω^0, *then* $\omega \ge \pi$.

Proof. Let p be a point of contact, let q_1, q_2 be points on the respective arcs equidistant from p. If $\omega < \pi$ and the points are close enough to p, then the new domain Ω^0 obtained by replacing the broken arc $\widehat{q_1 p q_2}$ by the segment $\overline{q_1 q_2}$ will yield a smaller Φ.

6.8. The First Variation

In view of Theorem 6.8, our principal interest will be to characterize those configurations for which the minimizing $\Omega^0 = \emptyset$, Ω. Even when that occurs, it can happen that extremal arcs Γ appear, which satisfy formal (first) variational conditions but do not minimize. We therefore wish to examine in further detail the interrelationship between the variational conditions and the boundary geometry. In the remaining sections of this chapter, we assume that Σ is piecewise smooth, in the sense that $\Sigma \in C^{(2)}$ in local coordinates, except possibly for a finite number of corner points, at each of which two uniformly smooth boundary arcs meet at an angle 2α ($\leq 2\pi$, measured within Ω), for which $\alpha+\gamma>\pi/2$. The condition (6.14) is then always satisfied. We then obtain from Lemmas 6.4 and 6.7–6.9:

Theorem 6.9. *If $\alpha+\gamma>\pi/2$ at each corner point, a minimizing set Ω^0 consists of a finite number of components bounded by a finite number of circular arcs $\Gamma\subset\Omega$ of radius $R_\gamma=\Omega/\Sigma\cos\gamma$, and by a finite number of subarcs $\Sigma^0\subset\Sigma$. The arcs Γ (and sets Σ^0) are mutually disjoint except perhaps for contact at corner points with opening angle $2\alpha>\pi$.*

Theorem 6.9 is essential in what follows, as it will permit us to introduce general variations of the extremal arcs Γ without creating inadmissible configurations, such as multiple coverings of Ω or Σ.

To express the variational condition on an arc Γ of $\{\Gamma^0\}$ it is convenient to adopt polar coordinates, with origin at the center of Γ. Then Γ and Σ^* may be represented in the forms (see Fig. 6.14)

$$\Gamma: \quad r=r(\theta)\equiv R_\gamma, \qquad \theta_1\leq\theta\leq\theta_2 \tag{6.29}$$

$$\Sigma^*: \quad \rho=\rho(t);\theta=\varphi(t), \qquad t_1\leq t\leq t_2. \tag{6.30}$$

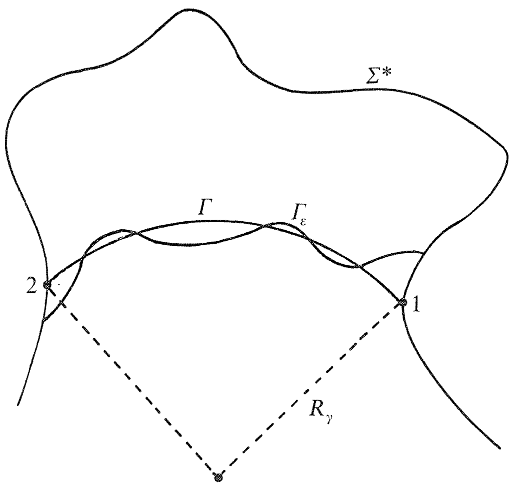


Figure 6.14. Coordinates for variational condition.

In the latter representation, we may clearly suppose $\rho'^2+\varphi'^2>\delta>0$ in $t_1 \le t \le t_2$, and that $t_1=\theta_1$, $t_2=\theta_2$.

Let us embed Γ in a one-parameter family of curves Γ_ε: $r=r(\theta;\varepsilon)$, with $r(\theta,0)\equiv R_\gamma$. The intersection points with Σ^* will be determined by the relations

$$r_i=r(\theta_i;\varepsilon)=\rho_i=\rho(t_i) \tag{6.31}$$

$$\theta_i=\varphi(t_i), \qquad i=1,2. \tag{6.32}$$

In terms of the notation

$$f(r;r')=\sqrt{r^2+r'^2}, \qquad r'=\frac{\partial r}{\partial\theta}, \tag{6.33}$$

$$g(\rho;\rho';\gamma)=\sqrt{\rho^2\varphi'^2+\rho'^2}\cos\gamma, \qquad \rho'=\frac{d\rho}{dt}, \tag{6.34}$$

we find, setting $\Phi(\varepsilon)=\Phi[\Gamma_\varepsilon]$,

$$\Phi(\varepsilon)=\int_{\theta_1}^{\theta_2}\left[f(r;r')-\frac{1}{2R_\gamma}r^2\right]d\theta-\int_{t_1}^{t_2}\left[g(\rho;\rho';\gamma)-\frac{1}{2R_\gamma}\rho^2\varphi'\right]dt$$

from which, denoting differentiation in ε by $(\dot{})$,

$$\dot\Phi(\varepsilon)=\int_{\theta_1}^{\theta_2}\left(f_r\dot r+f_{r'}\dot r'-\frac{r}{R_\gamma}\dot r\right)d\theta+\left\{\left(f-\frac{1}{2R_\gamma}r^2\right)\dot\varphi-\left(g-\frac{1}{2R_\gamma}\rho^2\varphi'\right)\dot t\right\}_1^2. \tag{6.35}$$

From (6.31) and (6.30) we find

$$\begin{aligned}\dot r_i&=(\rho_i'-r_i'\varphi_i')\dot t_i\\ \dot\varphi_i&=\varphi_i'\dot t_i, \qquad i=1,2.\end{aligned} \tag{6.36}$$

An integration by parts in (6.35) thus yields

$$\dot\Phi(\varepsilon)=\int_{\theta_1}^{\theta_2}\left(f_r-\frac{r}{R_\gamma}+\frac{d}{d\theta}f_{r'}\right)\dot r\,d\theta+\{[(\rho'-r'\varphi')f_{r'}+(f\varphi'-g)]\dot t\}_1^2 \tag{6.37}$$

since $r=\rho$ at the endpoints.

When $\varepsilon=0$, then $r\equiv R_\gamma$ and the integrand in (6.37) vanishes. We obtain, using (6.36),

$$\dot\Phi(0)=\{[R_\gamma\varphi'-\sqrt{\rho'^2+R_\gamma^2\varphi'^2}\cos\gamma]\dot t\}_1^2, \tag{6.38}$$

whenever the points P_1, P_2 lie interior to regular boundary arcs on Σ. If either P_1 or P_2 are corner points, then (6.38) still has a meaning, as a onesided derivative along any adjoining arc that makes with Γ an angle $\leq \pi$ (cf. Figs. 6.15 and 6.16).

It suffices to consider this latter case, as the former is obtained from it when the two adjacent arcs meet with angle π. Since the choice of variation is arbitrary, we may clearly assume $\dot{t}=1$. Since at a minimum there must hold $\dot{\Phi}(0) \geq 0$, we find first from (6.38) that

$$\varphi' R_\gamma \leq \sqrt{\rho'^2 + (\varphi' R_\gamma)^2} \cos \gamma$$

at the point P_1, and thus (see Fig. 6.15) that

$$\cos \beta' \leq \cos \gamma$$

at P_1; that is, $\beta' \geq \gamma$. A similar reasoning, applied to the adjacent arc Σ'' (see Fig. 6.16) yields $\beta'' \geq \pi - \gamma$. Thus, $\beta' + \beta'' \geq \pi$. The same relations clear-

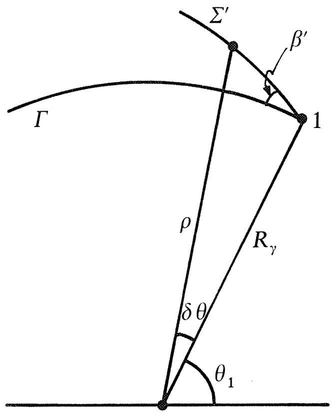


Figure 6.15. The first variation; case 1.

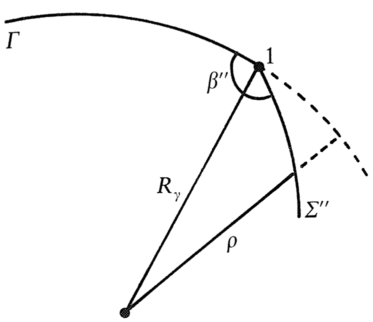


Figure 6.16. The first variation; case 2.

ly hold also at P_2. If P is a regular point, then $\beta' + \beta'' = \pi$ and hence $\beta' = \gamma$, $\beta'' = \pi - \gamma$.

We consider finally the case $\gamma = 0$. The relation $\beta' \geq \gamma$ now requires no proof. Suppose $\beta'' < \pi - \gamma = \pi$ as indicated in Fig. 6.16. Consider a sequence of points on $\Sigma \setminus \Sigma^*$ approaching the contact point and of distance l to P. If we join these points to points on Γ equidistant to P by straight segments and replace the corresponding part of Γ by the new segment (of length $d < 2l$), the change in Φ can be shown to be

$$d - 2l + O(l^2).$$

Thus the change in Φ would be negative for small l, contradicting the assumed minimizing property.

We summarize the result obtained thus far:

Theorem 6.10. *Let Γ be a component of a minimizing set $\{\Gamma\}$, lying in Ω. Then Γ is a circular arc of radius $R_\gamma = \Omega/\Sigma \cos\gamma$, meeting Σ in distinct points P_1, P_2. If $\gamma > 0$, Γ is isolated from all other components of $\{\Gamma\}$; if $\gamma = 0$, Γ can conceivably continue into other components across isolated points of Σ, as part of the same circular arc. On the side of Γ opposite to that into which its curvature vector points, it meets Σ in an angle $\beta' \geq \gamma$; on the remaining side, it makes with Σ an angle $\beta'' \geq \pi - \gamma$. If a contact point P is interior to a regular arc of Σ, then $\beta' = \gamma$, $\beta'' = \pi - \gamma$. No arc Γ of a minimizing set can meet Σ at a corner point P at which the interior angle 2α is less than π. If $\gamma > 0$ and $\alpha + \gamma > \pi/2$ at each corner, then $\{\Gamma\}$ contains at most finitely many components.*

6.9. The Second Variation

To obtain further information, we examine the second variation of Φ, in an extremal configuration at the value $\varepsilon = 0$. For simplicity we restrict attention to the situation in which neither contact point is a corner point of Σ. By Theorem 6.10, the extremal arc Γ meets Σ in equal angles γ at P_1, P_2. Since, as we have assumed, $0 \leq \gamma < \pi/2$, it is clear that we can choose $\theta \equiv t$ in neighborhoods of P_1, P_2. The relation (6.37) then takes the form

$$\dot{\Phi}(\varepsilon) = \int_{\theta_1}^{\theta_2} \left(f_r - \frac{1}{R_\gamma} r + \frac{d}{d\theta} f_{r'} \right) \dot{r}\, d\theta + \{[f - g + (\rho' - r') f_{r'}]\dot{\theta}\}_1^2. \qquad (6.39)$$

We differentiate (6.39) in ε, and set $\varepsilon = 0$. Using the variational conditions we have already found, and the boundary relations (6.36), we obtain after some cancellations

$$\begin{aligned}\ddot{\Phi}(0)=&\int_{\theta_1}^{\theta_2}\left[f_{rr}\dot{r}+f_{rr'}\dot{r}'-\frac{1}{R_\gamma}\dot{r}-\frac{d}{d\theta}(\dot{r}f_{rr'}+\dot{r}'f_{r'r'})\right]d\theta\\&+\{[(r'\dot{\theta}+\dot{r})f_r-\rho'\dot{\theta}g_\rho-\rho''\dot{\theta}g_{\rho'}+\rho''\dot{\theta}f_{r'}\\&+(\rho'-r')((r'\dot{\theta}+\dot{r})f_{rr'}+(r''\dot{\theta}+\dot{r}')f_{r'r'})]\dot{\theta}\}_1^2.\end{aligned}\tag{6.40}$$

Integration by parts in (6.40) leads to further cancellations; using again the boundary relations (6.36), we are led to the form

$$\begin{aligned}\ddot{\Phi}(0)=&\int_{\theta_1}^{\theta_2}\left[\left(f_{rr}-\frac{1}{R_\gamma}\right)\dot{r}^2+2f_{rr'}\dot{r}\dot{r}'+f_{r'r'}\dot{r}'^2\right]d\theta\\&+\{[(f_r-g_\rho)\rho'+(f_{r'}-g_{\rho'})\rho''+(\rho'-r')r'f_{rr'}\\&+(\rho'-r')r''f_{r'r'}]\dot{\theta}^2\}_1^2.\end{aligned}\tag{6.41}$$

On an extremal arc Γ, we have $r\equiv R$. Using the explicit forms (6.34) for f and g, we obtain

$$\ddot{\Phi}(0)=Q+\left\{\left[1-\frac{\rho+\rho''}{\sqrt{\rho^2+\rho'^2}}\cos\gamma\right]\dot{\theta}^2\rho'\right\}_1^2\tag{6.42}$$

with

$$Q=\frac{1}{R_\gamma}\int_{\theta_1}^{\theta_2}(\dot{r}'^2-\dot{r}^2)\,d\theta.\tag{6.43}$$

Let k denote the scalar curvature of Σ, considered positive in the direction of the curvature vector. A calculation yields

$$\rho+\rho''=\frac{\rho^2+\rho'^2}{\rho}(2-k\sqrt{\rho^2+\rho'^2}).\tag{6.44}$$

Placing (6.44) into (6.42), referring to Figs. 6.15 and 6.16 (with $\beta'=\beta''=\gamma$), and using again the boundary relations (6.36), we find after simplification

$$\ddot{\Phi}(0)=Q+\frac{\cot\gamma}{R_\gamma}\left[\left(1-k_1\frac{R_\gamma}{\cos\gamma}\right)\dot{r}_1^2+\left(1-k_2\frac{R_\gamma}{\cos\gamma}\right)\dot{r}_2^2\right]\tag{6.45}$$

when $\gamma\neq 0$. If $\gamma=0$, we restrict ourselves to variations that vanish at the endpoints and find

$$\ddot{\Phi}=Q.\tag{6.46}$$

6.10. Solution of the Jacobi Equation

We wish to determine conditions under which (6.45) or (6.46) are necessarily positive for any (nontrivial) choice of $\dot{r}$, and conditions under which they can be made negative. We thus consider the problem of minimizing (6.45), under a suitable normalization. Under obvious notational designations, we are led to a minimum problem for the functional

$$\mathscr{I}[\eta]=\int_{-\delta}^{\delta}(\eta'^2-\eta^2)\,d\theta+A_1\eta_1^2+A_2\eta_2^2. \tag{6.47}$$

Here we have rotated coordinates so that $\theta_1=-\delta$, $\theta_2=\delta$. We seek to minimize (6.47) under the constraint

$$\eta_1^2+\eta_2^2\le 1. \tag{6.48}$$

If $\theta_2-\theta_1\ge\pi$ (i.e., $\delta\ge\pi/2$), we observe that the choice

$$\eta=\begin{cases}\dfrac{\sqrt{2}}{2\cos\lambda\dfrac{\pi}{2}}\cos\lambda\theta, & |\theta|\le\dfrac{\pi}{2},\\[2ex] \dfrac{1}{2}\sqrt{2}, & |\theta|\ge\dfrac{\pi}{2},\quad 0<\lambda<1\end{cases}$$

leads to an estimate

$$\int_{-\delta}^{\delta}(\eta'^2-\eta^2)\,d\theta<-\frac{1}{\lambda}\tan\lambda\frac{\pi}{2}$$

which $\to-\infty$ as $\lambda\to 1$. Thus $\ddot{\Phi}(0)$ can be made negative, and we find

Theorem 6.11. *If* $0<\gamma<\pi/2$, *then on any minimizing arc* Γ *there holds* $\theta_2-\theta_1<\pi$.

That is, any minimizing arc is a strict subarc of a semicircle.

In general, we note that the Euler equation for (6.47) is

$$\eta''+\eta=0, \tag{6.49}$$

and thus any interval exceeding π contains a pair of conjugate points. Thus, in this case $\ddot{\Phi}(0)$ can be made negative by a variation that leaves the endpoints fixed. We obtain:

Theorem 6.11 (a). *If* $\gamma=0$, *then on any minimizing arc* Γ *there holds* $\theta_2-\theta_1\le\pi$.

If $\theta_2-\theta_1<\pi$, we have $\delta<\pi/2$, and the minimizing condition for (6.45) leads to a choice

$$\eta=a\cos(\theta-\sigma), \tag{6.50}$$

where σ, a satisfy the relations

$$\begin{aligned}\frac{2\sin 2\sigma}{\cos 2\delta+\cos 2\sigma}&=A_1-A_2\\&=(k_2-k_1)\frac{R_\gamma}{\sin\gamma}\end{aligned} \tag{6.51}$$

and

$$a^2\leq\frac{1}{1+\cos 2\delta\cos 2\sigma}. \tag{6.52}$$

For any choice of σ, $a>0$, we have

$$\frac{1}{a^2}\mathscr{I}[\eta]=-\sin 2\delta\cos 2\sigma + \cot\gamma\left\{\cos^2(\delta-\sigma)\left[1-k_2\frac{R_\gamma}{\cos\gamma}\right]+\cos^2(\delta+\sigma)\left[1-k_1\frac{R_\gamma}{\cos\gamma}\right]\right\}. \tag{6.53}$$

The desired minimum is provided by one of the two solutions of (6.51), (6.52).

We note that (6.50) defines a rigid motion of Γ in the direction σ. In view of the general nonexistence-existence principle Theorem 6.8, we have proved:

Theorem 6.12. *If $\gamma>0$ and if for one of the solutions of* (6.51), (6.52) *there holds $\mathscr{I}[\eta]<0$ for every strict subarc Γ of a semicircle of radius R_γ that meets Σ in equal angles γ (measured exterior to the semicircle), then there exists a solution of* (6.4), (6.2).

We may also obtain a condition expressed entirely in terms of boundary curvatures at the intersection points:

Corollary. *If $\gamma>0$ and if the curvatures k_1, k_2 at the intersections with Σ of every subarc Γ as above satisfy $(k_1+k_2)R_\gamma\geq 2\cos\gamma$, then there exists a solution of* (6.4), (6.2).

Proof. The minimum of $\mathscr{I}[\eta]$ cannot exceed its value when $\sigma=0$, which under the hypothesis is negative.

We remark that the proof of the Corollary yields also an extension to smooth boundary points of the result of Theorem 6.10 that a minimizing extremal cannot enter a corner point $P\in\Sigma$ at which $2\alpha<\pi$.

The following additional consequence of the nonexistence-existence principle makes only limited use of the second variation; we formulate it for the general case of piecewise smooth domains (see §6.8).

Theorem 6.13. *Suppose either Σ is smooth or that $\alpha+\gamma>\pi/2$ at every corner, and suppose there is no subarc $\Gamma\subset\Omega$, of a semicircle of radius $\Omega/\Sigma\cos\gamma$, that contacts Σ in two distinct points, at each of which there holds $\beta'\geq\gamma$, $\beta''\geq\pi-\gamma$ (see Theorem* 6.10). *Then there exists a solution of* (6.4), (6.2).

Proof. For any minimizing set $\{\Gamma\}\subset\Omega$, each $\Gamma\in\{\Gamma\}$ must by Theorem 6.11 or 6.11(a) be a subarc of a semicircle, while by Theorem 6.10 there must hold $\beta'\geq 0$, $\beta''\geq\pi$ at each of two distinct contact points with Σ. Under our hypothesis the minimizing set is empty; by the nonexistence-existence principle Theorem 6.8 a solution of (6.4), (6.2) must exist.

Whenever a solution of (6.4), (6.2) fails to exist, there must be a nontrivial minimizing configuration $\{\Gamma\}$ in Ω, for which $\Phi\leq 0$. We have:

Theorem 6.14. *If $\gamma>0$, $\theta_2-\theta_1<\pi$ and if $\mathscr{I}[\eta]$, as given by* (6.47), *is positive, then the given extremal Γ provides a strict strong relative minimum.*

Proof. If $\varepsilon>0$ is sufficiently small, then any pair of points in ε neighborhoods of P_1, P_2 can be joined by a unique strict subarc of a semicircle of radius R_γ, with the same orientation as Γ. The usual procedures of the calculus of variations (cf. the discussion in [2]) show that every such arc provides a strict strong minimum relative to all neighboring arcs through the same endpoints. Thus it suffices to restrict attention to these circular arcs.

We assert that ε can be chosen so that, for any arc $\hat{\Gamma}$ joining respective points of the two neighborhoods, there holds $\Phi[\hat{\Gamma}]>\Phi[\Gamma]$. If not, there would be a sequence $\varepsilon\to 0$ and a sequence of corresponding $\hat{\Gamma}_\varepsilon$, for which $\Phi[\hat{\Gamma}_\varepsilon]\leq\Phi[\Gamma]$. We can embed these arcs in a continuous family $r(\theta;\varepsilon)$, with $\dot{r}^2(\theta_1;0)+\dot{r}^2(\theta_2;0)=1$. We have $\dot{\Phi}(0)=0$, while by our hypothesis and continuity considerations there holds $\ddot{\Phi}(\varepsilon)>0$ for ε sufficiently small; thus it would be possible to choose ε so that $\Phi[\hat{\Gamma}_\varepsilon]>\Phi[\Gamma]$, contradicting the supposed construction.

Theorem 6.15. *If $\gamma=0$, then every extremal Γ joining regular points of Σ and for which $\theta_2-\theta_1<\pi$ provides a strict strong relative minimum.*

Proof. Since the conjugate points on Γ (determined by the integrand of Q) are separated by intervals of π, Γ can be embedded locally into a field of extremals (Fig. 6.17). Considering the variational problem for $\mathscr{I}$ as a problem in parametric form, one easily finds that the corresponding Legendre condition is satisfied (cf. [2, p. 63]). Thus, Γ provides a strict strong minimum for Φ, relative to all curves lying in a (physical) neighborhood $\mathscr{N}_\Gamma$ and having the same endpoints P_1, P_2.

Let $\mathscr{C}\subset\mathscr{N}_\Gamma$ be a curve joining points Q_1, Q_2 on Σ. We obtain a curve $\mathscr{C}^+\subset\mathscr{N}_\Gamma$ joining P_1, P_2 by adjoining the segments $\widehat{P_1Q_1}$, $\widehat{P_2Q_2}$ (see

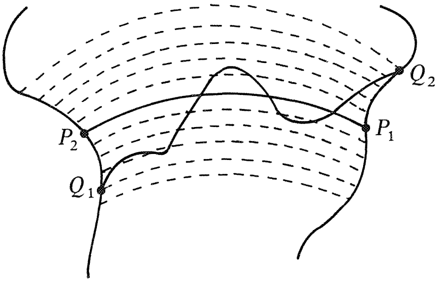


Figure 6.17. Local embedding.

Fig. 6.17). Since Ω^* is not changed and $\cos\gamma = 1$, we find $\Phi[\mathscr{C}^+] = \Phi[\mathscr{C}]$. But if $\mathscr{C} \subset \mathscr{N}$, then $\mathscr{C}^+ \subset \mathscr{N}$, hence $\Phi[\mathscr{C}] > \Phi[\Gamma]$ unless $\mathscr{C} = \Gamma$.

The diversity of situations that can occur is illustrated by the following result.

Theorem 6.16. *If k_1, $k_2 \le 0$ and if $\delta + \gamma < \pi/2$, then Γ provides a strict strong relative minimum. If $k_1 + k_2 \ge 0$ and if $\delta + \gamma > \pi/2$, then Γ cannot yield a minimum.*

Proof. The two situations are illustrated in Figs. 6.18(a), (b).

If k_1, $k_2 \le 0$ and $\delta + \gamma < \pi/2$, then from (6.53) we have, when $\gamma \neq 0$,

$$\begin{aligned} \frac{1}{a^2}\mathscr{I} &\ge -\sin 2\delta \cos 2\sigma + \cot\gamma\{\cos^2(\beta-\sigma) + \cos^2(\beta+\sigma)\} \\ &> -\sin 2\delta \cos 2\sigma + \tan\delta\{\cos^2(\delta-\sigma) + \cos^2(\delta+\sigma)\} \\ &= 2\tan\delta \sin^2\sigma \ge 0 \end{aligned}$$

and the result then follows from Theorem 6.14. If $\gamma = 0$, then Theorem 6.15 yields the result.

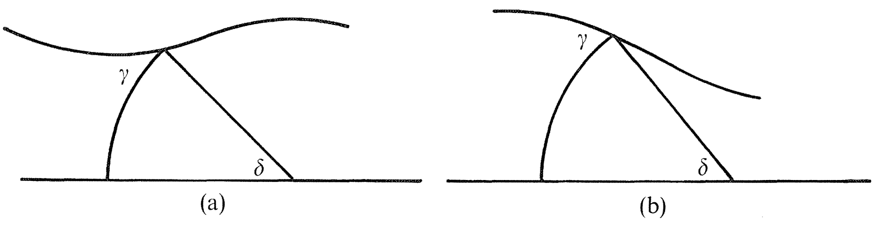


Figure 6.18. (a) $k \le 0$, $\delta + \gamma < \pi/2$. (b) $k_1 + k_2 \ge 0$, $\delta + \gamma > \pi/2$.

If $k_1+k_2\geq 0$ and $\delta+\gamma>\pi/2$, we set $\sigma=0$ in (6.53) to obtain

$$\begin{aligned}\frac{1}{a^2}\mathscr{I} &\leq -\sin 2\delta+2\cos^2\delta\cot\gamma\\ &< -\sin 2\delta+2\cos\delta\sin\delta=0\end{aligned}$$

so that Γ cannot minimize.

6.11. Convex Domains

In a general situation an extremal must be given before (6.53) can be applied, and in order to obtain information relative to existence it is necessary to examine all possible extremals. The number of extremals corresponding to differing configurations can vary from zero to infinity, as may easily be seen from explicit examples. However, if Ω is convex, the situation becomes much simpler, as then k_1, $k_2\geq 0$ and the angle δ can be estimated in terms of the maximal boundary curvature.

Lemma 6.10. *Let $\gamma>0$ and suppose the curvature k of Σ satisfies $0\leq k\leq k_M\leq\infty$. (The last equality can occur at a singular boundary point.) Let Γ be an interior extremal arc of a minimizing set, and suppose $\delta+\gamma\leq\pi/2$. Then $R_\gamma k_M>1$, and*

$$\tan\delta\geq\frac{\sin\gamma}{R_\gamma k_M-\cos\gamma}. \tag{6.54}$$

Proof. We recall (Theorem 6.10) that Γ cannot meet Σ at a corner point. Suppose $R_\gamma k_M\leq 1$. Construct a circular arc Σ_M of radius $R_M=k_M^{-1}$ tangent to Σ at P_1, as indicated in Fig. 6.19. Let L_1 be the (semi-infinite) line segment tangent to Σ at P_1, let L_1' be the segment parallel to L_1 and meeting Γ in the angle γ as indicated. Since $R_\gamma k_M\leq 1$, Σ_M meets L_1' on or exterior to Γ. We adopt a rectangular coordinate system with origin at the center 0 of Γ and with the y-axis directed parallel to L_1, L_1', as shown, and denote by ψ, ψ_M the inclination angles of Σ, Σ_M relative to the positively directed x-axis. The assumed curvature inequality gives, in the interval $[x_1,x_1']$ between L_1 and L_1',

$$(\sin\psi)_x\leq(\sin\psi_M)_x.$$

By our construction,

$$\sin\psi(x_1)=\sin\psi_M(x_1)$$

and thus

$$\sin\psi_M(x)\leq\sin\psi(x)$$

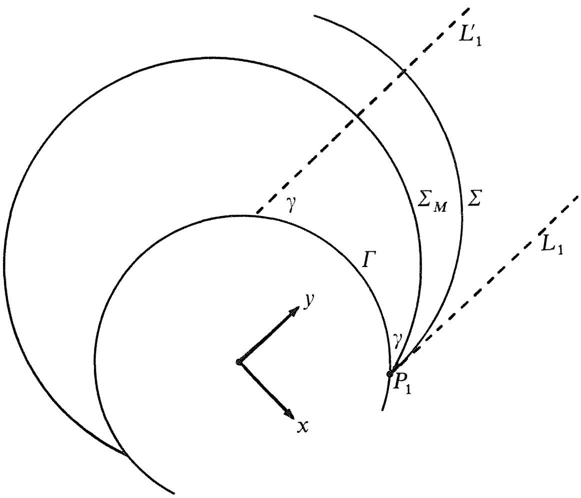


Figure 6.19. Construction for Lemma 6.10.

throughout $[x_1, x'_1]$. Thus, Σ lies above (or on) Σ_M in that interval, as shown in the figure. We now observe that P_2 must lie outside the interval. For otherwise, because of the assumed convexity of Σ, the tangent to Σ at P_2 would cross the line L'_1 at a non-zero angle, and thus the angle between Γ and Σ at P_2 would exceed γ. But if P_2 lies outside $[x_1, x'_1]$, then $\delta + \gamma > \pi/2$, which contradicts our hypothesis.

Thus, $R_\gamma k_M > 1$, and in view of the condition $\delta + \gamma \le \pi/2$, the construction of Σ_M appears as in Fig. 6.20. A repetition of the above reasoning now shows that Σ lies entirely outside the region bounded between Σ_M and Γ, and hence $\delta \ge \delta_M$.

We now observe that

$$R_M \sin(\gamma + \delta_M) = R_\gamma \sin \delta_M,$$

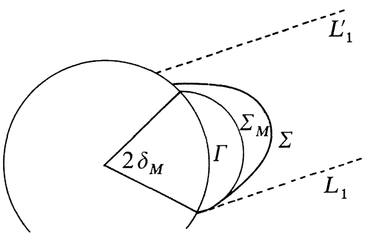


Figure 6.20. Construction for Lemma 6.10.

from which follows easily

$$\sin^2\delta_M = \frac{\sin^2\gamma}{1-2R_\gamma k_M\cos\gamma+(R_\gamma k_M)^2}$$

and finally

$$\tan\delta_M = \frac{\sin\gamma}{R_\gamma k_M - \cos\gamma}, \tag{6.55}$$

from which the lemma follows.

Simple examples show that the result of the lemma cannot be improved.

We may now prove:

Theorem 6.17. *Suppose* $0 \le k_m \le k \le k_M \le \infty$. *Then a solution of* (6.4), (6.2) *exists whenever either* $R_\gamma k_M \le 1$ *or*

$$\min\left\{\frac{\sin^2\gamma}{R_\gamma k_M - \cos\gamma}, \cos\gamma\right\} + \{R_\gamma k_m - \cos\gamma\} > 0. \tag{6.56}$$

Proof. We suppose first $\gamma>0$. Let Γ be an interior extremal arc, with $\delta+\gamma\le\pi/2$. Setting $\sigma=0$ in (6.52) and (6.53) and replacing k_1, k_2 by k_m, we find on Γ

$$\mathscr{I}[\eta] \le -\tan\delta + \left\{1 - \frac{k_m R_\gamma}{\cos\gamma}\right\}\cot\gamma.$$

By Lemma 6.10, $R_\gamma k_M > 1$ and

$$\tan\delta \ge \frac{\sin\gamma}{R_\gamma k_M - \cos\gamma}.$$

Thus, $\mathscr{I}[\eta]<0$ follows from (6.56) whenever $\delta+\gamma\le\pi/2$, $\gamma>0$. But if $\delta+\gamma\ge\pi/2$, then $\tan\delta\ge\cot\gamma$. Further, by Lemma 6.10, if $R_\gamma k_M\le 1$, then $\delta+\gamma>\pi/2$. Thus, the theorem is proved for all $\gamma \neq 0$.

If $\gamma=0$ and $R_\gamma k_M\le 1$, then clearly $\delta+\gamma=\delta>\pi/2$ for any interior extremal Γ. By Theorem 6.11(a) no such extremal can minimize, hence a solution to (6.4), (6.2) exists. The condition (6.56) is vacuous when $\gamma=0$, since in that case we find from Theorem 6.10 that $R_\gamma k_m \le 1$.

We remark that if $\gamma=0$ and if for some interior extremal Γ in a convex Ω there holds $\delta\le\pi/2$, then the configuration becomes a special case of the "tail domain" introduced by Chen [24], for which he was able to show that no solution exists. The considerations of Chen do not extend to the case $\gamma>0$, cf. the remarks in [61].

The case considered by Giusti-Weinberger [85] is subsumed under the conditions $\gamma=0$, $R_\gamma k_M \leq 1$ of Theorem 6.17, so that the present discussion offers as a special case another proof of their result. Also, Theorem 6.13 yields as special case a new proof of the sufficiency (rolling) criterion of Chen [24]. It should be remarked that all known sufficiency criteria depend ultimately for their success on the general existence results of Giusti [84, 85, 86], which we outline in the following chapter.

6.12. Continuous and Discontinuous Disappearance

It is apparent from the earlier material of this chapter, cf. Theorems 6.1 and 6.2; Corollaries 6.6 and 6.7, that to every domain Ω there corresponds a $\gamma_0[\Omega]$ such that a solution to (6.4), (6.2) exists if $\gamma_0 < \gamma \leq \pi/2$, while no solution exists if $0 \leq \gamma < \gamma_0$. We ask what happens when $\gamma = \gamma_0$.

of sets of bounded perimeter in Ω, hence there is a subsequence that converges to Ω_0^*, in the sense that the characteristic functions converge in $L^1(\Omega)$.

Suppose $\Omega_0^* = \emptyset$. We choose f in (6.14) to be the characteristic function φ_j of Ω_j^*, obtaining

$$\Phi[\Omega_j^*; \gamma_j] \geq (1 - \mu \cos\gamma_j)\Gamma - C\Omega_j^*$$

for some fixed constant C. By (6.15) we have, since $\Omega_j^* \to \emptyset$, that $\Omega_j^* < C\sqrt{\Omega_j^*}\,\Gamma$ for all large enough j. Since $\mu < 1/\cos\gamma_0$ there follows $\Phi[\Omega_j^*; \gamma_j] > 0$ for large enough j, contradicting the construction.

Suppose $\Omega_0^* = \Omega$. We introduce the "adjoint" functional

$$\Psi[\Omega^*; \gamma] \equiv \Gamma + \Sigma^* \cos\gamma - H_\gamma \Omega^*$$

and observe that $\Phi[\Omega^*; \gamma] = \Psi[\Omega \setminus \Omega^*; \gamma]$ since $H_\gamma = (\Sigma/\Omega)\cos\gamma$. Using (6.15), we thus find

$$\Phi[\Omega_j^*; \gamma_j] \geq (1 - C\sqrt{\Omega \setminus \Omega_j^*})\Gamma > 0$$

for large enough j, again a contradiction.

Thus $\Omega_0^* \neq \emptyset, \Omega$. We now observe that the conditions of Lemma 6.3 are satisfied, and thus $\Phi[\Omega_0^*; \gamma_0] \leq \liminf \Phi[\Omega_j^*; \gamma_j] \leq 0$. We conclude from Theorem 6.1 that no solution can exist at γ_0.

6.13. An Example

The case $\gamma_0 = 0$ is not covered by the above discussion. The varied behavior that can occur is illustrated by the example of Fig. 6.21, in which $\widehat{AB}$, $\widehat{CD}$ are circular arcs of radius 1 and ρ, respectively.

One determines easily that there exists $\rho_0 = 1.974\ldots$ such that for every $h > 0$ there holds $R_0 = \Omega/\Sigma < 1$, $R_0 = 1$, or $R_0 > 1$ according as $\rho < \rho_0$, $\rho = \rho_0$, $\rho > \rho_0$. We then have:

(a) If $\rho < \rho_0$ and $\gamma = 0$, there is no extremal arc for Φ in Ω, regardless of h. According to the nonexistence-existence principle, a solution surface exists at $\gamma = 0$ and *a fortiori*, for all $\gamma > 0$. For the given value of ρ, this holds for any h. (b) If $\rho = \rho_0$, then $\Omega/\Sigma = 1$ regardless of h. An extremal set Ω_0^*, for which $\Phi[\Omega_0^*; 0] = 0$, is obtained by inscribing a semicircular arc at any point in the strip, as indicated. Thus, no solution can exist at $\gamma_0 = 0$. If $\gamma > 0$, concentric circular arcs of radius $\Omega/\Sigma\cos\gamma$ provide again "extremal" domains, however in this case $\Phi > 0$, so that the domains do not minimize. Similarly, all other possible extremal arcs lead to $\Phi > 0$. Thus, regardless of the length of the strip, a solution exists for any $\gamma > 0$.

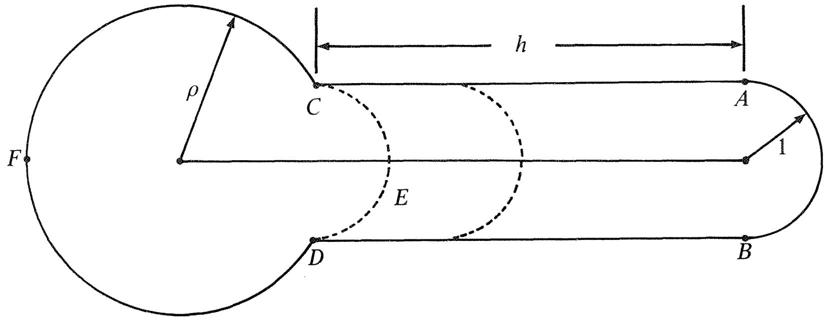


Figure 6.21. The "singular" case $\gamma_0 = 0$.

It is shown in Chapter 7 that the solutions $u^j(x)$ corresponding to a sequence $\gamma_j \downarrow 0$ can be normalized by additive constants, so that they converge throughout the domain $\widehat{CEDF}$ to a limit solution $u^0(x)$ corresponding to data $\gamma = 0$. The limit solution remains bounded for any approach to the open arc $\widehat{CFD}$, but tends to infinity on the arc $\widehat{CED}$. The solutions u^j tend to infinity throughout the extremal domain Ω_0^* that is determined in the strip by $\widehat{CED}$.

6.14. Another Example

An elaboration of the above ideas yields the example of Fig. 6.22. Here $\delta = 60°$, $\widehat{ABC}$ is a semicircle and $\widehat{AF}$ and $\widehat{CG}$ are circular arcs of unit radius.

If $\rho = 1.974$, a solution exists if $\gamma > 13.88°$. At $13.88°$ an "extremal" arc $\widehat{AC}$ with $\Phi = 0$ appears, and has radius 1.03. A sequence u of solutions

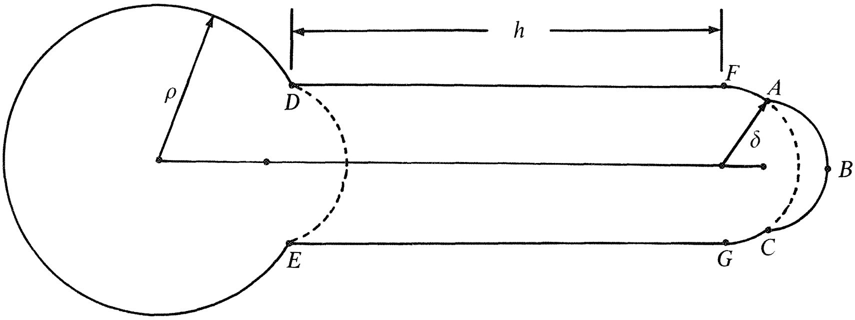


Figure 6.22. Configuration for which $\gamma_0 \neq 0$ but small change of data yields large change of solution.

corresponding to data $\gamma_j \downarrow 13.88°$ can be normalized so that u^j tends to a solution u^0 to the right of $\widehat{AC}$, and to infinity to the left of $\widehat{AC}$.

If $\rho = 1.995$, a solution exists if $\gamma > 14.15°$. At 14.15° an "extremal" arc $\widehat{DE}$ with $\Phi = 0$ appears, of radius 1.031. A sequence u^j of solutions corresponding to data $\gamma_j \downarrow 14.15°$ can be normalized so that u^j tends to a solution u^0 to the right of $\widehat{DE}$, and to infinity throughout the region to the left of $\widehat{DE}$.

Thus, the region in which the solution becomes infinite can be made to shift, essentially discontinuously, with small changes in the domain and data.

6.15. Remarks on the Extremals

We showed in §6.10 that whenever $\theta_2 - \theta_1 < \pi$, the (normalized) second variation $\mathscr{I}[\eta]$ of Φ is minimized by a rigid translational motion

$$\eta = a\cos(\theta - \sigma) \tag{6.57}$$

in a direction σ, which is one of two translational motions along which $\mathscr{I}$ is stationary.

In the original capillary problem, the boundary angle γ is prescribed. The extremal arcs Γ for the subsidiary problem must meet the boundary curve Σ with the same angle γ. In general it cannot be expected that a rigid motion (6.57) will leave γ unvaried; there are, however, situations in which that occurs, and these situations have a special interest.

Theorem 6.19 (Concus and Finn [36]). *The second variation $\mathscr{I}$ of Φ vanishes for any rigid motion of an extremal that leaves γ unvaried. Further, $\mathscr{I}$ is stationary (its first variation vanishes) in any such motion.*

Proof. The extremal is a circular arc of radius $R_\gamma = \Omega/(\Sigma\cos\gamma)$; since γ is unvaried, so is R_γ. The extremal meets Σ in two points, denoted as 1 and 2 in Fig. 6.23, with intersection angle γ as indicated.

We may characterize the motion as a composition of a rigid translation (6.57) of the center 0 in direction σ, and a rotation about 0. Since the rotation leaves everything invariant, it can be neglected.

We focus attention first on the point 1, and adopt as parameter to describe the motion the arc length s on Σ. Referring to Fig. 6.23, we find

$$\gamma + \xi + \tau + \delta = \pi/2 \tag{6.58}$$

$$P_s = -\cos\xi + \sin\xi\cot\sigma \tag{6.59}$$

$$P = \frac{\sin(\sigma + \tau + \delta)}{\sin\sigma} R_\gamma \tag{6.60}$$

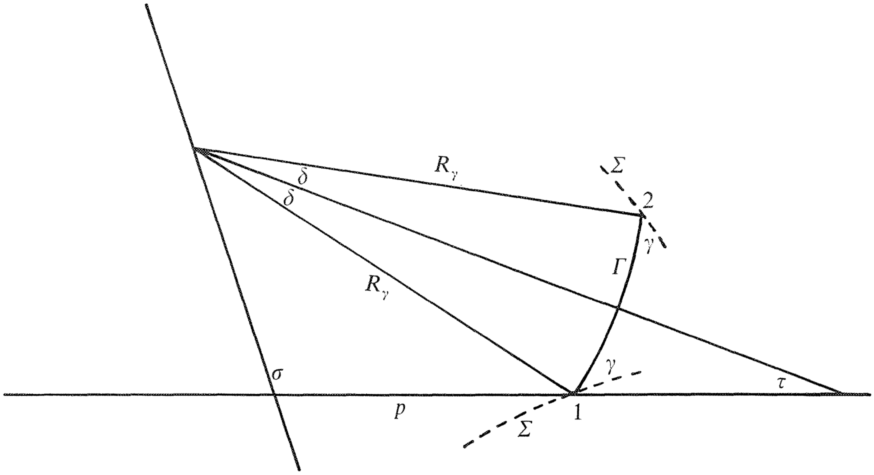


Figure 6.23. Configuration for Theorem 6.19.

and thus

$$\cos\xi - \sin\xi\cot\sigma = \frac{-R_\gamma}{\sin\sigma}\{(\tau+\delta)_s\cos(\sigma+\tau+\delta)+\gamma_s\tan\gamma\sin(\sigma+\tau+\delta)\}$$

so that

$$\sin(\xi-\sigma) = R_\gamma\{(\tau+\delta)_s\cos(\sigma+\tau+\delta)+\gamma_s\tan\gamma\sin(\sigma+\tau+\delta)\}. \tag{6.61}$$

For the curvature k_1 of Σ at 1 we find from (6.58)

$$k_1 = -\xi_s = \gamma_s + (\tau+\delta)_s. \tag{6.62}$$

We thus obtain from (6.61) and (6.58)

$$k_1 = \frac{\cos(\sigma+\gamma+\tau+\delta)}{R_\gamma\cos(\sigma+\tau+\delta)} + \gamma_s[1-\tan\gamma\tan(\sigma+\tau+\delta)]. \tag{6.63}$$

An analogous discussion now yields

$$k_2 = \frac{\cos(\sigma-\gamma+\tau-\delta)}{R_\gamma\cos(\sigma+\tau-\delta)} + \gamma_s[1+\tan\gamma\tan(\sigma+\tau-\delta)]. \tag{6.64}$$

These relations hold for any translation in the direction σ. In the special case that γ_s vanishes at both contact points, we obtain simplified expressions for k_1 and k_2, depending only on R_γ and on the angles σ, γ, τ, δ.

We now normalize (as in §6.10) by a rotation of coordinates so that $\tau=0$, and we place the resulting expressions for k_1 and k_2 into (6.53). A tedious but formal calculation then yields $\mathscr{I}[\eta]=0$, which was to be proved. Placing the indicated σ into (6.51), we verify directly that it provides one of the two solutions of that relation, and hence is an extremal direction for $\mathscr{I}$.

6.16. Example 1

The question remains, whether $\mathscr{I}$ is minimized by the above choice. We examine the question first in the particular case, for which Σ is a unit circle and Γ an interior circular arc (Fig. 6.24; we note that Γ always contains the center of Σ). The rigid motion for which the center of Γ moves on an arc concentric to Σ then yields $\gamma_s = 0$, $\mathscr{I} = 0$ (trivially).

Since $k_1 = k_2 = 1$, we find for a motion of the form (6.57), under the normalization $\eta_1^2 + \eta_2^2 = 1$,

$$\mathscr{I} = \cot\gamma\left(1 - \frac{R_\gamma}{\cos\gamma}\right) - \frac{\sin 2\delta \cos 2\sigma}{1 + \cos 2\delta \cos 2\sigma}, \tag{6.65}$$

which follows by formal calculation from (6.53). In the configuration indicated, (6.51) has the roots $\sigma = 0$ and $\pi/2$, the root $\pi/2$ being the one that leaves γ invariant. For the roots 0 and $\pi/2$ we obtain for the corresponding $\mathscr{I}_0$, $\mathscr{I}_{\pi/2}$

$$\mathscr{I}_{\pi/2} - \mathscr{I}_0 = \frac{2}{\sin 2\delta},$$

which is positive if $2\delta < \pi$, so that the "γ-invariant" direction fails to minimize. If $2\delta > \pi$, then $\mathscr{I}$ is in fact minimized by that direction among

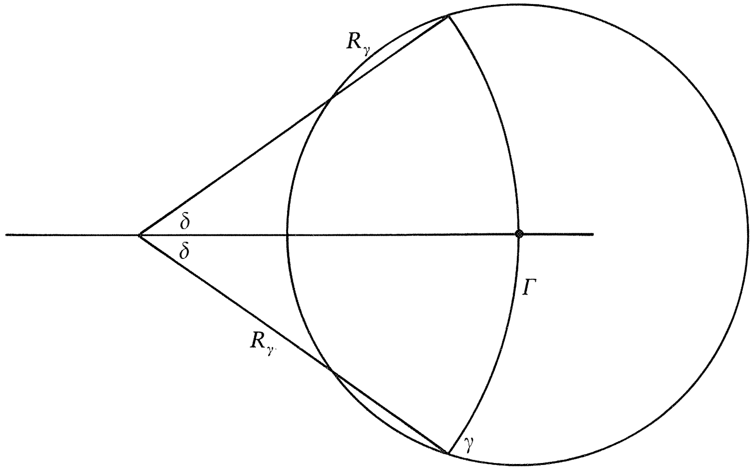


Figure 6.24. Example 1.

rigid motions, and thus $\mathscr{I} \geq 0$ for any rigid motion. Nevertheless, Γ contains a semicircle in this case and hence – as shown in §6.10 – there exist other variations for which $\mathscr{I} < 0$.

We remark that in the indicated configuration there holds $2\gamma + \delta = \pi$, hence for all situations that occur, we have $\delta + \gamma > \pi/2$ (cf. Theorem 6.16 and Lemma 6.10). Geometrically, this means that on the line L joining the centers of the two circles, the center of Γ lies between the intersection of L with Γ, and the intersection point of L with the line tangent to Σ at the intersection of Σ with Γ.

In all configurations considered, there holds $\Phi[\Gamma] > 0$. In fact, $\Phi[\Gamma] > 0$ is a necessary condition for existence of a solution, and in this case a solution can be obtained explicitly for any γ, as a lower spherical cap.

6.17. Example 2

In the configuration of Fig. 6.25, with the smaller circular arc on Σ of radius 1, there is a unique radius 1.974... for the larger arc so that $\Omega/\Sigma = 1$, independent of h_0. Corresponding to the arc Γ indicated, we have $R_\gamma = \Omega/\Sigma \cos\gamma$ and

$$\Phi = \frac{(\pi - 2\gamma)(2\cos\gamma - 1)}{2\cos^2\gamma} - \pi\cos\gamma + \tan\gamma + \frac{\pi}{2}, \tag{6.66}$$

which is independent of h. A horizontal translation of Γ thus yields $\mathscr{I} = 0$. Again we have $k_1 = k_2$ $(=0)$, so (6.51) yields once more the two roots 0 and $\pi/2$. However, in this case the roles of $\mathscr{I}_0$ and $\mathscr{I}_{\pi/2}$ are interchanged, and thus it is now the "γ-invariant" motion that minimizes $\mathscr{I}$.

One verifies easily that in $0 < \gamma < \pi/2$ the value Φ determined by (6.66) is positive. The only other extremal arcs are the reflections of the indicated ones, and for these Φ is still more positive; we thus conclude from the nonexistence-existence principle that for any $\gamma \in (0, \pi/2]$ a corre-

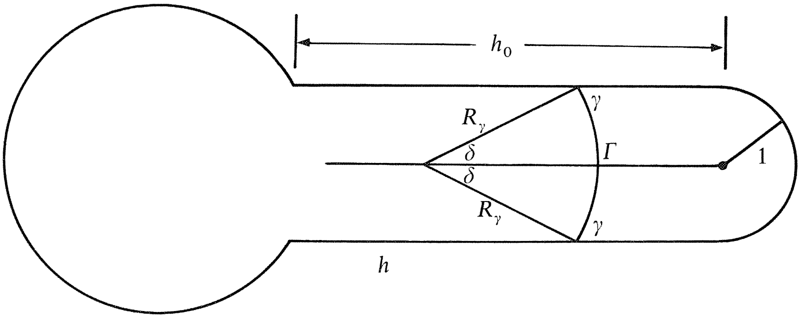


Figure 6.25. Example 2.

sponding capillary surface exists. (We note that Φ vanishes when $\gamma=0$, so that—regardless of h—no surface exists in that case.)

In the configuration indicated, one has $\delta+\gamma=\pi/2$.

6.18. Example 3

We consider finally the case of an ellipse. Computer calculations were made for the configuration for which the ratio of minor to major axis is 0.3 (see Fig. 6.26). It was found that for $\gamma \approx 25.2°$, there is an extremal Γ_0 such that $\gamma_s=0$ for horizontal displacement. Again we have $k_1=k_2$, we obtain the same two roots of (6.51), and we find that the γ-invariant motion minimizes $\mathscr{I}$. We again have $\Phi[\Gamma_0]>0$; we verify easily that all extremals are symmetric with respect to reflection in an axis of the ellipse, and that for the given γ the only other possibility is a shifted arc Γ_0' (as indicated in the figure) for which again $\Phi>0$. Thus, a solution of the capillary problem at this value of γ exists.

Corresponding to Γ_0, we have $\delta+\gamma<\pi/2$; however, for Γ_0' there holds $\delta+\gamma>\pi/2$.

The calculations for the ellipse have an independent interest extending beyond the above considerations. For each point p of the ellipse, those values of γ were sought, for which an arc Γ through p (not exceeding a semicircle), of radius $R_\gamma=\Omega/\Sigma\cos\gamma$, will meet the ellipse in two points with angle γ. The results are illustrated, qualitatively in Fig. 6.26 and quantitatively in Fig. 6.27, for an ellipse of semi-major axis $a=1$ and semi-minor axis $b=0.3$. For each p, a unique γ was found. A unique point p_0 yielded $\gamma=0$, corresponding to an inscribed circle of radius $R_0=\Omega/\Sigma$. From each side of this circle emanates a family of extremals with varying γ: on the left, γ increases from zero until $\pi/2$ is attained on the minor axis; on the right, γ increases to a maximum $\gamma_m \approx 25.2°$ at a corresponding extremal Γ_m and then decreases back to zero (which is not attained) at the right vertex. The entire configuration is repeated by reflection in the minor axis.

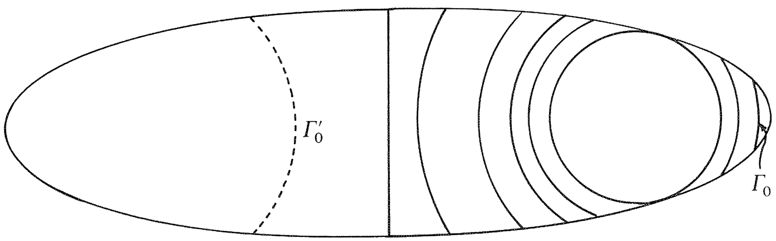


Figure 6.26. Example 3: Γ_0 and Γ_0' are extremal arcs for which $\gamma_s=0$.

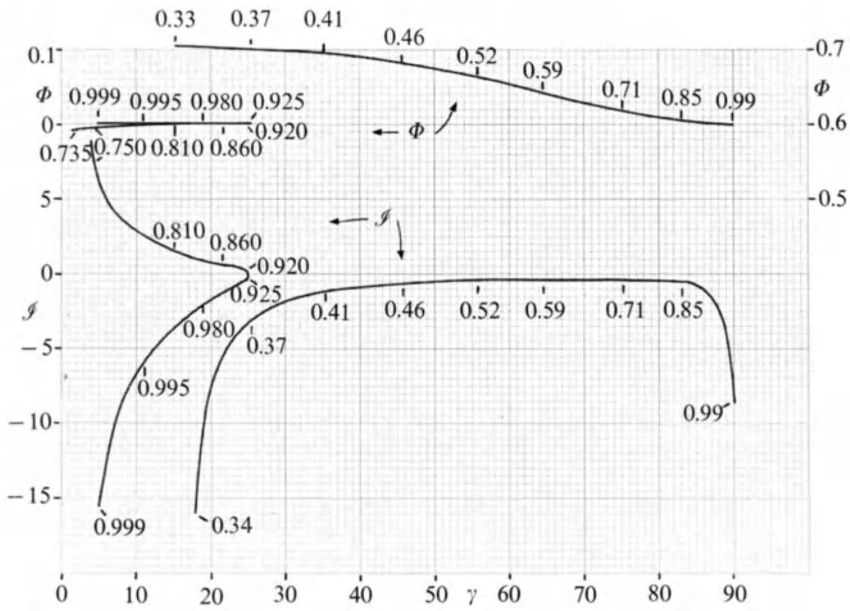


Figure 6.27. Analysis of the ellipse.

At Γ_m we have $d\gamma/ds=0$, hence also $dR_\gamma/ds=0$. The analysis of §6.15 thus applies at this point, and forms the basis for the discussion of this section.

Figure 6.27 shows $\mathscr{I}$ and Φ as functions of γ; the corresponding x-coordinates on the major axis are indicated on the curves.

6.19. The Trapezoid

Many of the considerations of this chapter are illustrated by the case in which Ω is a symmetric trapezoid; we already encountered that configuration in §6.4, where we pointed out a seemingly anomalous behavior of the associated solution set. We now wish to clarify that behavior, by examining the structure of the minimizing set $\{\Gamma\}$ of the subsidiary problem for $\Phi[\Gamma]$.

a) We consider a trapezoidal section T, of which a half (Ω) adjacent to the line of symmetry is illustrated in Fig. 6.28. It will be preferable to denote the smaller angle with α, rather than 2α as before. We ask for possible ways in which an extremal arc can appear.

i) The configuration of Fig. 6.29 can occur. Here $\tau=(\pi/2)-(\alpha^*+\gamma)$, with $\alpha^*=\alpha/2$ and we find

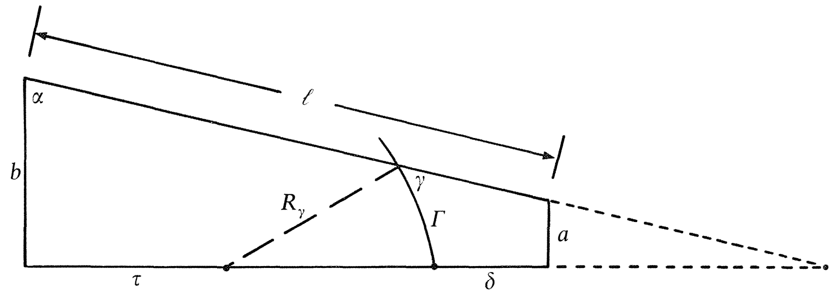


Figure 6.28. Symmetric trapezoid (upper half): notational designations.

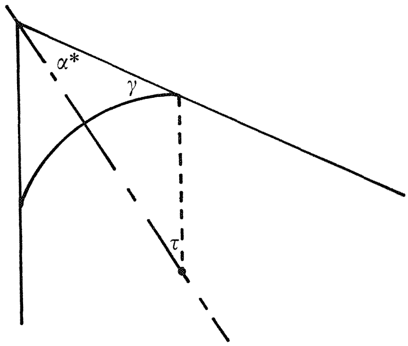


Figure 6.29. Extremal configuration; case i).

$$\frac{1}{R}\Phi[\Gamma]=\tau-2\frac{\sin\tau}{\sin\alpha^*}\cos\gamma+\frac{\sin 2\tau}{2}+\frac{\sin^2\tau}{\tan\alpha^*}$$

$$=\frac{1}{\sin\alpha^*}(\tau\sin\alpha^*-\sin\tau\cos\gamma)=\frac{1}{\sin\alpha^*}f(\tau).$$

We have $f(0)=0$, and

$$f'(\tau)=-\sin\gamma\cos(\alpha^*+\gamma)<0$$

since $\alpha^*+\gamma<\pi/2$. Thus $\Phi[\Gamma]<0$, so that no solution of the capillary problem can exist in T. (This result is local, depending only on the geometry at the corner; it was established another way in §6.2.)

ii) We consider the configuration of Fig. 6.30, in which Ω^* is the complement of the domain surrounding the corner. We find

$$\Phi[\Gamma]=\Gamma-\left(\Sigma^*-\frac{\Sigma}{\Omega}\Omega^*\right)\cos\gamma$$

$$=\Gamma+\left(\Sigma\setminus\Sigma^*-\frac{\Sigma}{\Omega}\Omega\setminus\Omega^*\right)\cos\gamma,$$

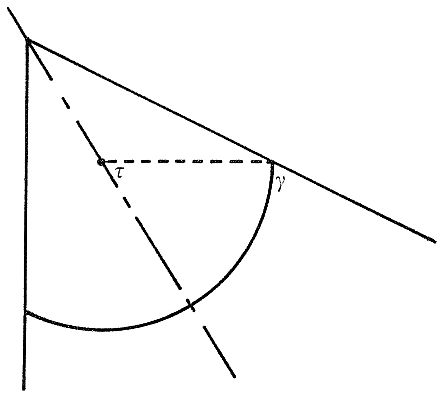


Figure 6.30. Extremal configuration; case ii).

thus reducing the problem to the previous one, but with two of the signs reversed. We calculate, with $\tau=(\pi/2)+(\alpha-\gamma)$,

$$\frac{1}{R}\Phi[\Gamma]=\tau+\frac{\sin\tau\cos\gamma}{\sin\alpha^*}>0,$$

so that in this case the existence of an extremal does not exclude the existence of a solution to the original problem.

iii) The identical considerations apply at the larger angle $\pi-\alpha$. However, we note that whenever existence is excluded by the larger angle it is also excluded by the smaller one; thus, no new information is provided.

iv) An extremal might conceivably occur as indicated in Fig. 6.31. We now observe that $\Phi[\Gamma]$ is unchanged by rigid translation of Γ in the direction of the parallel sides of T; thus, Γ can be moved until it contacts (tangentially) one of the nonparallel sides. At this point the problem is reduced to two of those of the type already considered. Either the one at the smaller corner excludes a solution or neither of them does. (A more careful analysis shows that this case is in fact vacuous; it cannot occur.)

v) One verifies easily that if Γ passes through a vertex, then $\Phi[\Gamma]>0$. This is in agreement with Theorem 6.10.

vi) We have shown till now that, if $2\gamma<\pi-\alpha$, the existence of a solution is excluded by the corner condition; if $2\gamma\geq\pi-\alpha$, then existence is not excluded by any extremal that joins two adjacent sides or passes through a vertex, or by any extremal that joins the parallel sides. There remains the possibility of an extremal joining the nonparallel sides, and that can happen in two ways. We consider

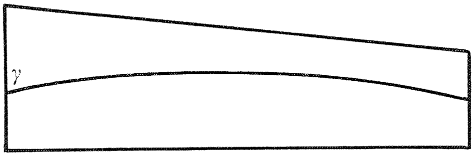


Figure 6.31. Extremal configuration; case iv).

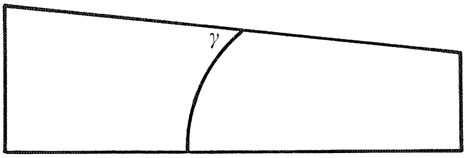


Figure 6.32. Extremal configuration; case vi).

first the configuration of Fig. 6.32 for which we find, with $\tau=\pi-(\alpha+\gamma)$,

$$\Phi[\Gamma]=R\tau+\frac{1}{R\cos\alpha}\{R^2\sin\tau\cos\gamma-2aR(1-\cos\alpha)\cos\gamma+a^2\sin\alpha\}.$$

The discriminant of the quadratic expression in brackets is

$$\cos^2\gamma\sin^2\alpha\left[1-\frac{\sin^2\alpha}{(1+\cos\alpha)^2}\right]>0$$

and thus $\Phi[\Gamma]>0$, so that existence is in no case excluded.

b) The only remaining possibility is the configuration of Fig. 6.28. For any $\gamma<\alpha$ and corresponding $R=R_\gamma=\Omega/\Sigma\cos\gamma$, an extremal Γ of the form indicated always exists (uniquely) in the figure obtained by extending the nonparallel sides. It follows from Theorem 6.16 that if Γ lies in T, it yields a strict minimum relative to all (piecewise smooth) neighboring curves.

If $2\gamma<\pi-\alpha$, then existence is excluded by the corner condition. If $\alpha>\pi/3$, then the interval $\mathscr{I}$: $\pi-\alpha/2\leq\gamma<\alpha$ is nonnull. We seek to determine conditions on the geometry and on $\gamma_0\in\mathscr{I}$, under which the choice $R=R_{\gamma_0}$ will yield an arc Γ_0 interior to T, for which $\Phi[\Gamma_0]=0$. If such an arc can be found, then a solution to the original problem will exist whenever $\gamma_0<\gamma\leq\pi/2$, and existence will fail when $\gamma\leq\gamma_0$.

We compute easily

$$\Omega=\frac{a+b}{2}l\sin\alpha \qquad\qquad \Sigma=a+b+l \tag{6.67}$$

$$R_\gamma = \frac{1}{2}\frac{(a+b)l\sin\alpha}{a+b+l}\frac{1}{\cos\gamma} \qquad \Gamma = R_\gamma(\alpha-\gamma)$$

$$l = \frac{b-a}{\cos\alpha} \qquad \Sigma^* = \frac{R_\gamma\sin(\alpha-\gamma)-a}{\cos\alpha}+a$$

$$\Omega^* = [R_\gamma\sin(\alpha-\gamma)-a]\tan\alpha\frac{R_\gamma\sin(\alpha-\gamma)+a}{2} \qquad (6.67\text{ bis})$$

$$-\frac{1}{2}R_\gamma^2(\alpha-\gamma)+\frac{1}{2}R_\gamma^2\cos(\alpha-\gamma)\sin(\alpha-\gamma).$$

For fixed γ, a, b, we find

$$\lim_{\alpha\to\pi/2} R_\gamma = \frac{b+a}{2\cos\gamma}. \qquad (6.68)$$

We also have

$$\delta = [R_\gamma\sin(\alpha-\gamma)-a]\tan\alpha - R_\gamma[1-\cos(\alpha-\gamma)] \qquad (6.69)$$

so that, using (6.46),

$$\delta \sim \frac{b-a}{2\cos\alpha} = \frac{1}{2}l \qquad (6.70)$$

for fixed γ, a, b, as $\alpha\to\pi/2$. (The symbol $\sim$ denotes "asymptotic to".)
We have

$$\Phi[\Gamma] \equiv \Gamma - \Sigma^*\cos\gamma + \Omega^*\frac{\Sigma}{\Omega}\cos\gamma$$
$$\equiv \frac{1}{2}R_\gamma\left[(\alpha-\gamma)-\frac{\cos\gamma}{\cos\alpha}\sin(\alpha-\gamma)\right]+a\frac{1-\cos\alpha}{\cos\alpha}\cos\gamma - \frac{a^2}{2R_\gamma}\tan\alpha. \qquad (6.71)$$

If $\delta \sim \frac{1}{2}l$, then from (6.68)

$$\Phi[\Gamma] \sim -\frac{1}{4}\frac{(b-a)^2}{b+a}\frac{\cos\gamma}{\cos\alpha} \to -\infty. \qquad (6.72)$$

The choice

$$\cos\gamma = \frac{b+a}{b-a}\cos\alpha + \frac{2a}{b-a}\left(\frac{b+a}{b-a}\right)^2\cos^2\alpha \qquad (6.73)$$

yields $\delta \sim 0$. For this choice, we find after some manipulation

$$\Phi[\Gamma] \sim a + 2a^2\frac{b+a}{(b-a)^2}\left(-\frac{1}{2}\cos\gamma + \frac{3a-b}{b-a}\cos\alpha\right) - a\left(\frac{b+a}{b-a}\right)\cos\alpha > 0 \qquad (6.74)$$

for $|(\pi/2)-\alpha|$ sufficiently small. From (6.72)-(6.74) we conclude *there is a value* $\gamma \sim \pi/2$ *for which* $\Phi[\Gamma]=0$. It remains to characterize γ, and also the associated value of δ.

A disagreable calculation yields, from (6.71),

$$\begin{aligned}\Phi[\Gamma] \sim \frac{b+a}{2} - \frac{1}{4}\frac{(b+a)^2}{b-a}\cos\alpha - \frac{1}{4}(b+a)\frac{\cos\alpha}{\cos\gamma} \\ + \frac{1}{4}\frac{(b+a)^2}{b-a}\frac{\cos^2\alpha}{\cos\gamma} - a\cos\gamma \\ - \frac{1}{4}\frac{(b-a)^2}{b+a}\frac{\cos\gamma}{\cos\alpha} + \frac{1}{4}(b+a)(\cos\gamma-\cos\alpha).\end{aligned} \tag{6.75}$$

Setting $b+a=x$, $b-a=y$, $\cos\alpha=\varepsilon$, $\cos\gamma=\eta$, we are led to the equation

$$\left(2y-x-\frac{y^2}{\varepsilon x}\right)\eta^2 + \left(2x-\varepsilon\frac{x^2}{y}-\varepsilon x\right)\eta + \left(\varepsilon^2\frac{x^2}{y}-\varepsilon x\right)=0 \tag{6.76}$$

for η as function of ε. Equation (6.76) has the asymptotic solution

$$\eta \sim A\varepsilon + B\varepsilon^2 \tag{6.77}$$

with

$$A=t^2(1+\sqrt{1-t^{-2}}), \tag{6.78a}$$

$$B=\frac{At(1-2t)+t}{2t(1-At^{-2})}(1-t), \tag{6.78b}$$

and $t=x/y$. Placing this result into (6.69) and using the formula for R_γ in (6.67), we find the desired estimate for δ. (We note that the choice of a negative sign before the root in (6.78a) would lead to a negative δ.)

We collect—in terms of the original variables—the principal features of our result.

Theorem 6.20. *For fixed* a, $b>a$, *there exists* $\varepsilon>0$ *such that whenever* $|(\pi/2)-\alpha|<\varepsilon$ *there is a* γ_0 *in the interval* $\pi-\alpha/2\le\gamma_0<\alpha$ *and unique circular arc* Γ_0 *interior to* Ω, *with radius* R_{γ_0}, *meeting the nonparallel sides in equal angles* γ_0, *and such that* $\Phi[\Gamma_0]=0$. *A solution of the capillary problem* (6.4), (6.2) *exists for all* γ *in the range* $\gamma_0<\gamma\le\pi/2$; *if* $\gamma\le\gamma_0$ *no solution exists. As* $\alpha\to\pi/2$, *there hold*

$$\cos\gamma_0 \sim \left(1+\sqrt{1-\left(\frac{b-a}{b+a}\right)^2}\right)\left(\frac{b+a}{b-a}\right)^2\cos\alpha \tag{6.79}$$

$$
\begin{aligned}
\delta &\sim \frac{\sqrt{ab}-a}{\cos\alpha} - \sqrt{ab}\,\frac{b+a}{b-a} \\
&= \frac{\sqrt{ab}-a}{b-a}\, l - \sqrt{ab}\,\frac{b+a}{b-a}
\end{aligned}
\tag{6.80}
$$

$$
R_{\gamma_0} \sim \frac{1}{2}\,\frac{a+b}{\cos\gamma_0} \sim \frac{a+b}{2\cos\alpha}\left(1 - \sqrt{1 - \left(\frac{b-a}{b+a}\right)^2}\right). \tag{6.81}
$$

Thus, asymptotically, δ is proportional to l. From the inequality $\sqrt{ab} < \frac{1}{2}(a+b)$ we see that, asymptotically, $\delta < \frac{1}{2}l$. The proportionality constant approaches $\frac{1}{2}$ as $b \searrow a$. However, if $b = a$ the solution once more exists for all $\gamma \geq \pi/4$.

Theorem 6.21. *The value γ_0 is uniquely determined.*

Proof. If there were two such values $\gamma_0' > \gamma_0''$, then by Theorem 6.20 the solution would exist for all $\gamma > \gamma_0''$ and would fail to exist for all $\gamma \leq \gamma_0'$.

c) We also wish to consider the trapezoid from another point of view, in which α is kept fixed and b allowed to increase.

We set

$$
f(\gamma) = (\alpha - \gamma) - \frac{\cos\gamma}{\cos\alpha}\sin(\alpha - \gamma)
$$

and observe $f(\alpha) = 0$ and

$$
f'(\gamma)\cos\alpha = -\cos\alpha + \cos(\alpha - 2\gamma),
$$

thus $f'(\gamma) > 0$ in $0 < \gamma < \alpha$, and it follows that $f(\gamma) < 0$ in that range. Since

$$
R_\gamma = \frac{l}{2}\,\frac{\sin\alpha}{1+\cos\alpha}\left[\cos\alpha + \frac{2a}{2a + l(1+\cos\alpha)}\right]\frac{1}{\cos\gamma} \tag{6.82}
$$

we have

$$
\lim_{l\to\infty} R_\gamma = \infty.
$$

We thus conclude from (6.71) that if l is large and if $\Phi[\Gamma] = 0$ on an extremal Γ, then $(\alpha - \gamma)$ must be correspondingly small. Conversely, given l sufficiently large, we can arrange to have $\Phi[\Gamma] = 0$ by choosing γ close enough to α.

To make these estimates precise and to establish that the corresponding Γ lies in Ω, we may write, for γ close to α,

$$
(\alpha - \gamma) - \frac{\cos\gamma}{\cos\alpha}\sin(\alpha - \gamma) = -(\alpha - \gamma)^2\tan\alpha + o(\alpha - \gamma)^2; \tag{6.83}
$$

thus, from (6.71), (6.82), and (6.83) we find that as $l \to \infty$ there holds, for the extremals Γ_0 with $\Phi[\Gamma_0]=0$ and corresponding angles γ_0,

$$(\alpha-\gamma_0)^2 l \to 4a\cos\alpha. \tag{6.84}$$

Similarly, from the relation

$$\delta = R_\gamma \frac{\cos\gamma - \cos\alpha}{\cos\alpha} - a\tan\alpha, \tag{6.85}$$

we obtain

$$\frac{1}{l}\delta_0^2 \to \frac{\sin^4\alpha}{\cos\alpha}\,\frac{a}{(1+\cos\alpha)^2}. \tag{6.86}$$

Thus, the distance of Γ_0 from the smaller base now increases as the square root of the side length, instead of as the side length itself, as occurred in the previous construction.

We have proved:

Theorem 6.22. *For fixed* a, $\alpha>\pi/3$, *there exists* $\varepsilon>0$ *such that whenever* $l>\varepsilon^{-1}$ *there is a* γ_0 *in the interval* $\pi-\alpha/2 \le \gamma_0 < \alpha$ *and unique circular arc* Γ_0 *interior to* Ω, *with radius* R_{γ_0}, *meeting the nonparallel sides in equal angles* γ_0, *and such that* $\Phi[\Gamma_0]=0$. *A solution of the capillary problem* (6.4), (6.2) *exists for all* γ *in the range* $\gamma_0<\gamma\le\pi/2$; *if* $\gamma\le\gamma_0$ *no solution exists. As* $l\to\infty$, γ_0 *and* δ_0 *are determined asymptotically by* (6.84) *and* (6.86).

d) For purposes of reference, we mention without explicit proof the following additional properties, which can be established within the context of the discussion above.

i) If $\delta_0>0$, then Γ_0 lies interior to Ω.

ii) For an arc Γ of radius R_γ, meeting the nonparallel sides in equal angles γ as in Fig. 6.28 there holds $\delta>0$ if and only if both inequalities

$$\cos\gamma > \frac{l(2a+l\cos\alpha)}{(l-2a)\,l\cos\alpha-4a^2}\cos\alpha \tag{6.87}$$

$$(l-2a)\,l\cos\alpha-4a^2>0 \tag{6.88}$$

are satisfied.

iii) The angle γ_0 is uniquely determined.

iv) For fixed α, both γ_0 and δ_0 increase monotonically with l.

e) The existence of the curves Γ_0 is the key to clarification of the apparently singular properties of the trapezoid. Consider a fixed configuration in which a curve Γ_0 exists in Ω, corresponding to the angle γ_0 in the interval $\pi-\alpha/2 \le \gamma_0 < \alpha$. Let γ_j be a sequence of values, with $\gamma_j \searrow \gamma_0$. For each γ_j a solution u_j exists, unique up to an additive constant. The solutions are conveniently normalized by requiring that all $u_j=0$ at some fixed point in $\Omega \setminus \Omega^*$.

Since $\varphi(\Gamma_0)=0$, there must hold

$$\lim_{j\to\infty} \int_{\Gamma_0} \nu \cdot \mathrm{T} u_j \, ds = -\Gamma_0, \tag{6.89}$$

from which we conclude that $|\nabla u_j| \to \infty$ a.e. on Γ_0; further ∇u_j is directed, asymptotically, normal to Γ_0 into Ω^0 a.e. on Γ_0 (in this respect, cf. Roytburd [159]). Thus, the solutions u_j cannot be bounded in compacta, as the general gradient estimate [16, 113] would then imply a gradient bound over Γ_0.

More precisely, we may apply at this point the material of §§7.8–7.10, which in conjunction with the present considerations shows that the solutions u_j tend to infinity throughout the minimizing domain Ω^0, and that the u_j tend to a strict solution $u^0(x,y)$ in the interior of the complementary domain, corresponding to data γ_0 on $\Sigma \setminus \Sigma_0^*$ and to data $\gamma=0$ on Γ_0.

We summarize:

Theorem 6.23. *Let γ_0, Γ_0 be such that $\Gamma_0 \subset \Omega$, $\varphi(\Gamma_0)=0$, as described above. Let u_j be a sequence of solutions corresponding to data γ_j, with $\gamma_j \searrow \gamma_0$, and suppose $u_j(p)=0$ for a fixed $p \in \Omega \setminus \Omega^0$. Then there is a subsequence that converges throughout $\Omega \setminus \Omega^0$ to a solution $U(x,y)$ of* (6.4), (6.2) *corresponding to data γ_0 on $\Sigma \setminus \Sigma_0^*$ and which converges to infinity throughout Ω^0.*

Geometrically, $U(x,y)$ is a surface of constant mean curvature $(\Sigma/\Omega)\cos\gamma_0$ in $\Omega \setminus \Omega^0$, corresponding to boundary angle γ_0 on $\Sigma \setminus \Sigma_0^*$ and angle zero on Γ_0. As Γ_0 is approached from within $\Omega \setminus \Omega^0$ the surface tends asymptotically to the vertical cylinder (with the same constant mean curvature) that lies over Γ_0.

Alternatively, the solutions u_j could be normalized to converge to a solution in Ω^0 and to $-\infty$ in $\Omega \setminus \Omega^0$.

f) We examine the singular behavior in the trapezoid from the point of view of deformation of a rectangle, which is the context in which it was first encountered. For a rectangle of side lengths $2a$ and l, we find

$$R_\gamma = \frac{\Omega}{\Sigma \cos\gamma} = \frac{2al}{2(2a+l)} \frac{1}{\cos\gamma} < \frac{a}{\cos\gamma}.$$

But a circular arc meeting the sides of length l in equal angles γ has radius $R=a/\cos\gamma$. We conclude that (except for the trivial case $\gamma=\pi/2$) *in a rectangle there is no extremal arc meeting opposite sides in equal angles γ, for any γ.*

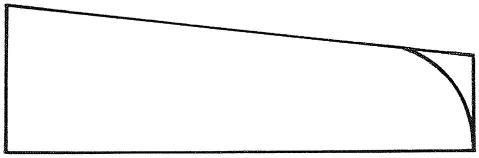


Figure 6.33. Entering extremal.

Let us now pick a, l, γ_0, α_0, such that an extremal for the corresponding trapezoid exists with $\Phi[\Gamma_0]=0$, as described in Theorem 6.20. We examine what happens when a rectangle with the side lengths $2a$ and l is deformed continuously into the trapezoid with smaller base $2a$ and adjacent side l, by decreasing the opposite angles from $\pi/2$ to α_0.

According to a) ii, initially the situation will remain unchanged, and no extremal arcs meeting opposite sides at angle γ_0 will appear. But as α decreases, eventually (6.88) will hold, and with still further decrease of α the right side of (6.87) decreases to 1. At this point, an extremal Γ corresponding to $\gamma=0$ just appears in the trapezoid at the smaller base (Fig. 6.33).

We verify that *on this initial extremal* $\hat{\Gamma}$, *there holds* $\Phi[\hat{\Gamma}]<0$. In fact, we have from (6.71)

$$\Phi[\hat{\Gamma}]=\frac{1}{2}R(\alpha-\tan\alpha)+a\frac{(1-\cos\alpha)}{\cos\alpha}-\frac{a^2}{2R}\tan\alpha, \tag{6.90}$$

while the condition that $\hat{\Gamma}$ contact three sides of T yields

$$a=R\tan\frac{\alpha}{2}.$$

We thus find after some manipulation that

$$\frac{1}{R}\Phi[\hat{\Gamma}]=\frac{1}{2}\alpha-\tan\frac{\alpha}{2}=-\frac{1}{2}\int_0^\alpha \tan^2\frac{\theta}{2}\,d\theta<0,$$

as we have stated.

If we continue to decrease α, $\hat{\Gamma}$ moves into T and new extremals appear at the smaller base. The range of boundary angles that appear is determined by (6.87), the right side of which will achieve a minimum equal to

$$\cos\gamma_m=2a\frac{2a+l+2\sqrt{2la}}{(l-2a)^2} \tag{6.91}$$

at a value

$$\cos\alpha_m=2a\frac{2a+\sqrt{2la}}{l(l-2a)}. \tag{6.92}$$

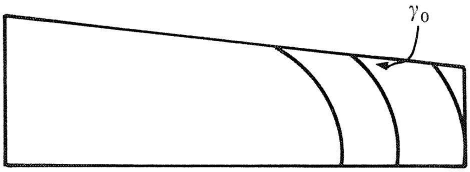


Figure 6.34. Behavior with decreasing α.

For each $\gamma<\gamma_m$ there is, for some $\alpha>\alpha_m$, a corresponding arc Γ in T that meets the nonparallel sides in equal angles γ; for $\gamma\geq\gamma_m$ there is no such arc (Fig. 6.34). Thus the value γ_0 initially chosen, for which $\Phi[\Gamma_0]=0$ and whose (unique) existence is assured by Theorem 6.22, satisfies $\gamma_0<\gamma_m$, and γ_0 will be found at an angle α_0 satisfying $\cos\alpha_0<\cos\alpha_m$.

Denoting again by $\hat{\Gamma}$ the extremal corresponding to $\gamma=0$, *we show now that* $\Phi[\hat{\Gamma}]$ *continues to be negative.*

Lemma 6.11. *If l is sufficiently large, then on an extremal $\hat{\Gamma}$ for which $\gamma=0$ there holds $\partial\Phi/\partial R<0$.*

Proof. From (6.90) we find

$$\frac{\partial\Phi}{\partial R}=\frac{1}{2}(\alpha-\tan\alpha)+\frac{a^2}{2R^2}\tan\alpha, \tag{6.93}$$

and thus in the initial configuration, for which $a=R\tan(\alpha/2)$,

$$4\frac{\partial\Phi}{\partial R}\cos^2\frac{\alpha}{2}=\alpha(1+\cos\alpha)-2\sin\alpha\equiv f(\alpha).$$

We have

$$f(0)=0,\qquad f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}-2<0,$$

$$f'(\alpha)=1-\cos\alpha-\alpha\sin\alpha,$$

and thus

$$f'(0)=0,\qquad f'\left(\frac{\pi}{2}\right)=1-\frac{\pi}{2}<0$$

while $f''(\alpha)=-\alpha\cos\alpha<0$ in $0<\alpha<\pi/2$. Hence $f'(\alpha)<0$, and finally $f(\alpha)<0$. We have shown

$$\left.\frac{\partial\Phi}{\partial R}\right]_{R=a/\tan(\alpha/2)}<0. \tag{6.94}$$

From (6.82) we now calculate, for $\gamma=0$,

$$\lambda\frac{\partial R}{\partial\alpha}=4a^2\cos\alpha+2la(2\cos^2\alpha+\cos\alpha-\sin^2\alpha+\sin\alpha)$$
$$+l^2(\cos^3\alpha+\cos^2\alpha-\sin^2\alpha),$$

where λ is a positive factor. We may assume that $\alpha > \pi/3$ as otherwise there are no extremals of the type considered. Thus the coefficient of l^2 is negative, and it follows that R is decreasing in α when l is large enough. With such a choice for l, the R corresponding to $\hat{\Gamma}$ increases with decreasing α, thus by (6.93), $\partial\Phi/\partial R$ will decrease. Since in the initial configuration $\partial\Phi/\partial R < 0$ by (6.94), there must hold $\partial\Phi/\partial R < 0$ on $\hat{\Gamma}$, as was to be proved.

To prove the assertion just preceding the lemma, we note that for any α, when $R = a/\tan\frac{1}{2}\alpha$ is inserted into (6.90) the corresponding Φ will be negative, as we have shown. But decreasing α increases R, and by the lemma $\partial\Phi/\partial R < 0$. Hence the value of Φ corresponding to $\hat{\Gamma}$ is again negative, q.e.d.

The extremal Γ_0 is thus seen to be embedded in a family of extremals $\Gamma(\gamma)$, each meeting opposite sides in equal angles γ, with $\Gamma_m = \Gamma(\gamma_m)$ contacting the smaller base, and such that $\Phi(\hat{\Gamma}) < 0$, $\Phi(\Gamma_m) > 0$.

If l is sufficiently large, this entire process can be made to occur with an arbitrarily small change of α; in that sense, the solutions necessarily become unstable relative to boundary perturbations. We note that $\lim_{l\to\infty} \cos\gamma_m = 0$, as is necessary for consistency with Theorem 6.21.

g) We examine the above discussion in the context of Lemma 6.5 and Theorem 6.8. In accordance with those results, a solution to (6.4), (6.2) exists if and only if there is no extremal Γ for which $\Phi[\Gamma] \le 0$. The case $\alpha + \gamma \le \pi/2$ does not occur, as the solution has already ceased to exist at larger values of γ. Therefore, in agreement with Theorem 6.18, the solution disappears continuously as γ decreases through γ_0 (i.e., no solution exists at γ_0). In that respect the behavior is quite different from what can happen when there is a corner for which $\alpha + \gamma_0 = \pi/2$.

This latter case can, however, be retrieved by a limiting transition from the one we have just considered. Consider a general domain Ω with a

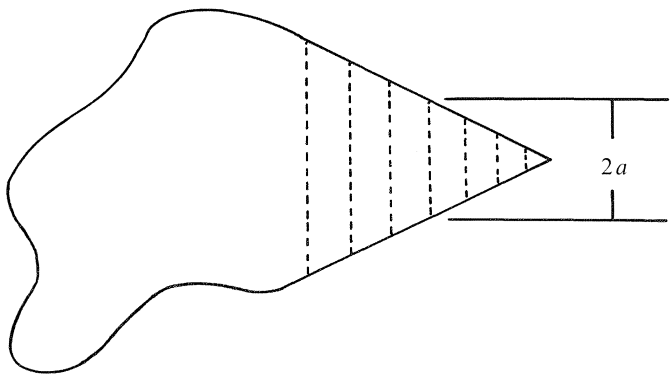


Figure 6.35. Trapezoidal approximation of corner.

corner as shown in Fig. 6.35. By adjoining two vertical lines as indicated, we obtain (essentially) a trapezoidal figure. We keep the left-hand line fixed and vary the right one toward the corner; a similarity transformation converts the figure into one in which the shorter base has fixed length $2a$ and the larger base moves to infinity, with fixed angle α.

An examination of the proof of (6.86) shows that the larger base has asymptotically no effect on the determination of δ_0 (note that b appears in (6.86) only as a common factor), and thus the same asymptotic relation holds. Transforming back to the original coordinates, we find

$$\delta_0 \sim \lambda\sqrt{a}$$

for a fixed $\lambda > 0$, as $a \to 0$. We therefore also find

$$\gamma_0 \sim \frac{\pi}{2} - \alpha_0,$$

where α_0 is the half-angle at the vertex. We thus obtain a sequence of extremals Γ_0 moving toward the vertex, with $\Phi[\Gamma_0]=0$, and in the limit one finds an "extremal" passing through the vertex, for which $\Phi=0$ and for which $\alpha_0+\gamma_0=\pi/2$. This extremal is not in Ω, hence it does not serve to exclude a solution; however, there is no solution for $\gamma<\gamma_0$, since an extremal near the vertex could then be found, for which $\Phi<0$.

If corresponding to γ_0 there is an extremal $\hat{\Gamma}_0 \subset \Omega$ such that $\Phi[\hat{\Gamma}_0]=0$, then no solution can exist at γ_0. If there is no such extremal (as happens, e.g., for a regular polygon) then the situation is dependent on the local geometry near the vertex. This question will be discussed further in Chapter 7. We mention here only that in the special case for which the corner is formed by two straight segments meeting at the vertex, a bounded solution continues to exist at $\gamma=\gamma_0$, but disappears discontinuously as γ_0 is crossed (§7.9). In that sense, the theory of behavior at a corner appears as a special (limiting) case of the behavior in a trapezoid.

6.20. Tail Domains; A Counterexample

From the point of view of general existence criteria, the domain Ω^* of Fig. 6.28 provides a formal analogue, for $\gamma>0$, of the tail domain of Chen (see §6.4). However, we have seen in the preceding section that (unique) such extremal configurations can be found for which $\Phi>0$, and in these cases a solution will exist. We thus see that the result of Chen does not extend to the case $\gamma \neq 0$.

6.21. Convexity

The following result, for a zero-gravity capillary surface, is analogous to Korevaar's theorem (§5.5) for surfaces in a gravity field, and was found independently.

Theorem 6.24 (Chen and Huang [25]). *Let $u(x)$ satisfy* (6.4) *in a convex Ω, and suppose $\nu\cdot Tu=1$ on $\Sigma=\partial\Omega$. Then $u(x)$ determines a surface S of positive Gaussian curvature over Ω.*

The proof is similar to a procedure used by Finn in [51] and by Finn and Giusti in [69]. We observe first from the boundary condition that the minimum u_0 of u is attained at an interior point x_0 of Ω, and that the Gaussian curvature K must be nonnegative at (x_0, u_0). Thus K cannot be everywhere negative on S, and it suffices to exclude the possibility of a point at which $K=0$.

If there were such a point, we could construct a comparison surface $v(x)$ as a lower half cylinder Z (cut along two generators) of the same mean curvature H, tangent to S at the point and oriented so that the generator at the point of contact has the zero-curvature principal direction on S. The other principal curvatures must then be equal, so that S and Z will have, at the point, a contact of (at least) second order. The projection Ω' of Z onto the plane of Ω is an infinite strip of width $2H^{-1}$, and in view of the convexity of Ω, the common domain $\Omega\cap\Omega'$ is bounded by (at most) two straight segments of $\partial\Omega'=\Sigma'$ and by arcs of $\partial\Omega=\Sigma$, as in Fig. 6.36.

Since $u-v=w$ satisfies an elliptic equation without zero-order term, there is a finite number ≥ 6 of domains emanating from the contact point, in which $w \lessgtr 0$ (for a proof see, e.g., [51] or [25]). Thus, either there is a domain $\mathscr{D}^+$ in which $w>0$, whose boundary contains no points of $\Sigma\backslash(\Sigma\cap\Sigma')$, or there is a domain $\mathscr{D}^-$ in which $w<0$, whose boundary

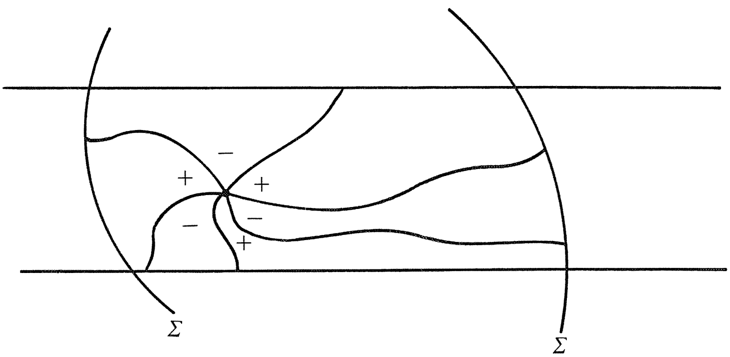


Figure 6.36. Proof of Theorem 6.24; one of the $(-)$ regions leads to a contradiction.

contains no points of $\Sigma' \setminus (\Sigma \cap \Sigma')$. In either case, we obtain a contradiction by using Theorem 5.1.

6.22. A Counterexample

The behavior of solutions in a trapezoid as $\gamma_j \searrow \gamma_0$ (§6.19) provides an example to show that Theorem 6.24 is in general false if $\gamma \neq 0$. In fact, consider the subsequence u_j of §6.19e; let $p \in \Omega \setminus \bar{\Omega}^0$ and $q \in \Omega^0$. Let L_{pq} denote a segment joining $(p, u_j(p))$ to $(q, u_j(q))$. By the indicated convergence properties, p and q lie in disks interior to which $|\nabla u_j| < M < \infty$, independent of j. This for j sufficiently large, there will be points near p at which L_{pq} lies above the surface, and points near q at which L_{pq} lies below the surface. Hence L_{pq} must meet the surface at least three times, so that the surface cannot be convex.

6.23. Transition to Zero Gravity

In view of the qualitatively different kind of behavior that can occur, depending on whether or not a gravitational field is present, it is important to examine the transition that occurs as gravity tends to zero. The first to do so was D. Siegel, who considered the fluid in a capillary tube of general section closed at the bottom. Writing the equation nondimensionally, as in §1.10, and taking account of the volume constraint, we obtain for the height u in the presence of a gravity field directed vertically downwards

$$\operatorname{div} Tu = 2H + Bu, \qquad H \equiv \text{const.},\; B > 0, \tag{6.95a}$$

in Ω, with

$$\nu \cdot Tu = \cos\gamma \tag{6.95b}$$

on $\Sigma = \partial\Omega$. Here u is considered to be normalized by an additive constant so that

$$\int_\Omega u\,dx = 0; \tag{6.96}$$

this condition serves to determine $2H = (\Sigma/\Omega)\cos\gamma$.

For the corresponding height v in the absence of gravity, we have

$$\operatorname{div} Tv = 2H \tag{6.97a}$$

in Ω, and

$$\nu \cdot Tv = \cos\gamma \tag{6.97b}$$

on Σ. The normalization (6.96) will be assumed also for v.

In the following chapter it is shown that under quite general conditions (6.95) admits a unique solution. In the present chapter, we have seen that solutions of (6.97) may or may not exist, depending on the geometry of Ω.

Theorem 6.25 (Siegel [165]). *Let Σ be (sufficiently) smooth, $0<\gamma<\pi/2$, and suppose* (6.97) *admits a bounded solution. Then there exists a constant $C(\Omega;\gamma)$ such that*

$$|u-v|<CB \tag{6.98}$$

throughout Ω.

With regard to the existence of a bounded solution to (6.97), see Theorem 7.1.

We indicate the initial (underlying) steps in Siegel's proof. We first observe that u is bounded, independent of B. In fact, letting $v_m=\inf_\Omega v$, we have

$$\operatorname{div} \mathsf{T} u - B = H$$

$$\operatorname{div} \mathsf{T}(v-v_m) - B(v-v_m) \le H$$

in Ω, with

$$\nu\cdot \mathsf{T}(v-v_m)=\nu\cdot \mathsf{T} v=\nu\cdot \mathsf{T} u$$

on Σ, and thus

$$u<v-v_m$$

by Theorem 5.1. Similarly, setting $v_M=\sup_\Omega v$, we have

$$u>v-v_M.$$

The next observation is that $|Du|$ is bounded in Ω, independent of B. This step follows from general gradient estimates in [16, 113]. From this bound we conclude (cf. the discussion in §5.1)

$$(Du-Dv)\cdot(\mathsf{T} u-\mathsf{T} v)>C|Du-Dv|^2.$$

We have, however,

$$\int_\Omega (Du-Dv)\cdot(\mathsf{T} u-\mathsf{T} v)\,dx=-B\int_\Omega u(u-v)\,dx.$$

Combining these relations and using the bound on u, we find

$$\int_\Omega |D(u-v)|^2\,dx<CB\int_\Omega |u-v|\,dx.$$

Using Poincaré's Inequality, we thus obtain

$$\int_\Omega |u-v|^2\,dx < C\int_\Omega |D(u-v)|^2\,dx < C_1 B\int_\Omega |u-v|\,dx$$
$$< C_1 B|\Omega|^{1/2}\Big(\int_\Omega |u-v|^2\,dx\Big)^{1/2}$$

from which

$$\Big(\int_\Omega |u-v|^2\,dx\Big)^{1/2} < CB$$

for some constant C independent of B, thus providing a basic global estimate. For the remaining details, we refer the reader to [165] or [174].

The following further results were obtained by Tam [174]:

1) Theorem 6.25 continues to hold if Ω has a finite number of corners of opening 2α, at each of which $\alpha+\gamma>\pi/2$.
2) Let Ω be smooth; suppose $\gamma=0$ and (6.97) has a solution, which may be unbounded (no normalization is imposed). If $v\in L1(\Omega)$, then there is a constant C such that $\lim_{B\to 0} u(x)=v(x)+C$ in Ω. If $v\notin L^1(\Omega)$, then $\lim_{B\to 0} u(x)=-\infty$ at each $x\in\Omega$. Nevertheless, there exist $C(B)$ such that $\lim_{B\to 0}[u(x)+C(B)]=v(x)$ in Ω.
3) In a configuration for which there is a unique minimizing set Ω^0 for which $\Phi=0$, there exist constants $C_1(B)$, $C_2(B)$ such that

$$\lim_{B\to 0}[u(x)+C_1(B)]=\begin{cases}\infty & \text{in } \Omega^0\\ \text{solution} & \text{in } \Omega\setminus\bar{\Omega}^0\end{cases}$$

$$\lim_{B\to 0}[u(x)+C_2(B)]=\begin{cases}\text{solution} & \text{in } \Omega^0\\ -\infty & \text{in } \Omega\setminus\bar{\Omega}^0.\end{cases}$$

The considerations of 3) apply to the trapezoid (§6.19).

Notes to Chapter 6

1. §6.2ff. Initially it is advantageous to consider Φ as functional of the set Ω^*, and thus we write $\Phi[\Omega^*]$ (or $\Phi[\Omega^*;\gamma]$). Once the boundary sets $\{\Gamma\}$ of extremal domains have been characterized, local variations are most naturally interpreted in terms of their effect on Γ; thus in later sections the notation $\Phi[\Gamma]$ (or $\Phi[\Gamma;\gamma]$) is adopted. Both expressions refer to the identical object.

2. Theorem 6.7. The result does not appear in this form in Giusti's papers, nor is it strictly contained in his published work. It can be obtained however by pursuing the line of reasoning he initiated. An outline of the procedure is given in Chapter 7.

3. Theorem 6.7. Ultimately the existence proof can be based on a hypothesis of the form

$$\inf_R \sup_{E \subset \Omega \cap B_R(x_0)} \frac{\int_{\partial\Omega} \varphi_E \, ds}{\int_\Omega |D\varphi_E|} = \mu(x_0) < \frac{1}{\cos\gamma}$$

for all $x_0 \in \bar{\Omega}$. An inequality of this form is implied by (6.14) or, alternatively, by the conditions of Lemma 6.1; see Giusti [84], Anzellotti and Giaquinta [3]. In the latter reference conditions for existence of a trace with the requisite properties are given.

4. §6.7. In the two-dimensional case considered here, the sophisticated theorem of Massari is not strictly necessary; the result can be obtained alternatively from the classical "parametric" theory of Calculus of Variations, cf. [2, Chapter 4].

5. Theorem 6.24. Korevaar later improved the result of §5.5, so as to include the case of zero gravity. For the particular case considered the hypotheses of Chen and Huang are significantly weaker; their method however does not seem to extend easily to the case of nonconstant curvature, nor does it extend to higher dimensions.

The statement of the boundary hypothesis in Theorem 6.24 is intentionally vague. It suffices that the boundary data be achieved in the sense of the variational condition, see §7.7.

Chapter 7

Existence Theorems

7.1. Choice of Venue

A considerable part of modern literature on capillary surfaces has been devoted to establishing the existence of solutions of (6.4), (6.2) under varying conditions. We have already cited some of the references in §1.9. Some authors make considerable demands on boundary regularity; such an approach has the advantage that the solution appears in a class that possesses an *a priori* smoothness up to the boundary, so that the prescribed data are achieved strictly. There is however a disadvantage, in that only those solutions are found whose behavior emulates that of solutions to a (linear) Neumann problem. Thus the kind of discontinuous behavior discussed in the two preceding chapters, which is characteristic for the nonlinearity in the problem, is not seen in the results.

Three authors – Emmer [46], Giusti [84–86] and Tam [172, 174] – have studied the problem variationally in the context of Lipschitzian base domains Ω. Their methods, and the conditions they have had to impose, contact closely with the earlier material of this text; we outline some of their results in this chapter. Finn and Gerhardt [68] developed the relationship further, while Gerhardt [74–77] developed the variational approach from other points of view.

The variational approach uses ideas that can be traced at least to Hilbert. They were developed by Tonelli, Evans, Morrey, Caccioppoli, and others; in the generality needed for this problem, the underlying theory was created by De Giorgi and his pupils. The structure of the theory is measure-theoretic; the solution is sought in a class of competing functions sufficiently broad that compactness of a minimizing sequence can be guaranteed; the variational condition then leads, by "bootstrap" reasoning, to the uniqueness and regularity of the limit function. Such an approach might seem at first glance to be hazardous for the present problems, since the boundary condition (6.2) involves derivatives on the boundary, where differentiability of weak solutions is usually difficult to prove. In fact, the minimizing function produced by the variational procedure is known initially only to have a (generalized) L^1 trace on the boundary, so that (6.2) can be defined only in a very weak sense. Nevertheless, the minimizing property suffices for showing the uniqueness of the limit function, and for identifying it with the smooth solution whenever a smooth solution exists. In fact, the local smoothness to the boundary can be shown wherever the boundary is locally smooth by using results of other authors [76, 77, 163, 167, 181], however from the point of view of the variational calculus these steps in the theory cannot yet be said to be on a satisfactory footing. We shall however be able to obtain boundedness in some cases.

A natural further development of the theory is the notion of *generalized solution* as introduced by Miranda [140]. Such solutions can conceivably be infinite on sets of significant size, although as it turns out the nature of these sets is severely restricted by the geometry of the domain. Giusti [86] and later Tam [172–174] applied the method with striking success to the capillary problem, and we adopt that approach in what follows with a view toward completing in certain directions the program initiated by Giusti. We wish to present so far as possible a unified discussion covering either the presence or absence of a gravity field, and we shall allow boundary data that vary on Σ. We therefore study the variational problem for a functional

$$\mathscr{E}[u]=\int_\Omega \sqrt{1+|Du|^2}+\int_\Omega \lambda(u)\,dx-\oint_\Sigma \beta(s)u\,ds, \tag{7.1}$$

for which the Euler equation takes the form

$$\operatorname{div} \mathsf{T}u=2H(u) \tag{7.2}$$

in Ω, with $2H(u)=\lambda'(u)$, and

$$\nu\cdot \mathsf{T}u=\beta(s) \tag{7.3}$$

on $\Sigma=\partial\Omega$. Thus, $\lambda'(u)$ is twice the mean curvature of a solution surface, and $\beta(s)=\cos\gamma(s)$ on Σ, $\gamma(s)=$contact angle. We assume $H'(u)\geq 0$, thus $\lambda''(u)\geq 0$. Hence there exist $H_\infty=\lim_{t\to\infty}H(t)\leq\infty$, $H_{-\infty}=\lim_{t\to-\infty}H(t)\geq -\infty$.

Let us replace the general (zero-gravity) necessary condition (6.7) with the two relations

$$\Phi[\Omega^*]\equiv\Gamma-\int_{\Sigma^*}\beta\,ds+2H_\infty\Omega^*>0 \tag{7.4}$$

$$\Psi[\Omega^*]\equiv\Gamma+\int_{\Sigma^*}\beta\,ds-2H_{-\infty}\Omega^*>0, \tag{7.5}$$

which are clearly necessary, for any $\Omega^*\subset\Omega$, $\Omega^*\neq\emptyset$, Ω. If $H_\infty\neq H_{-\infty}$ (that is, if $H\neq$const.), then both inequalities are strict also for $\Omega^*=\Omega$, whenever a nontrivial solution of (7.2), (7.3) exists. If $H \equiv$ const., then (7.4), (7.5) imply $\Phi[\Omega]=\Psi[\Omega]=0$, $2H\Omega=\int_\Sigma \beta\, ds$, from which one finds easily (on replacing Ω^* by $\Omega \setminus \Omega^*$) that (7.4) and (7.5) are equivalent. As with (6.7), we observe that the equation for u does not appear in (7.4) or (7.5), the inequalities express a property of the geometry of Ω.

We intend to prove:

Theorem 7.1. *Suppose there is a decomposition* $\{\Sigma_j\}$ *as in Lemma* 6.1, *such that* $\sqrt{1+L_j^2}\max_{\Sigma_j}\beta(s)<\mu<1$, *all* j (*cf. Note* 3, *Chapter* 6). *Suppose further that* (7.4), (7.5) *hold for every Caccioppoli set* $\Omega^* \neq \emptyset$, Ω *if* $H \equiv$ const., *otherwise that they hold for every* $\Omega^* \neq \emptyset$. *Then there is a minimizing function* $u(x) \in BV(\Omega)$ *for* $\mathscr{E}[u]$. *The surface* $u(x)$ *is regular and bounded in* Ω, *has finite area, satisfies* (7.2) *in* Ω, *and also the variational condition*

$$\int_\Omega (W_{p_i}\zeta_i+\lambda'(u)\eta)\,dx-\int_\Sigma \beta\eta\, ds=0 \tag{7.6}$$

for any $\eta \in H^{1,1}(\Omega)$, $\zeta_i=\eta_{x_i}$; *here* $W=\sqrt{1+p^2}$, $p=(p_1,p_2)$, $p_i=u_{x_i}$. *If* $H \not\equiv$ const. *on the surface* $u(x)$, *then* $u(x)$ *is uniquely determined; otherwise,* $u(x)$ *is determined up to an additive constant. In either case,* $u(x)$ *is a strict solution of* (7.2), (7.3) *whenever such a solution exists.*

We note that the physical cases, with or without gravity, are encompassed in a single statement. When gravity is present, $H_{-\infty}=-\infty$, $H_\infty=\infty$, so that the relations (7.4), (7.5) are satisfied *ipso facto* for any Ω.

7.2. Variational Solutions

The conditions of Theorem 7.1 exclude the data $\gamma=0$. In this case it can occur that (unique) solutions of (7.2), (7.3) appear for which individual energy terms in (7.1) are infinite. We will study this and other particular cases separately, although still within the same framework of ideas. In order to do so, we introduce a more general notion of variational solution than has been common in the literature.

Definition 7.1. A function $u(x)$ will be called a *variational solution* for $\mathscr{E}$ in Ω if i) $u(x)$ is twice differentiable and satisfies (7.2) in Ω, and ii) the relation (7.6) holds for any $\eta \in H^{1,1}(\Omega)$.

This extension of the minimizing requirement first appears in a more limited context in [68]. A function $u(x)$ can be a variational solution even though $\mathscr{E}[u]$ may not be defined. No requirement of differentiability up to Σ is made, however any solution of (7.2) in Ω that is differentiable up to Σ and satisfies (7.3) on Σ will be a variational solution for $\mathscr{E}$. We shall show (Theorem 7.9) that under general conditions a variational solution can be constructed, that it is the unique such solution, and that it equals the smooth solution of (7.2), (7.3) whenever such a solution exists. For a class of problems including the zero-gravity solutions, we obtain this result as a necessary and sufficient condition (Theorem 7.10).

Our procedure will be to construct a generalized solution under very weak conditions, and then to show that under the hypotheses the solution must be the one that is sought.

7.3. Generalized Solutions

The underlying observation [140] is that a function $u \in BV(\Omega)$ minimizes $\mathscr{E}$ in Ω if and only if its subgraph

$$U = \{(x, t) \in \Omega \times \mathbb{R} : t < u(x)\}$$

minimizes the functional

$$\mathscr{F}[U] = \int_Q |D\varphi_U| + 2\int_Q H\varphi_U \, dx\, dt - \int_{\delta Q} \beta\varphi_U \, ds \tag{7.7}$$

with $Q = \Omega \times \mathbb{R}$. Minimization is here to be understood in the following sense: for $T > 0$ set

$$Q_T = \Omega \times [-T, T]$$

$$\delta Q_T = \partial\Omega \times [-T, T],$$

and for $U \subset Q$ we write

$$\mathscr{F}_T[U] = \int_{Q_T} |D\varphi_U| + 2\int_{Q_T} H\varphi_U \, dx\, dt - \int_{\delta Q_T} \beta\varphi_U \, ds. \tag{7.8}$$

We say that U *minimizes* (or U is a *solution* of) $\mathscr{F}_T$ in Q_T if $\mathscr{F}_T[U] \le \mathscr{F}_T[S]$ for any $S \subset Q_T$. We say that U *minimizes* (or U is a *local solution* of) $\mathscr{F}$ in Q if U minimizes $\mathscr{F}_T$ in Q_T for every $T > 0$.

Definition 7.2 (Miranda [140]). A function $u(x)$: $\Omega \to [-\infty, +\infty]$ is a *generalized solution* for the functional $\mathscr{E}$ if its subgraph U is a local solution for $\mathscr{F}$.

We note that a generalized solution can assume the values $\pm\infty$ on sets of positive measure. However, if a generalized solution $u(x)$ can be modified on a set of zero measure to be locally bounded, then $u(x)$ is a classical solution of (7.2) in Ω. Generalized solutions have the basic compactness property that, if $\{u_k\}$ is any sequence of such solutions in Ω, then a subsequence of the corresponding subgraphs U_k will converge in $L^1_{\text{loc}}(Q)$ to a set $U=\{(x,y)\in Q: y<u(x)\}$ and $u(x)$ will be a generalized solution for $\mathscr{E}$ (see Lemma 7.1).

We introduce the sets $\mathsf{P}=\{x\in\Omega: u(x)=+\infty\}$; $\mathsf{N}=\{x\in\Omega: u(x)=-\infty\}$ and $\mathsf{G}=\Omega\setminus(\mathsf{P}\cup\mathsf{N})$. Then $u(x)$ can be modified on a null set so that for any ball $B(x;\rho)$ of radius ρ centered at x, there holds $|\mathsf{P}\cap B(x;\rho)|\neq 0$ for all $x\in\mathsf{P}$, $|\mathsf{N}\cap B(x;\rho)|\neq 0$ for all $x\in\mathsf{N}$.

7.4. Construction of a Generalized Solution

We begin by solving the obstacle problem: for given $j>0$, to minimize $\mathscr{E}$ in the class

$$V_j=\{u\in BV(\Omega): |u|\le j\}. \tag{7.9}$$

Clearly $\mathscr{E}$ is bounded from below in V_j, and every minimizing sequence $\{u_k^j\}$ is bounded in $BV(\Omega)$. From the general compactness theorem for BV functions (cf. [89, p. 17]) we may extract a subsequence converging in $L^1(\Omega)$ to $u^j(x)\in BV(\Omega)$. By a theorem of Miranda [136], $u^j(x)$ has an L^1 trace on Σ, and thus $\mathscr{E}[u^j]$ is defined. To show that u^j minimizes, it suffices to prove lower semicontinuity; with later purposes in mind, we show this property in more generality than is needed at this point. The following result is an extended form of Lemma 6.3.

Lemma 7.1. *Let $v_k\in BV(\Omega)$, $\int_\Omega\sqrt{1+|Dv_k|^2}<M<\infty$, and suppose v_k converges in $L^1(\Omega)$ to $v(x)\in BV(\Omega)$. Suppose that for given $\varepsilon>0$ there is a covering $\{\Sigma_j\}\supset\Sigma$ as in Lemma 6.1, with $\mu<1+\varepsilon$. Then $\mathscr{E}[v]\le \liminf_k\mathscr{E}[v_k]$.*

Proof. a) We consider first $\int_\Omega\lambda(u)\,dx$. Set

$$\lambda_T(u)=\begin{cases}\lambda(T)+\lambda'(T)(u-T), & u\ge T\\ \lambda(u), & -T\le u\le T\\ \lambda(-T)+\lambda'(-T)(u+T), & u\le -T.\end{cases}$$

Since $\lambda''(u)\ge 0$ by hypothesis, we have always $\lambda_T(u)\le\lambda(u)$ if T is large enough. Now

$$\int_\Omega\lambda_T(v)\,dx-\int_\Omega\lambda_T(v_k)\,dx\le M\int_\Omega|v-v_k|\,dx$$

where $M=\max_{|t|\le T}|\lambda'(t)|$, and thus

$$\int_\Omega\lambda_T(v)\,dx\le\liminf_k\int_\Omega\lambda_T(v_k)\,dx$$

because of the L^1 convergence of v_k to v. Hence

$$\int_\Omega \lambda_T(v)\,dx \le \liminf_k \sup_T \int_\Omega \lambda_T(v_k)\,dx = \liminf_k \int_\Omega \lambda(v_k)\,dx$$

and finally

$$\int_\Omega \lambda(v)\,dx = \sup_T \int_\Omega \lambda_T(v)\,dx \le \liminf \int_\Omega \lambda(v_k)\,dx,$$

which was to be shown.

b) We consider the area integral $\mathfrak{A}[u]=\int_\Omega \sqrt{1+|Du|^2}\,dx$. We write $\mathfrak{A}_\delta[u]=\int_{\Omega_\delta}\sqrt{1+|Du|^2}\,dx$ over a subdomain $\Omega_\delta \subset \Omega$, to be chosen. We have by definition

$$\mathfrak{A}_\delta[v]-\mathfrak{A}_\delta[v_k]=\sup\int_{\Omega_\delta}\left\{g_0+v\sum_1^2 D_i g_i\right\}dx-\sup\int_{\Omega_\delta}\left\{h_0+v_k\sum_1^2 D_i h_i\right\}dx$$

among vector functions g, $h\in C_0^\infty(\Omega_\delta)$, with $|g|$, $|h|\le 1$. Thus

$$\begin{aligned}\mathfrak{A}_\delta[v]-\mathfrak{A}_\delta[v_k] \quad &\le \sup\int_{\Omega_\delta}\left\{g_0+v\sum_1^2 D_i g_i\right\}dx-\sup\int_{\Omega_\delta}\left\{g_0+v_k\sum_1^2 D_i g_i\right\}dx\\ &=\sup\int_{\Omega_\delta}\left\{(v-v_k)\sum_1^2 D_i g_i\right\}dx=0\end{aligned}$$

because of the L^1 convergence of v_k to v.

c) We consider the expression $Q[u]\equiv\mathfrak{A}[u]-\int_\Sigma \beta u\,dx$. We set $\Omega_\delta=\Omega\setminus\mathscr{A}_\delta$, $\mathscr{A}_\delta=\{x\in\Omega\colon d(x,\Sigma)<\delta\}$ being a boundary strip of width δ in Ω. Using Lemma 6.1, we find

$$\begin{aligned}Q[v]-Q[v_k]<(\mathfrak{A}_\delta[v]-\mathfrak{A}_\delta[v_k])+(2+\varepsilon)\int_{\mathscr{A}_\delta}\sqrt{1+|Dv|^2}\,dx\\ +\varepsilon\int_{\mathscr{A}_\delta}\sqrt{1+|Dv_k|^2}+\Upsilon(\Omega;\delta)\int_\Omega |v-v_k|\,dx.\end{aligned}$$

Given $\hat{\varepsilon}>0$, we first choose δ so that

$$\int_{\mathscr{A}_\delta}\sqrt{1+|Dv|^2}\,dx<\hat{\varepsilon};$$

we next observe from b) that for the given δ we have

$$\sup_{k\to\infty}(\mathfrak{A}_\delta[v]-\mathfrak{A}_\delta[v_k])\le 0;$$

finally, we obtain

$$\lim_{k\to\infty}\int_\Omega |v-v_k|\,dx=0$$

because of the L^1 convergence of the $\{v_k\}$. Thus, $\sup_{k\to\infty}(Q[v]-Q[v_k]) \leq (2+\varepsilon)\hat{\varepsilon}+\varepsilon M$. Since $\hat{\varepsilon}$, ε are arbitrary, the lemma is proved.

The above lemma assures us of the existence of a minimizing u^j for $\mathscr{E}$ in V_j. Since $u^j \in BV(\Omega)$, its subgraph U^j is a solution for $\mathscr{F}$ in Q_T, for any $T\leq j$.

Lemma 7.2. *There exists $C(T)$ such that if A is a solution for $\mathscr{F}$ in Q_T, then*

$$\int_{Q_T}|D\varphi_A|<C(T). \tag{7.10}$$

Proof. We have, by the minimizing property,

$$\begin{aligned}\mathscr{F}_T[A]&=\int_{Q_T}|D\varphi_A|+\int_{Q_T}H\varphi_A\,dx\,dt+\int_{\delta Q_T}\beta\varphi_A\,ds\,dt\\ &\leq \mathscr{F}_T[A\backslash Q_T]\\ &=\int_{Q_T}|D\varphi_{A\setminus Q_T}|\leq 2|\Omega|\end{aligned}$$

so that

$$\int_{Q_T}|D\varphi_A|\leq 2|\Omega|+\int_{Q_T}|H|\,dx\,dt+2|\partial\Omega|T=C(T).$$

We are now free to let $j\to\infty$. For any fixed T we have

$$\int_{Q_T}\varphi_{U^j}\,dx\,dt+\int_{Q_T}|D\varphi_{U^j}|\leq 2T|\Omega|+C(T);$$

applying again the general compactness theorem for BV functions, we find $U^j\to U$ in $L^1(Q_T)$ for any T, and

$$\int_{Q_T}\varphi_U\,dx\,dt+\int_{Q_T}|D\varphi_U|\leq 2T|\Omega|+C(T).$$

The set U, as a limit of subgraphs, is again a subgraph, for a function $u(x)$ over Ω that may assume the values $\pm\infty$. The lower semicontinuity of the functional $\mathscr{F}$ is obtained as in Lemma 7.1, thus $\mathscr{F}_T[U]\leq F_T[V]$ for any Caccioppoli set V. We have proved:

Theorem 7.2. *Under the hypotheses of Theorem 7.1, but without any requirement of the form (7.4), (7.5), there exists a generalized solution $u(x)$ for the functional $\mathscr{E}$.*

7.5. Proof of Boundedness

Under the hypotheses that have been used thus far, it must be expected that the sets P or N can be of significant size; Theorem 7.2 applies, for example, to the trapezoid of §6.19. The extent to which (7.4), (7.5) restrict the behavior of the solution can be seen from the following results.

Lemma 7.3. *Suppose U^j is a solution for*

$$\mathscr{F}_T^j[A] = \int_{Q_T} |D\varphi_A| + 2\int_{Q_T} H^j \varphi_A \, dx\, dt - \int_{\delta Q_T} \beta^j \varphi_A \, ds\, dt.$$

Suppose $H^j \to H$ uniformly in Q_T, $\beta^j \to \beta$ uniformly on δQ_T. Then U^j is a minimizing sequence for $\mathscr{F}_T[A]$.

Proof. If not, there would exist a subsequence $\hat{U}_j$ and a subgraph $\hat{U} \in BV(Q_T)$ with

$$\begin{aligned}
\mathscr{F}_T[\hat{U}] &< \inf \mathscr{F}_T[\hat{U}^j] \\
&= \inf\left\{\mathscr{F}_T^j[\hat{U}^j] + \int_{Q_T} (H - H^j)\varphi_{\hat{U}^j} \, dx\, dt + \int_{\delta Q_T} (\beta - \beta^j)\varphi_{\hat{U}^j} \, ds\, dt\right\} \\
&= \inf \mathscr{F}_T^j[\hat{U}^j] \le \inf \mathscr{F}_T^j[\hat{U}]
\end{aligned}$$

since $\hat{U}^j$ is minimizing for $\mathscr{F}_T^j$. Letting $j \to \infty$, we obtain a contradiction.

Lemma 7.4. *The set* P *minimizes the functional* $\Phi[A]$; *the set* N *minimizes* $\Psi[A]$.

Proof. For a sequence $j \to \infty$, let

$$U_j = \{(x,t) \in Q : t < u(x) - j\}.$$

Then U_j is a solution for

$$\mathscr{F}^j[A] = \int_Q |D\varphi_A| + \int_Q H(u+j)\varphi_A \, dx\, dt - \int_{\delta Q} \beta \varphi_A \, ds\, dt.$$

We have $U^j \to \cap U^j = U = \mathsf{P} \times \mathbb{R}$. According to Lemma 7.3, U^j is a minimizing sequence, hence by semicontinuity $\mathsf{P} \times \mathbb{R}$ minimizes $\mathscr{F}_T^\infty[A] \equiv \int_{-T}^T \Phi[A]\, dt$. Since $\mathsf{P} \times \mathbb{R}$ and $\Phi[A]$ do not contain t, the result for P follows directly. The corresponding result for N is proved analogously.

Lemma 7.5. *Let u be a generalized solution for $\mathscr{E}$, and suppose* (7.4) *holds for every $\Omega^* \neq \emptyset$. Then $\mathsf{P} = \emptyset$. If* (7.5) *holds for every $\Omega^* \neq \emptyset$, then $\mathsf{N} = \emptyset$.*

Proof. By Lemma 7.4, P minimizes Φ, N minimizes Ψ.

The conditions of Lemma 7.5 exclude the case $H \equiv \text{const.}$, for which (7.4) and (7.5) fail when $\Omega^* = \Omega$. To study this case, set $H_j = H + \varepsilon_j \tan^{-1} u$, for some sequence $\varepsilon_j \to 0$, and let u_j be the corresponding generalized solution for $\mathscr{E}$. Choose λ_j such that

$$\begin{aligned} &\operatorname{meas}\{x \in \Omega : u_j(x) - \lambda_j \geq 0\} \geq \tfrac{1}{4}|\Omega| \\ &\operatorname{meas}\{x \in \Omega : u_j(x) - \lambda_j \leq 0\} \geq \tfrac{1}{4}|\Omega|, \end{aligned} \tag{7.11}$$

and set

$$U_j = \{(x,t) \in Q : t < u_j(x) - \lambda_j\}.$$

Then U_j is a solution for

$$\mathscr{F}^j[A] = \int_Q |D\varphi_A| + \int_Q H_j(t + \lambda_j)\varphi_A \, dx\, dt - \int_{\delta Q} \beta \varphi_A \, ds\, dt.$$

By Lemma 7.2, $\int_{Q_T} |D\varphi_{U_j}| < C(T)$, hence there is a subsequence $U_j \to U$ in $L^1(Q_T)$. Since $H_j \to H$ uniformly in Q, U_j is a minimizing sequence for $\mathscr{F}_T[A]$, and, by semicontinuity, U is a solution for $\mathscr{F}_T[A]$ for any $T > 0$, hence U is a local solution for $\mathscr{F}[A]$ and the corresponding $u(x)$ a generalized solution for $\mathscr{E}$. By Lemma 7.4, P minimizes Φ, N minimizes Ψ. If the relations (7.4), (7.5) hold, then $\mathsf{P} = \emptyset$ or Ω, $\mathsf{N} = \emptyset$ or Ω. But (7.11) holds in the limit as $j \to \infty$, hence neither P nor N can be Ω. We have proved:

Theorem 7.3. *Suppose the hypotheses of Theorem* 7.1 *hold except that the only requirement on μ is that for any $\varepsilon > 0$ a covering $\{\Sigma_j\}$ as in Lemma* 6.1 *can be found, with $\mu < 1 + \varepsilon$. Then there exists a generalized solution $u(x)$ of $\mathscr{E}$ in Ω that is locally finite in Ω.*

We may now state:

Theorem 7.4. *The solution $u(x)$ is locally real analytic and satisfies* (7.2) *in Ω.*

For the proof of Theorem 7.4 we refer the reader to an extensive literature, see, e.g., [80, 81, 83, 123, 135, 137]; see also the expositions in [89, 126]. Much of the cited material was developed specifically for minimal surfaces, but applies as well to the case considered here.

Under the conditions given, it can still happen that u will have infinite limits with approach to some boundary sets. That eventuality is however excluded when the constant μ of Lemma 6.1 satisfies $\mu<1$. It suffices to make that assumption with respect to those sets Σ_j (see Lemma 6.1) such that a given point $p\in\Sigma$ lies in the support of the corresponding φ_j. The point p will then have a distance $>\sigma$ to the remaining points of Σ, in the sense that any ball $B_\sigma(p)$ of radius $\sigma>0$ about p (in the metric indicated in Lemma 6.1) will meet Ω in a component for which all boundary points on Σ lie in the indicated Σ_j. This hypothesis permits, for example, inward cusps and even boundary segments that may physically coincide but are adjacent to differing parts of Ω (see Fig. 6.9).

Lemma 7.6 (cf. de Giorgi [201]; see also [123, 86]). *Let $p\in\Sigma$. We suppose $\max|\beta_j|\max\sqrt{1+L_j^2}<1$ in the indicated $\{\Sigma_j\}$. Let $u(x)$ be a generalized solution for $\mathscr{E}$ in Ω with subgraph U. Let $Z_r(p,\tau)$ denote the cylinder $|x-p|<r$, $|t-\tau|<r$, and let $U_r=U\cap Z_r$. There exist constants $C>0$ and $r_0>0$ such that if the measure $|U_r|>0$ for all $r>0$, then $|U_r|>Cr^3$ for all $r<r_0$.*

Proof. Since U minimizes $\mathscr{F}$ in any Q_T, we may compare U with $U\backslash Z_r$ to obtain

$$\int_{Q\cap Z_r}|D\varphi_U|+\int_{Q\cap Z_r}H\varphi_U\,dx\,dt-\int_{\delta Q\cap Z_r}\beta\varphi_U\,ds\,dt\le\int_{\partial Z_r}\varphi_U$$

so that for almost all $r>0$

$$\int_Q|D\varphi_{U_r}|+\int_Q H\varphi_{U_r}\,dx\,dt-\int_{\delta Q}\beta\varphi_{U_r}\le 2\int_{\partial Z_r}\varphi_U. \tag{7.12}$$

Using (6.16) and summing over the appropriate j, we find

$$\begin{aligned}\int_{\delta Q}\varphi_{U_r}\,ds\,dt&\le\max\sqrt{1+L_j^2}\int_Q|D\varphi_{U_r}|+C|U_r|\\&\le\max\sqrt{1+L_j^2}\int_Q|D\varphi_{U_r}|+C|Z_r|^{1/3}\int_{\mathbb{R}^3}|D\varphi_{U_r}|\end{aligned}\tag{7.13}$$

for some constants C; here we have used the isoperimetric inequality (6.15).

We have

$$\int_{\mathbb{R}^3}|D\varphi_{U_r}|=\int_Q|D\varphi_{U_r}|+\int_{\delta Q}\varphi_{U_r}\,ds\,dt;$$

for $r<\sigma$ small enough that $C|Z_r|^{1/3}<\frac{1}{2}$ we obtain from (7.13)

$$\int_{\delta Q}\varphi_{U_r}\le\frac{\max\sqrt{1+L_j^2}+C|Z_r|^{1/3}}{1-C|Z_r|^{1/3}}\int_Q|D\varphi_{U_r}| \tag{7.14}$$

$$\int_{\mathbb{R}^3} |D\varphi_{U_r}| \leq \frac{\max\sqrt{1+L_j^2}}{1-C|Z_r|^{1/3}} \int_Q |D\varphi_{U_r}|$$
$$\leq 2(1+\max\sqrt{1+L_j^2}) \int_Q |D\varphi_{U_r}|. \tag{7.15}$$

We have next

$$\int_Q H\varphi_{U_r}\,dx\,dt = \int_{\tau-r}^{\tau+r} dt \int_{B_r(p)} H\varphi_{U_r}\,dx$$
$$\geq -\int_{\tau-r}^{\tau+r} \{\int_{B_r(p)} |H|^2 \varphi_{U_r}\,dx \int_{B_r(p)} \varphi_{U_r}\,dx\}^{1/2}\,dt \tag{7.16}$$
$$\geq -C|H_0^-|\,r\int_{\tau-r}^{\tau+r} |U_r(t)|^{1/2}\,dt$$

where $H_0^- = \min\{0, H(\tau-r)\}$ and $U_r(s) = U_r \cap \{(x,t): t=s\}$. Using again the isoperimetric inequality, we obtain

$$\int_Q H\varphi_{U_r}\,dx\,dt \geq -C|H_0^-|\,r\int_{\mathbb{R}^3} |D\varphi_{U_r}|. \tag{7.17}$$

Combining (7.14), (7.15), and (7.17) we find now, for a suitable constant C,

$$\int_Q H\varphi_{U_r}\,dx\,dt - \int_{\partial Q} \beta\varphi_{U_r}\,ds\,dt$$
$$\geq -\left\{-2C(1+\max_j\sqrt{1+L_j^2})|H_0^-|\,r\right.$$
$$\left.+(\max_{\Sigma_j}\beta_j)\frac{\max_j\sqrt{1+L_j^2}+C|Z_r|^{1/3}}{1-C|Z_r|^{1/3}}\right\}\int_Q |D\varphi_{U_r}|, \tag{7.18}$$

where the maxima are taken among the Σ_j under consideration.

The expression in brackets tends to $(\max_j \beta_j)\max_j\sqrt{1+L_j^2}<1$ by hypothesis, as $r\to 0$. Therefore, for all sufficiently small r we will have by (7.12), (7.15), and (7.18)

$$\frac{d}{dr}|U_r| = \int_{\partial Z_r} \varphi_U \geq \varepsilon_0 \int_Q |D\varphi_{U_r}|$$

for some $\varepsilon_0 > 0$. Using once more the isoperimetric inequality, we find

$$\frac{d}{dr}|U_r| \geq \varepsilon_1 |U_r|^{2/3} \tag{7.19}$$

for some $\varepsilon_1>0$, for almost all r in an interval $0<r<r_1$. If $|U_r|>0$ for all $r>0$, we may integrate (7.19) to obtain the stated result.

The values of H enter into (7.19) only through the factor $|H_0^-|$ in (7.16), and affect (7.19) only in the size of the interval $[0,r_1]$. For given t_0, $|H_0^-|$ is uniformly bounded for all $t>t_0$, thus the estimate (7.19) holds with fixed ε_1 and r_1 for all $t>t_0$.

Suppose there would exist a point $p_0\in\Sigma$ and a sequence $x_i\in\Omega$, with $x_i\to p_0$ and $u(x_i)\to+\infty$. Then for all t there would hold $|U_r(p_0,t)|>0$ for all $r>0$, hence by Lemma 7.6, for any fixed τ, there exist C, $R>0$ such that $|U_R(p_0,t)|>CR^3$ for all $t>\tau$. We conclude immediately that $\mathsf{P}\neq\emptyset$, thus contradicting Lemma 7.5.

To discuss the set N, we make the transformation $u=-v$, $\hat{H}(v)=-H(-u)$, $\beta=-\hat{\beta}$. Then

$$\mathscr{F}[U]\equiv\hat{\mathscr{F}}[V]\equiv\int_Q|D\varphi_V|+\int_Q\hat{H}\varphi_V\,dx\,dt-\int_{\delta Q}\hat{\beta}\varphi_V\,ds\,dt$$

and U is a local solution for $\mathscr{F}$ if and only if V is a local solution for $\hat{\mathscr{F}}$. The above reasoning shows $\mathsf{P}_V\neq\emptyset$, which is equivalent to $\mathsf{N}_U\neq\emptyset$. In view of the freedom in the way the partition of unity used in Lemma 6.1 can be constructed, we obtain

Theorem 7.5. *Let $p\in\Sigma$ be in a neighborhood on Σ that can be covered by intervals Σ_k of Σ as in Lemma* 6.1, *each of which can be represented locally by Lipschitz functions with Lipschitz constants L_k such that* $\max\beta_k\max\sqrt{1+L_k^2}<1$. *Then any generalized solution $u(x)$ of $\mathscr{E}$ for which* $\mathsf{P}=\mathsf{N}=\emptyset$ *is (essentially) bounded for any approach to p. If the above condition holds for all $p\in\Sigma$, then $u(x)$ is bounded in Ω, describes a surface with finite area, minimizes $\mathscr{E}$ in $BV(\Omega)$, and is a variational solution for $\mathscr{E}$ in Ω.*

We have proved all but the last statement. If u were (essentially) unbounded, there would be a sequence of sets $\Omega_j\subset\Omega$ of positive measure, in which $u>j\to\infty$. A limit point p_0 on Σ would satisfy the conditions of Lemma 7.6, and we would obtain a contradiction as above. Since u is bounded and its subgraph U is a local solution for $\mathscr{F}$, Lemma 7.2 yields that $\int_Q|D\varphi_U|=\int_\Omega\sqrt{1+|Du|^2}\,dx<\infty$. Thus $u(x)\in BV(\Omega)$, and since U minimizes $\mathscr{F}_T$ for all sufficiently large T, we conclude that u minimizes $\mathscr{E}$; Theorem 7.4 then implies that u is a strict solution of (7.2) in Ω.

Let $\eta\in H^{1,1}(\Omega)$. Since $u(x)$ minimizes, we have

$$\delta\mathscr{E}=\mathscr{E}[u+\varepsilon\eta]-\mathscr{E}[u]\geq 0 \tag{7.20}$$

for any real ε. Set $\zeta_i=\eta_{x_i}$, $p=Du=(p_1,p_2)$, and $W(p)=\sqrt{1+p^2}$. We have

$$\delta\mathscr{E}=\int_\Omega(A_i(\varepsilon)\zeta_i+A(\varepsilon)\eta)\,dx-\varepsilon\int_\Sigma\beta\eta\,ds$$

with

$$A_i = \int_0^\varepsilon W_{p_i}(p+\tau\zeta)\,d\tau, \qquad \Lambda = \int_0^\varepsilon \lambda'(u+\tau\eta)\,d\tau.$$

There holds

$$\lim_{\tau\to 0} W_{p_i}(p+\tau\zeta) = W_{p_i}(p)$$

for almost all $x\in\Omega$, while $|W_{p_i}(p+\tau\zeta)|<1$ for all τ. Thus, the bounded convergence theorem implies the existence of

$$\begin{aligned}\left.\frac{\partial\mathscr{E}}{\partial\varepsilon}\right|_{\varepsilon=0} &= \int_\Omega (W_{p_i}\zeta_i + \lambda'\eta)\,dx - \int_\Sigma \beta\eta\,ds \\ &= \lim_{\varepsilon\to 0}\frac{\mathscr{E}[u+\varepsilon\eta]-\mathscr{E}[u]}{\varepsilon} \qquad (7.21)\\ &= 0\end{aligned}$$

since the numerator is nonnegative for both positive and negative ε.

7.6. Uniqueness

Theorem 7.6. *Let u, $\hat{u}$ be variational solutions for $\mathscr{E}$, in the sense of Definition* 7.1, *corresponding to $\beta\le\hat{\beta}$. Then either $u<\hat{u}$ in Ω, or else $\lambda'(u)\equiv\lambda'(\hat{u})$, $\beta\equiv\hat{\beta}$, and $u\equiv v+C$ in Ω.*

Proof. Analogously to (7.21), we obtain for any real ε and $\eta\in H^{1,1}(\Omega)$

$$\frac{\partial\mathscr{E}}{\partial\varepsilon} = \int_\Omega [W_{p_i}(p+\varepsilon\zeta)\zeta_i + \lambda'(u+\varepsilon\eta)]\,dx - \int_\Sigma \beta\eta\,ds. \qquad (7.22)$$

Suppose there were to exist two variational solutions $u(x)$, and $\hat{u}(x)$, with $u(x)>\hat{u}(x)$ on some open set $\hat{\Omega}\subset\Omega$. We choose M sufficiently large that the set Ω_M: $0<u-\hat{u}<M$ is nonnull, and we set

$$\eta(x) = \begin{cases} 0, & u-\hat{u}\le 0\\ u-\hat{u}, & 0<u-\hat{u}<M\\ M, & M\le u-\hat{u}.\end{cases}$$

Then $\eta\in H^{1,1}(\Omega)$, and since u, $\hat{u}$ are variational solutions, we have

$$\int_\Omega [W_{p_i}(p)\zeta_i + \lambda'(u)\eta]\,dx - \int_\Sigma \beta\eta\,ds = 0$$

$$\int_\Omega [W_{p_i}(\hat{p})\zeta_i + \lambda'(\hat{u})\eta]\,dx - \int_\Sigma \hat{\beta}\eta\,ds = 0$$

and hence, setting $\hat{\eta}=u-\hat{u}$, $\hat{\zeta}_i=\hat{\eta}_{x_i}$,

$$\int_\Omega Q\,dx=\int_\Sigma(\beta-\hat{\beta})\eta\,ds \tag{7.23}$$

with

$$Q=\int_0^1[W_{p_ip_j}(\hat{p}+\tau\zeta)\zeta_i\hat{\zeta}_j+\lambda''(\hat{u}+\tau\hat{\eta})\eta\hat{\eta}]\,d\tau. \tag{7.24}$$

By assumption, $\lambda''\geq 0$. Further, the discriminant of the form $W_{p_ip_j}(p)\zeta_i\zeta_j$ is $(1+p^2)^{-2}>0$.

On Ω_M we have $\eta\equiv\hat{\eta}$, and hence $Q\geq 0$, the equality holding only if $u\equiv\hat{u}$ or else $\lambda''\equiv 0$ and $u\equiv\hat{u}+c$. On the set Ω_M^+: $u-\hat{u}\geq M$, we have $\zeta_i=0$ wherever it is defined, and $\eta\hat{\eta}\geq M^2$, hence the same conclusion holds. On the set $u-\hat{u}\leq 0$ we find $Q\equiv 0$ wherever it is defined (up to a set of measure zero). Since $\beta\leq\hat{\beta}$, we obtain a contradiction from (7.23) unless either $u\leq\hat{u}$ in Ω or else $\lambda'(u)\equiv\lambda'(\hat{u})$ and $\beta\equiv\hat{\beta}$. The stronger statement $u<\hat{u}$ follows as in the proof of Theorem 5.1.

From the above theorem we find immediately:

Corollary 7.6. *The result just proved implies the uniqueness statement in Theorem* 7.1.

For later reference, we point out the following extension of Theorem 7.6, which can be obtained without essential change in the proof.

Theorem 7.7. *Let u, $\hat{u}$ satisfy* $\mathrm{N}u\geq\mathrm{N}\hat{u}$ *in the sense of the variational relation*

$$\int_\Omega\{[W_{p_i}(p)-W_{p_i}(\hat{p})]\zeta_i+[\lambda'(u)-\lambda'(\hat{u})]\eta\}\,dx-\int_\Sigma(\beta-\hat{\beta})\eta\,ds\leq 0, \tag{7.25}$$

all $\eta\in H^{1,1}(\Omega)$. Suppose $\Sigma=\Sigma_\alpha\cup\Sigma_\beta\cup\Sigma_0$, with $u\leq\hat{u}$ on Σ_α, $\beta\leq\hat{\beta}$ on Σ_β, and such that Σ_0 has one-dimensional Hausdorff measure zero (*cf. Theorem* 5.1; *here no differentiability hypothesis need be introduced on Σ_β*). *Then either*

i) *$u\leq\hat{u}$ in Ω, equality holding at any point if and only if equality holds throughout Ω; or else*
ii) *$\lambda'(u)\equiv\lambda'(\hat{u})$, $\beta\equiv\hat{\beta}$, equality holds in* (7.25), *and $u\equiv\hat{u}+c$ in Ω.*

Theorem 7.8. *Let $\Omega\in C^{(1)}$, and let $u(x)$ be a minimizing function for $\mathscr{E}$ in Ω in the sense of Theorem* 7.1. *Let $v(x)\in C^{(1)}(\bar{\Omega})\cap C^{(2)}(\Omega)$ be a solution of* (7.2) *in Ω, with* (7.3) *holding on Σ. Then either $u\equiv v$ in Ω, or else $\lambda'(u)=\lambda'(v)$ and $u\equiv v+c$ in Ω.*

Proof. The relations (7.2), (7.3) are the Euler equations for $\mathscr{E}[u]$; because the smoothness of $v(x)$ up to Σ and since $u(x)$, $Du(x)\in L^{(1)}(\Omega)$, we may set

$\eta = u - v$ and find that

$$\left.\frac{\partial \mathscr{E}[v+\varepsilon\eta]}{\partial \varepsilon}\right|_{\varepsilon=0} = 0.$$

The remainder of the proof follows as in Theorem 7.6.

Thus, the solution we have constructed is identical to the smooth solution of (7.2), (7.3) whenever such a solution exists.

7.7. The Variational Condition; Limiting Case

We suppose as before that to any $\varepsilon > 0$ there is a covering $\{\Sigma_j\}$ for which $\mu < 1 + \varepsilon$; however we now permit configurations for which the improved estimate $\mu < 1$ cannot be attained at all points of Σ. This situation arises, for example, if Σ is smooth but $\gamma = 0$ on all or part of Σ, also if a corner appears at which $\alpha + \gamma = \pi/2$. We are assured by Theorem 7.3 of the existence of a generalized solution for which $\mathsf{P} = \mathsf{N} = \emptyset$. However, we have not determined whether the solution is variational.

Theorem 7.9. *Under the above conditions, there exists a variational solution for $\mathscr{E}$ in Ω. If $u(x)$, $v(x)$ are any two such solutions, then either $u(x) \equiv v(x)$ in Ω, or else $\lambda'(u) \equiv \lambda'(v)$ and $u \equiv v + c$.*

Proof. We approximate β by functions β^j, for each of which the condition $\mu < 1$ is satisfied. By Theorem 7.1, each of the corresponding u^j will be a bounded variational solution for $\mathscr{E}$. We perform the approximation in two steps: having chosen β^0 with $|\beta^0| \le |\beta|$, we take a sequence β^{0j} such that $\beta^{0j} = \beta^0$ on the set $\beta < 0$, $\beta^{0j} \nearrow \beta$ on the set $\beta > 0$. The corresponding sequence u^{0j} will then be nondecreasing (Theorem 7.6). Following the proof of Theorem 7.3, we may obtain convergence to a generalized solution u^+ corresponding to the data

$$\beta^+ = \begin{cases} \beta, & \beta > 0 \\ \beta^0, & \beta < 0, \end{cases}$$

and for which $\mathsf{P}, \mathsf{N} = \emptyset$. For each j, we have the variational condition

$$\int_\Omega (W_{p_i}(p^{0j})\zeta_i + \lambda'(u^{0j})\eta) = \int_\Sigma \beta^{0j}\eta\, ds \tag{7.26}$$

for any $\eta \in H^{1,1}(\Omega)$.

At this point, we need

Lemma 7.7. *For any compact $K \subset \Omega$, there is a constant $C(K)$ such that $|u^{0j}| < C(K)$.*

Proof. If not, there would exist points $x_k \to x_0 \in K$ such that $u^{0j}(x_j) \to \pm\infty$; we may suppose $u^{0j} \to +\infty$. Then for any t and any $r>0$ the set $(U^{0j})_r = \{U^{0j} \cap Z_r(x_0;t)\}$ would be nonnull for all sufficiently large j, hence following the proof of Lemma 7.6 we would find $|U_r^{0j}| > Cr^3$ for all $r<r_0$ for some constant C, for all large enough j. We conclude $|U_r^0| \geq Cr^3$, independent of t, which would imply $\mathsf{P} \neq \emptyset$, a contradiction.

Lemma 7.8. *There exists $C(M;\rho)$ such that if $u(x)$ is a solution of (7.2) in a disk B_ρ of radius ρ and center x_0, and $|u(x)|<M$ in B_ρ, then $|Du(x_0)| + |D^2 u(x_0)| < C(M;\rho)$.*

The bound for the first derivative follows from [16, 113]. For the second derivative the result follows from the general theory of uniformly elliptic equations in the plane, see, e.g., [38].

We return to the proof of Theorem 7.9. In any compact $K \subset \Omega$ we have $|u^{0j}| < M(K)$, $|Du^{0j}| < M(K)$, $|D^2 u^{0j}| < M(K)$. Hence there is a subsequence for which $u^{0j} \to u^+$, $Du^{0j} \to Du = p^+$. Since $|W_{p_i}| < 1$, $\zeta_i \in L^1(\Omega)$, we conclude from the bounded convergence theorem

$$\int_\Omega W_{p_i}(p^{0j})\zeta_i\,dx \to \int_\Omega W_{p_i}(p^+)\zeta_i\,dx.$$

Setting $\eta \equiv 1$ in (7.6), we find

$$\int_\Omega \lambda'(u^{0j})\,dx = \int_\Sigma \beta^{0j}\,ds. \tag{7.27}$$

Letting $\beta^{0j} \nearrow \beta^+$, we obtain $u^{0j} \nearrow u^+$ (Theorem 7.6). Since $\lambda'' \geq 0$ by hypothesis, there follows $\lambda'(u^{0j}) \nearrow \lambda'(u^+)$. Since the right side of (7.27) converges to $\int_\Sigma \beta^+\,ds$, we conclude there exists

$$\int_\Omega \lambda'(u^+)\,dx = \lim_{j\to\infty} \int_\Omega \lambda'(u^{0j})\,dx$$

and there follows immediately, since λ' is bounded below,

$$\int_\Omega (W_{p_i}(p^+)\zeta_i + \lambda'(u^+)\eta) = \int_\Sigma \beta^+\eta\,ds$$

for any $\eta \in H^{1,1}(\Omega)$.

We now keep β fixed in the set $\beta>0$ and replace β^+ by a sequence $\beta^j \searrow \beta$. Observing that now $u^j \searrow u$, $\lambda'(u^j) \searrow \lambda'(u)$, we conclude finally that

$u(x)$ satisfies the variational condition (7.6). The second part of the statement of the theorem now follows from Theorem 7.6.

7.8. A Necessary and Sufficient Condition

When $-\infty < H_{-\infty} \leq H_\infty < \infty$, it is unnecessary to use Lemma 6.1 to obtain Theorem 7.9. We consider any domain Ω having the properties that each $f \in BV(\Omega)$ admits a trace on Σ permitting partial integration (cf. Anzellotti and Giaquinta [3]), and that an inequality of the form (6.15) holds for Ω. We then find

Lemma 7.9. *If* (7.4), (7.5) *hold for any* $\Omega^* \subset \Omega$ *with* $\Omega^* \neq \emptyset, \Omega$, *then for any* $\varepsilon > 0$ *an inequality* (6.16) *holds for every* $f \in BV(\Omega)$, *with* $\mu \leq 1 + \varepsilon$.

Proof. We suppose first that an annulus $\mathscr{A}_\delta$ of width δ adjacent to Σ contains supp$f$. Set $F_t = \{x \in \Omega : f(x) > t\}$. We have by (7.4)

$$\begin{aligned}\int_\Sigma \beta \varphi_{F_t} ds &\geq -\int_\Omega |D\varphi_{F_t}| - H_\infty |F_t| \\ &\geq -\int_\Omega |D\varphi_{F_t}| - H_\infty |F_t|^{1/2} |\mathscr{A}_\delta|^{1/2}\end{aligned}$$

since $F_t \subset \mathscr{A}_\delta$; thus, using (6.15) we find

$$\int_\Sigma \beta \varphi_{F_t} ds \geq -(1+\varepsilon) \int_\Omega |D\varphi_{F_t}|$$

if δ is sufficiently small. Integrating in t, we obtain

$$\int_\Sigma \beta f\, ds \geq -(1+\varepsilon) \int_\Omega |Df|.$$

Replacing (7.4) by (7.5), we are led to

$$\int_\Sigma \beta f\, ds \leq (1+\varepsilon) \int_\Omega |Df|;$$

thus the stated result holds under the indicated restriction supp$f \subset \mathscr{A}_\delta$.

Let $\eta(x) \in C^\infty(\Omega)$, $\eta = 1$ on Σ, $0 \leq \eta \leq 1$, supp$\eta \subset \mathscr{A}_\delta$. If $f \in BV(\Omega)$, then $\eta f \in BV(\Omega)$ and supp$\eta f \subset \mathscr{A}_\delta$. The lemma follows by applying to ηf the result already proved.

Returning to the proof of Theorem 7.9 and using Lemma 7.9, we obtain immediately

Theorem 7.10. *If* $-\infty < H_{-\infty} \leq H_{\infty} < \infty$, *a variational solution exists in* Ω *if and only if* (7.4), (7.5) *hold for every* $\Omega^* \subset \Omega$ *with* $\Omega^* \neq \Omega, \emptyset$.

7.9. A Limiting Configuration

The question arises, under what circumstances will the solution $u(x)$ be bounded? We have already considered this question for the case $2H = \lambda' \equiv \text{const.}$ in §6.2, where we showed that there is no solution $u(x)$ with $\gamma = 0$ adjacent to an arc of Σ with curvature $k > 2H$, and also that if $\gamma = 0$ and $k \equiv 2H$, then $u(x)$ is unbounded, while if $k < 2H$, then $u(x)$ is bounded at the arc. A related question occurs for configurations for which a boundary angle appears, of interior opening 2α. If in a neighborhood of the vertex V there holds $\alpha + \gamma < \pi/2$, then no solution can exist (Theorem 6.2). If $\alpha + \gamma > \pi/2$, then locally the conditions of Theorem 7.5 are fulfilled, and any generalized solution will be bounded at V. If the value γ at V satisfies $\alpha + \gamma = \pi/2$, then differing kinds of behavior may be possible. The following hypothesis is a slightly weakened version of one that was introduced in [65].

Hypothesis $\alpha(\gamma)$. It is possible to place a lower hemisphere $v(x;\gamma)$ of radius $(2H)^{-1} = R_\gamma$, with equatorial circle Q passing through V, in such a way that at each point of Σ interior to Q and to some neighborhood $\mathcal{N}_V$ of V there holds $\nu \cdot \mathsf{T}v \geq \cos\gamma$.

Theorem 7.11 (Finn [65]; Tam [172]). *Suppose the hypotheses of Theorem* 7.5 *hold except at the single point* V, *at which they are replaced by Hypothesis* $\alpha(\gamma)$. *Then the conclusion of Theorem* 7.5 *holds throughout* Ω.

The proofs of this result in [65] and in [172] can be considerably shortened with the aid of the material above. By Theorem 7.9 there exists a unique variational solution $u(x)$, and by Theorem 7.5 the solution is bounded except perhaps at V. We introduce an arc Γ within $Q \cap \mathcal{N}_V$, joining points P_1 and P_2 on Σ on opposite sides of V and cutting off the domain $\mathcal{M}_V$ at V, and we place the hemisphere $v(x)$ as in Hypothesis $\alpha(\gamma)$, with $v(x) \geq u(x)$ on Γ. Since $u(x)$, $v(x)$ are both variational solutions of the same equation in $\mathcal{M}_V$, we may apply Theorem 7.7 to obtain $u(x) < v(x)$ in $\mathcal{M}_V$, hence $u(x)$ is bounded above in Ω. Next we place the hemisphere with center at V, and observe that then $\nu \cdot \mathsf{T}v < \cos\gamma$ at points of Σ close enough to V. If we move the hemisphere vertically downwards so that $v(x) \leq u(x)$ on Γ, the same procedure yields $v(x) < u(x)$ in $\mathcal{M}_V$. Thus, $|u(x)|$ is bounded in Ω, as was to be shown.

7.10. The Case $\mu > \mu_0 > 1$

If there are portions of Σ on which for every covering $\{\Sigma_j\}$ there will hold $\mu > \mu_0 > 1$, then if $\lambda'(u)$ is bounded, there will in general be no variational solution. Such a situation occurs, for example, at a corner at which $\alpha + \gamma < \pi/2$ (cf. §6.2). If $\lambda'(u)$ is unbounded, then solutions may exist. In the (gravitational) case $\lambda = \kappa u$, $\kappa > 0$, we have already given estimates (Theorem 5.5) on the behavior of a solution under the assumption that a solution exists. We now proceed to show that in the configuration studied in Theorem 5.5 a variational solution of the Euler equation (7.2) under the boundary condition (7.3) does in fact exist (cf. Finn and Gerhardt [68]). For simplicity, we assume an angle 2α formed by two straight segments, and constant data γ_0, $0 < \gamma_0 < \pi/2$, with $\alpha + \gamma_0 < \pi/2$. It will be clear that more general configurations and data are amenable to the method. The single hypothesis we shall need on $\lambda(u)$, besides $\lambda''(u) \geq 0$, is that $\lim_{u \to \infty} \lambda'(u) = \infty$.

We begin by completing the segments in any way to form a smooth domain; we then smooth the corner with an inscribed circular arc Γ_δ of radius δ, obtaining a domain Ω_δ without corners, bounded by Γ_δ and by $\Sigma_\delta \subset \Sigma$. By Theorem 7.9, there is a unique variational solution u^δ in Ω_δ, corresponding to boundary data γ_0.

Let us fix δ and consider the solution $u^{\delta'}$, $\delta' < \delta$, in the disk B_δ obtained by completing Γ_δ to a full circle. We set

$$\Lambda(t) = \sup\{u : \lambda'(u) < t\}.$$

The procedure (essentially) of §5.2 yields

$$u^{\delta'} < \Lambda\left(\frac{2}{\delta}\right) + \delta < \infty$$

in B_δ, all $\delta' < \delta$, according to the hypothesis $\lim_{u \to \infty} \lambda'(u) = \infty$. A similar estimate holds throughout Ω_δ. Further comparison of u^δ with the solution $u \equiv 0$ of (7.2) yields $u^\delta > 0$ in Ω_δ. Thus, the $\{u^\delta\}$ are bounded above and below in any fixed Ω_δ, and the general *a priori* derivative bounds (cf. [16, 113]) imply the convergence of a subsequence to a solution $u(x)$ of (7.2) in the wedge domain Ω.

As in the proof of Theorem 7.9, we may now obtain the absolute integrability of $\lambda'(u^\delta)$, independent of δ, and the proof that the limit function $u(x)$ is variational follows as in the proof of that theorem. As in Theorem 7.9, we obtain also the uniqueness of the solution $u(x)$.

7.11. Application: A General Gradient Bound

Consider a "moon domain" Ω^0 as indicated in Fig. 7.1; here $r_0 = 1/2$, R_0 is the unique positive root of

$$R\sqrt{1-R}(\pi\sqrt{1-R}-\sqrt{1+R})-(2R^2-2R+1)\sin^{-1}R=0. \quad (7.28)$$

The choice R_0 expresses the necessary condition (corresponding to $\Omega^* = \Omega^0$ in (7.4), (7.5)) that a surface $u(x,y)$ of mean curvature $H \equiv 1$ can be defined in Ω^0, with boundary data $\gamma = 0$ ($\beta = 1$) on Σ^0, $\gamma = \pi$ ($\beta = -1$) on Γ^0. The choice (7.28) is equivalent to the relation $2\Omega^0 = \Sigma^0 - \Gamma^0$, obtained by integration of the equation

$$\operatorname{div} \mathsf{T} u = 2 \quad (7.29)$$

over Ω^0, under the given boundary conditions.

We have

$$R_0 = 0.5654062332 \ldots. \quad (7.30)$$

A solution with the indicated properties does in fact exist in Ω^0. We may see that by observing that the extremal curves (cf. Chapter 6) for the functionals Φ and Ψ of (7.4) and (7.5) are circular arcs of radius 1/2, that must meet Γ^0 in an angle π, or Σ^0 in an angle zero (measured on the side of Γ^0 opposite to that into which the curvature vector points) or else terminate in one or both of the points $\Gamma^0 \cap \Sigma^0$. Clearly there is no such arc interior to Ω^0, hence (7.4) and (7.5) are satisfied, hence by Theorem 7.9 there is a variational solution $v(x,y)$ in Ω^0, unique up to an additive constant. Using the variational condition (7.6), we obtain easily that if the normals to Γ^0 and Σ^0 are extended slightly into Ω^0 so as to

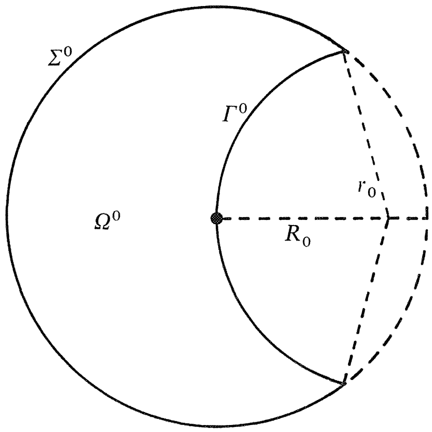


Figure 7.1. Moon domain.

permit definition of $\nu \cdot \mathrm{T} v$, then $\nu \cdot \mathrm{T} v \to 1$ a.e. on Σ^0, $\nu \cdot \mathrm{T} v \to -1$ a.e. on Γ^0.

Let $u(x, y)$ be a solution of (7.29) in a disk $B_R(0)$ of radius $R = R_0 + 2\varepsilon > R_0$, and center at the origin. Let $M = \min |\nabla v|$ in a disk $B_\varepsilon(P)$. If there were a point $p \in B_\varepsilon(0)$ at which $|\nabla u(p)| \geq M$, we could then position the surface $v(x, y)$ so that $v(p) = u(p)$, $\nabla v(p) = \nabla u(p)$, and so that $\Omega^0 \subset B_R(0)$. The reasoning we have already described in the proof of Theorem 6.24 shows that at least four components $\Omega_j \subset \Omega^0$ emanate from p, with $u - v > 0$ in Ω_{2j+1}, $u - v < 0$ in Ω_{2j}. We conclude that either there is a region with $u - v < 0$ whose boundary does not meet Σ^0, or else there is a region with $u - v > 0$ whose boundary does not meet Γ^0. In either case, Theorem 5.1 leads to a contradiction. We have proved (Finn and Giusti [69]):

Theorem 7.12. *Let $u(x, y)$ define a surface of constant mean curvature $H \equiv 1$ (solution of (7.29)) in a disk $B_R(0)$. If $R > R_0$, then there exists $C(R - R_0)$ such that $|\nabla u(0)| < C$.*

Thus the gradient at the center is bounded, depending only on the radius of the circle of definition and in no other way on the solution.

In Theorem 7.12, *the value R_0 cannot be improved; the radius $R = R_0$ cannot be achieved.* It suffices to show that solutions exist in a disk $B_{R_0}(0)$, for which the gradient becomes unbounded at points in a neighborhood of the origin. To do so we construct, essentially, the solution $v(x, y)$ of (7.29) in Ω^0, as a limit of solutions defined throughout B_{R_0}.

Let us extend the arc Σ^0 to a full circle Σ, and write $\hat{\Sigma} = \Sigma \setminus \Sigma^0$. It is clear that, regardless of the data on $\hat{\Sigma}$, there is no solution of (7.29) in B_{R_0} with data $\beta \equiv 1$ on Σ^0. For in such a case one verifies $\Phi[\Omega^0] = 0$, in conflict with the necessary condition (7.4).

We may, however, choose $\beta = 1 - \varepsilon$ on Σ^0, $0 < \varepsilon < 1$. A simple calculation then shows there is a unique $\hat{\beta}$, $-1 < \hat{\beta} < 0$, such that the data

$$\beta = \begin{cases} 1 - \varepsilon & \text{on } \Sigma^0 \\ \hat{\beta} & \text{on } \hat{\Sigma} \end{cases}$$

yield $H \equiv 1$. Corresponding to this choice, any minimizing arc Γ for Φ will be a subarc of a semicircle of radius 1/2 which, if it meets Σ^0, must do so with angle $\gamma_\varepsilon = \cos^{-1}(1 - \varepsilon)$. There is clearly no such arc, and we see that the only candidate for a minimizing configuration is the arc Γ^0 of Fig. 6.36, which yields $\Phi > 0$ when $\varepsilon > 0$. It follows that (7.4), (7.5) hold for any $\Omega^* \subset B_{R_0}$ and hence, by Theorem 7.9, there exists a variational solution in B_{R_0}, corresponding to the given data, for any $\varepsilon > 0$.

Now let $\varepsilon \to 0$. We cannot have $|\nabla u|$ bounded in ε on any subarc of Γ, as that would imply $\Phi[\Omega^0] < 0$ for small enough ε. Thus we see that if $R = R_0$, no gradient bound of the form indicated is possible.

It is shown in [69] that if $R>(1+R_0)/2$, then all solutions of (7.29) are (after a normalization) uniformly bounded above in B_R and below in B_{R^*}, with $\lim_{R\to 1} R^*=1$. As $R\to 1$, any family $v(x,y;R)$ of corresponding solutions tends to a lower hemisphere which in turn is the only solution (up to an additive constant) in the unit disk.

Notes to Chapter 7

1. With regard to the recurring use of Lemma 6.1 in the results of this chapter, cf. Note 3 of Chapter 6.

2. Theorem 7.7. Under the hypotheses of Theorem 5.1 it is not difficult to show that the functions u, v satisfy a variational relation of the form (7.25). Thus, Theorem 5.1 is subsumed by Theorem 7.7.

3. Theorem 7.12. It was shown by Bernstein [10] that there is no surface $z(x,y)$ of mean curvature $H\equiv 1$ over a domain Ω that strictly contains the unit disk $B_1(0)$. Finn [52] later showed that if $\Omega\equiv B_1$, then $z(x,y)$ describes a lower hemisphere of unit radius. Finn then conjectured a bound on gradient at the origin if $\Omega\supseteq B_R$ with $R>1/2$. The result of [69] shows that the conjecture was false in detail but correct in substance.

4. Theorem 7.12. The result has been extended by Giusti [85] to n dimensions, however an explicit estimate for R_0 is not known for $n>2$.

5. In all results of this chapter, the question of local boundary regularity has been left open. By adjoining work of Siegel [163] to that of Ural'tseva [181] and of Gerhardt [76, 77], it can be shown that in those cases for which a (local) bound on $|u|$ up to Σ can be obtained, there follows $u\in C^{2+\alpha}$ up to Σ locally in any neighborhood in which $\Sigma\in C^{2+\alpha}$.

6. Chapters 6 and 7 dealt with the nonparametric problem, for a fluid in a cylindrical container with simple projection of the surface interface onto the base. In that case the solution surface, when it exists, is uniquely determined up to an additive constant. The corresponding parametric problem, for a fluid of prescribed volume in a general closed container, was studied by Massari and Pepe [127] and by Taylor [177]. The methods of the former paper fall within the purview of those we have discussed. The result of Taylor is remarkable in that she shows an energy minimizing surface to consist of a finite number of components, with finite smooth intersection set on the boundary (which is assumed smooth). There is a conceptual analogy with the subsidiary variational problem of Chapter 6; however, in that latter problem the Lagrange parameter, rather than the volume, is prescribed. For the parametric case uniqueness fails in general.

7. Theorem 7.10 (added in proof). In view of the remark following Theorem 7.1, a variational solution exists under the hypotheses preceding Lemma 7.9 whenever $H_{-\infty} = -\infty$, $H_{\infty} = \infty$.

8. Theorem 7.12 (added in proof). Fei-Tsen Liang has characterized, for a solution of (7.29) in a disk of radius $R > R_0$, the supremum of radii of disks about the origin in which $|\nabla u|$ is *a priori* bounded.

Chapter 8

The Capillary Contact Angle

8.1. Everyday Experience

It was recognized by the initiators of capillarity theory that the underlying hypotheses leading to the notion of contact angle are satisfied only under very particular circumstances, usually related to purity and cleanliness of the materials and smoothness of the bounding surfaces. Even so, reproducible experimental determinations of contact angle are notoriously difficult to obtain. There remains considerable disagreement as to the root causes of the difficulty, but a body of literature has arisen describing a resistance to motion of the contact line (or "hysteresis") as evidenced by a discrepancy between "advancing" and "receding" contact angles. Even on this point there remains disagreement. For an expository survey, see [42]; experiments with discordant, seemingly conflicting, results are reported in [1, 45, 104, 121, 188]. We remark also the comments in [4, p. 609].

In some sense, the presence of resistance forces in capillarity is a matter of everyday experience. A drop of water on a horizontal glass plate can be distended by a thin rod into a variety of equilibrium shapes, while according to the theorem of Wente [186] the only such surface with constant contact angle is rotationally symmetric. It might be said that there are impurities on the supporting surface; if so, they would have to be distributed uniformly, as the same distension (from the symmetric configuration) can be attained in any direction. Thus the impure surface, even though it may not be glass, must be considered as a homogeneous surface to which the theory developed in Chapter 1 does not apply.

Similar comments apply to raindrops on a windowpane ([132]) or to water drops on vertical tiles in a bathroom. We show in §8.9 that in a gravity field there can be no equilibrium configuration for a drop on an inclined plane, with a constant contact angle.

8.2. The Hypothesis

It is apparent that, to obtain a theory in good agreement with reality under general conditions, some way must be found to account for resistance forces. Although several attempts have been made to provide theoretical descriptions for liquids in motion, the only such studies appropriate to equilibrium configurations seem to be those of Dussan V and Chow [44] and of Finn and Shinbrot [71, 72] with regard to drops on planar surfaces. We report here on the latter work, which is based on a phenomenological approach.

Hypothesis. *Associated with any equilibrium configuration, there is an energy φ per unit area of wetted surface, from which the areal density $\mathscr{F}$ of the resistance force is determined from the relation*

$$\mathscr{F} = -\nabla\varphi. \tag{8.1}$$

The motivation for the hypothesis is twofold. First, the introduction of the potential φ allows the problem to be studied by means of the principle of virtual work, that is, within the context of the variational procedures of Chapter 1. Second, and more specifically, φ admits a dual interpretation as a line distribution of normal force density on the contact line. We see this heuristically by integrating (8.1) over the wetted area Ω; we then find for the net resistive force

$$\int_\Omega \mathscr{F}\,dx = -\oint_\Sigma \varphi\,\nu\,ds, \tag{8.2}$$

where ν is unit exterior normal on Σ.

The relation (8.2) displays the indicated identification in an average sense; we obtain the result precisely and rigorously by making use of the first observation, and introducing φ as an energy density in addition to those already described in §1.4. The principle of virtual work then yields, in place of (1.46), the modified relation

$$\cos\gamma = \beta - \frac{1}{\sigma}\varphi \tag{8.3}$$

which can be viewed as a local (normal) force balance on Σ (see [71] for further discussion and interpretation).

We proceed to test the hypothesis in two configurations of general interest.

8.3. The Horizontal Plane; Preliminary Remarks

Consider a symmetric drop in equilibrium on a horizontal plane, as in Chapter 3. We imagine the volume $\mathscr{V}$ of the drop to be increased by the very slow addition of liquid through a small hole in the plane under the drop. If resistance to motion is present, we must expect that initially the wetted area Ω does not change. We prove below (§8.5) that in order to maintain equilibrium the contact angle γ must increase.

In accordance with (8.3), this increase can be accounted for by a potential φ, which is a function only of the radial distance r. Many choices are possible (presumably depending on the materials), as γ is determined entirely by the values of β and of φ on Σ.

8.4. Necessity for φ

It might be argued that there is formally no need to introduce φ, as the same effect could be obtained by a change in the adhesion coefficient β. We assumed initially (§1.4) that β is a constant depending only on the materials; however, since the pressure in the fluid changes with changing $\mathscr{V}$, it is conceivable that β would change with pressure. If so, β would have to be a function *only* of the pressure, as for a slow volume change the temperature would be constant, and no other parameter could affect the local adhesion between fluid and solid surface Π.

Consider a configuration in the absence of gravity, for which the fluid surface $\mathscr{S}$ is necessarily a spherical cap that meets Π in an angle γ, and suppose initially $\gamma<\pi/2$. With increasing volume, the pressure increases until $\gamma=\pi/2$ and then decreases, since the pressure change across $\mathscr{S}$ is proportional to the curvature of $\mathscr{S}$. But γ increases monotonically, hence $\cos\gamma$ decreases monotonically, with volume. Thus we see that the phenomenon cannot be described within the context of the classical theory.

8.5. Proof that γ is Monotone

Consider a liquid drop of volume $\mathscr{V}$ that wets a disk of radius a on a horizontal homogeneous plane surface Π. We introduce the dimensionless volume $\mathscr{V}_0=\mathscr{V}/a^3$ and the Bond number $B=\kappa a^2$, $\kappa=$capillarity constant (§1.9).

Theorem 8.1. *There is at most one symmetric sessile drop with given B and $\mathscr{V}_0$, for which $0<\gamma\leq\pi$. For any fixed $B>0$, there is an interval $0<\mathscr{V}_0<\overline{\mathscr{V}}_0<\infty$, in which a drop exists and in which γ increases monotonically, as a function of $\mathscr{V}_0$, from 0^+ to π. As B varies from 0 to ∞, so does $\overline{\mathscr{V}}_0$. To each sessile drop there corresponds exactly one pair of values $(B,\mathscr{V}_0)$.*

Proof. We use the result of §3.1 that each symmetric sessile drop $u(r)$ can be transformed by a uniquely determined rigid motion into a symmetric "capillary" surface $v(r)$, and that these surfaces can be parametrized in terms of the height v_0 on the axis of symmetry. The set of all vertical sections of such surfaces (taken through the symmetry axis) is a family of curves having the general appearance indicated in Fig. 8.1. The inclination angle ψ increases monotonically in arc on each curve and varies from 0 to π. It is clear that there is a biunique correspondence between $(a, \mathscr{V})$ and $(B, \mathscr{V}_0)$. The set of all symmetric sessile drops whose wetted surfaces have given radius a is obtained by cutting the curves of the figure with a vertical through $\sqrt{B}$, as indicated. The drop then appears – inverted – as the shaded area. The case shown is for $0<\gamma\leq\pi/2$. If $\gamma>\pi/2$, one continues the vertical until it cuts the appropriate curve of the figure at the second (higher) point.

By Theorem 3.1, each curve in Fig. 8.1 can be parametrized in its entirety by its inclination angle ψ, in the interval $0\leq\psi\leq\pi$. Notice that, when $r=a$, then $\sqrt{\kappa}\,r=\sqrt{B}$ and $\psi=\gamma$, the contact angle. In terms of ψ, the curves in Fig. 8.1 are determined by the differential equations

$$\frac{dr}{d\psi}=\frac{r\cos\psi}{\kappa r v-\sin\psi}, \qquad \frac{dv}{d\psi}=\frac{r\sin\psi}{\kappa r v-\sin\psi} \tag{8.4}$$

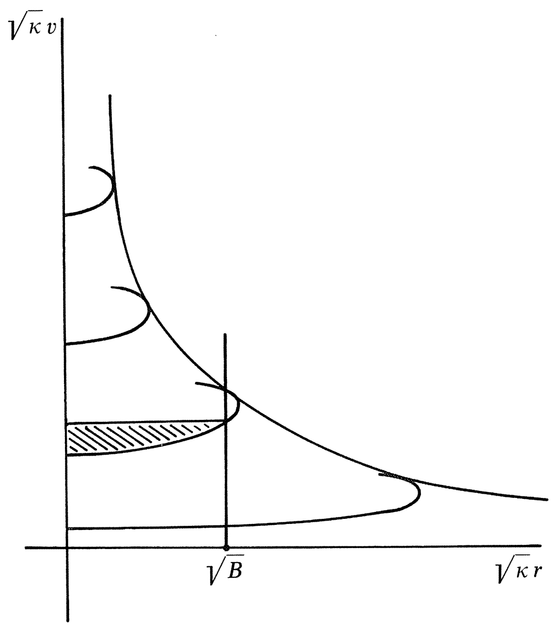


Figure 8.1. Proof of Theorem 8.1; the case $0<\gamma\leq\pi/2$.

under the initial conditions

$$r(0)=0, \qquad v(0)=v_0. \tag{8.5}$$

According to Theorem 3.2, there is exactly one symmetric sessile drop with given $\mathscr{V}_0$ and γ.

Except at the vertical points, the equation

$$(r\sin\psi)_r=\kappa r v \tag{8.6}$$

holds on each trajectory.

We consider first the case $0<\gamma\leq\pi/2$. We have from (8.6), in the interval $0<\psi\leq\gamma$,

$$r\sin\psi=\kappa\int_0^r \rho v\,d\rho, \tag{8.7}$$

and for any two solutions $v^{(1)}$, $v^{(2)}$ corresponding to initial values $v_0^{(1)}<v_0^{(2)}$

$$r(\sin\psi^{(2)}-\sin\psi^{(1)})=\kappa\int_0^r (v^{(2)}-v^{(1)})\rho\,d\rho. \tag{8.8}$$

We conclude from (8.8) that for sufficiently small r there holds $\psi^{(2)}>\psi^{(1)}$ and thus $v^{(2)}-v^{(1)}$ is, initially, increasing. Continuing the integration, we thus find $\psi^{(2)}>\psi^{(1)}$, $v^{(2)}-v^{(1)}$ increasing in r (and hence positive) until the vertical point $\psi^{(2)}=\pi/2$ is reached. The particular case $v^{(1)}\equiv 0$, $v^{(2)}\equiv v$ yields that v is also increasing in r, for any initial value $v_0>0$.

Let us now move upwards from $v=0$ along the line $r=a$ as indicated in Fig. 8.1. From Theorem 2.7, we see that for small v_0 the trajectories extend beyond $r=a$ and that on $r=a$ their inclinations tend to zero with v. Using the above remarks, we thus find that on $r=a$, $\psi=\gamma$ increases continuously from zero with v.

Since on any trajectory v increases with r, (8.7) yields $2\sin\psi>\kappa r v_0$. Thus, if $v_0\geq 2/\kappa a$, the trajectory does not extend to $r=a$. From the relation (2.45b) we obtain that in the interval $0<\psi\leq\pi/2$ on any trajectory there holds

$$v<\sqrt{\frac{4}{\kappa}+v_0^2}.$$

It follows that as we move upward on $r=a$ the value $\gamma=\pi/2$ must be attained at a height

$$v_M<\frac{2}{\sqrt{\kappa}}\sqrt{1+B}; \tag{8.9}$$

we have thus shown that the entire interval $0<\psi\leq\pi/2$ is covered monotonically as v moves from 0 to v_M.

To prove the monotonicity of $\mathscr{V}_0$, we consider again $v^{(2)}$, $v^{(1)}$. Since $v^{(2)}-v^{(1)}$ is increasing in r, the function $q(r)=v^{(1)}(r)+v_0^{(2)}-v_0^{(1)}$ satisfies

$$\begin{aligned} q(0) &= v_0^{(2)} \\ q(r) &< v^{(2)}(r) \end{aligned} \tag{8.10}$$

whenever $v^{(1)}$, $v^{(2)}$ are defined in $0<r<a$. Thus, the upper curve defines the greater volume, as was to be shown.

It remains to study the interval $\pi/2<\gamma<\pi$. To do so, we come downwards on $r=a$ from v_M, focusing attention now on the upper intersection point with each curve of the family, until $\psi=\pi$ at the point of intersection. From the relations (2.45a) and (3.82) we see that the entire vertical segment must lie above the height $v_m>2/\kappa a$.

We now have recourse to the inequalities

$$\frac{\partial r}{\partial v_0}<0, \qquad \frac{\partial \mathscr{V}_0}{\partial v_0}<0 \tag{8.11}$$

for any fixed ψ in $0<\psi\leq\pi$, proved in §3.3.

Consider again two curves

$$v=v^{(i)}(\psi; v_0^i); \qquad i=1,2; \qquad v_0^{(1)}<v_0^{(2)}$$

at the value $\psi=\gamma$, $\pi/2\leq\gamma\leq\pi$, see Fig. 8.2. By (8.11) we have $r^1>r^2$, also $\mathscr{V}_0^1>\mathscr{V}_0^2$. Choosing $r^1=a$, we see that if the curve v^2 is continued on its

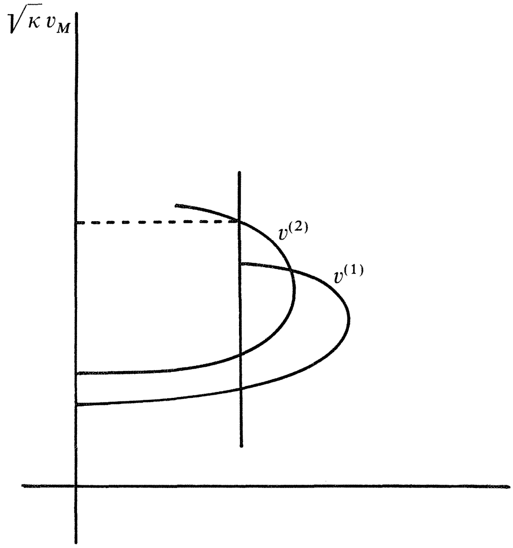


Figure 8.2. Proof of Theorem 8.1; the case $\pi/2<\gamma\leq\pi$.

upper branch only back to the value a, we will have $\psi^2<\gamma=\psi^1$, $\mathscr{V}_0^2(\psi^2)<\mathscr{V}_0^2(\gamma)<\mathscr{V}_0^1(\gamma)$ as above. The proof is complete.

8.6. Geometrically Imposed Stability Bounds

An examination of the above proof shows that *in every situation the addition of sufficient volume of liquid will increase γ beyond π.* This is however physically not possible, as the fluid would then have to penetrate the supporting plane. When γ exceeds π, a continued equilibrium configuration with the same Ω becomes impossible. Regardless of the resistive force, any further addition of fluid must result in an increase in the wetted area. We present here explicit estimates for the critical volume. The considerations are entirely geometrical and require only a knowledge of the dependence of γ on B and on $\mathscr{V}_0$.

We observe first, from the formula (3.9) for the volume of a drop,

$$\frac{1}{\pi}\mathscr{V}_0(\gamma)=\frac{1}{a}v(\gamma)-\frac{2}{B}\sin\gamma \tag{8.12}$$

in our notation. Also, denoting by (R, v_R) the coordinates of the vertical point on a solution curve, the relation (3.57c) yields

$$v(\psi)<\frac{1}{\kappa R}+\sqrt{\frac{2}{\kappa}(1-\cos\psi)+\frac{1}{\kappa^2R^2}}, \qquad \frac{\pi}{2}\le\psi\le\pi. \tag{8.13}$$

On the other hand, we have from (3.23b)

$$3\mathscr{V}_0<4\pi\left(\frac{R}{a}\right)^3 \tag{8.14}$$

and hence, from (8.13)

$$B\frac{v(\pi)}{a}<\left(\frac{4\pi}{3\mathscr{V}_0}\right)^{1/3}+\sqrt{\left(\frac{4\pi}{3\mathscr{V}_0}\right)^{2/3}+4B}$$

so that (8.12) now gives

$$\tfrac{4}{3}BR_0^4<1+\sqrt{1+4BR_0^2} \tag{8.15}$$

where R_0 is defined by $3\mathscr{V}_0=4\pi R_0^3$.

We have also from (8.12), when $\psi=\pi$,

$$\frac{4}{3}R_0^3=\frac{1}{a}v(\pi)>\frac{1}{a}v_R>\frac{2^{2/3}}{BR_0}$$

by (3.24). Hence, $\frac{4}{3}BR_0^4 > 1$ and we obtain from (8.15)

$$B < \frac{3}{2R_0^4}\left(1 + \frac{3}{2R_0^2}\right), \tag{8.16}$$

which gives a bound for the Bond number in terms of volume. When (8.16) is violated, $\gamma > \pi$ and the drop cannot exist without penetrating the plane Π. Thus, the wetted surface expands.

The interpretation of (8.16) is facilitated by introducing the dimensionless measure of volume $\mathscr{B} = \kappa \mathscr{R}^2$ with $\mathscr{R}$ defined by $3\mathscr{V} = 4\pi\mathscr{R}^3$. Equation (8.16) then takes the form

$$\mathscr{B}^3 - \tfrac{3}{2}\mathscr{B}B - \tfrac{9}{4}B^2 \le 0 \tag{8.17}$$

which displays an absolute upper bound for the volume in terms of the wetted area.

The estimate (8.17) is asymptotically exact both for large and small B; we may obtain however an improved version for larger B by noting in (8.13) that $R > a$. We then obtain

$$\mathscr{B}^{3/2} < \tfrac{3}{4}\sqrt{B}\,(1 + \sqrt{1 + 4B}). \tag{8.18}$$

Both (8.17) and (8.18) are correct in all cases. If $B = 15/4$, then (8.17) and (8.18) both yield the same $\mathscr{B} = 15/4$; if $B < 15/4$, then (8.17) yields the better estimate, otherwise (8.18) is preferable. Still more precise estimates, and also lower bounds for the critical $\mathscr{B}$, can be obtained by analogous reasoning using the appropriate estimates of Chapter 3.

8.7. A Further Kind of Instability

We consider the case of a drop of small volume, wetting a small disk on Ω. In this case, resistive forces can be expected to be small relative to surface tension and adhesion terms, so that the geometry should, essentially, be determined by the classical Young-Gauss theory (Chapter 1). We recall the asymptotic relation (3.79 b)

$$B \sim \tfrac{2}{3}\mathscr{B}^2 \tag{8.19}$$

in the limit as $\mathscr{B} \to 0$, when $\gamma = \pi$. This shows that the radius of the wetted disk vanishes as the square of the radius of the ball of equivalent volume. Thus, *for small configurations when γ is close to π the drop rests, essentially, on a point, about which small disturbances could cause it to pivot and to move with a kind of rolling motion.*

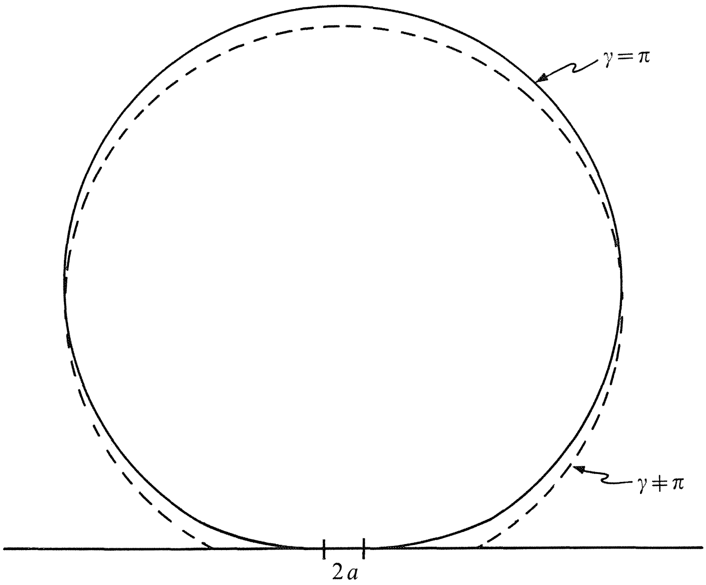


Figure 8.3. Nonuniformity of wetted area.

If $\gamma < \pi$, then the estimate (3.79a) governs the geometry. The two radii are then proportional, and a more stable configuration can be expected.

The indicated behavior is illustrated in Fig. 8.3. See also §3.6, Fig. 3.9 for quantitative relations.

8.8. The Inclined Plane; Preliminary Remarks

If the plane Π of the preceding sections is slowly tilted relative to the gravity field, then a rotationally symmetric drop can no longer be expected, although we may expect that resistive forces will, at least initially, suffice to maintain the drop in equilibrium with the wetted area a disk. In the following section we show that there can be no equilibrium drop for which the contact angle γ is constant on the triple interface Σ; that is in fact the case regardless of the shape of the wetted disk. Thus, according to (8.3), φ cannot be constant on Σ, and our underlying hypothesis will be subjected to a different kind of test than was imposed in the above sections.

It will be most convenient for us to carry out the procedure somewhat differently than was just indicated: we assume a drop initially on an inclined plane in the absence of gravity and in spherical configuration, and we then allow gravity to increase. We attempt (§8.11) to characterize a form for φ that will be asymptotically correct for small gravity, and we will verify the result by a formal asymptotic solution of the governing equation. This latter step will for simplicity be carried out here only in the special case of a vertical plane; the case of general inclination is treated in [72].

8.9. Integral Relations, and Impossibility of Constant Contact Angle

We consider a drop with free surface $\mathscr{S}$ resting on a plane Π whose normal is inclined at angle ψ to the vertical. Let Ω denote the wetted area on Π, $\Sigma=\partial\Omega$ the triple interface (contact line). We introduce a coordinate system such that Π is the xy-plane and the x-axis is horizontal. We may choose the origin of coordinates so that

$$\int_\Omega y\,dx=0. \tag{8.20}$$

The potential Υ of §1.4 can be written

$$\Upsilon=g(z\cos\psi+y\sin\psi), \tag{8.21}$$

and thus the underlying equation (1.44) takes the form

$$2H=\kappa(z\cos\psi+y\sin\psi)-\lambda. \tag{8.22}$$

Let $\mathbf{x}$ denote the position vector of $\mathscr{S}$, $\mathbf{N}$ the unit normal, as in Chapter 1. If we integrate the relation (1.34)

$$\Delta\mathbf{x}=2H\mathbf{N} \tag{8.23}$$

over $\mathscr{S}$, we obtain

$$\int_{\mathscr{S}}\mathbf{H}\,d\mathscr{S}=\tfrac{1}{2}\oint_\Sigma \mathbf{n}\,ds; \tag{8.24}$$

here $\mathbf{H}$ is the mean curvature vector, $\mathbf{n}$ a unit vector directed out of $\mathscr{S}$ on Σ and orthogonal to $\mathbf{N}$ and to Σ.

The relation (8.22) gives the value of the (scalar) mean curvature H on $\mathscr{S}$. Let us extend H by the formula (8.22) to a function defined throughout $\mathbb{R}^3$. Letting $(\mathbf{i},\mathbf{j},\mathbf{k})$ be unit vectors in the coordinate directions, we may write

$$\int_{\mathscr{S}}\mathbf{H}\,d\mathscr{S}=\int_{\mathscr{S}}H\mathbf{N}\,d\mathscr{S}=\int_{\mathscr{S}\cup\Omega}H\mathbf{N}\,d\mathscr{S}-\tfrac{1}{2}\lambda\Omega\mathbf{k} \tag{8.25}$$

by (8.20) and (8.22), since $z=0$ on Π. By the divergence theorem we obtain

$$\int_{\mathscr{S}\cup\Omega}H\mathbf{N}\,d\mathscr{S}=\int_{\mathscr{V}}\nabla H\,d\mathscr{V}=\frac{\kappa}{2}\mathscr{V}(\mathbf{j}\sin\psi+\mathbf{k}\cos\psi). \tag{8.26}$$

Now let $\boldsymbol{\nu}$ denote unit exterior normal to Σ in Π; we have

$$\mathbf{n} = (\mathbf{n}\cdot\boldsymbol{\nu})\,\boldsymbol{\nu} + (\mathbf{n}\cdot\mathbf{k})\,\mathbf{k} = \boldsymbol{\nu}\cos\gamma - \mathbf{k}\sin\gamma$$

and thus

$$\oint_\Sigma \mathbf{n}\,ds = \oint_\Sigma \boldsymbol{\nu}\cos\gamma\,ds - \mathbf{k}\oint_\Sigma \sin\gamma\,ds. \tag{8.27}$$

We may thus equate the **i**, **j**, **k** components in (8.24) to obtain

Theorem 8.2. *For a drop resting on a plane inclined at angle ψ to a gravitational field as above, there holds for the contact angle distribution γ*

$$\oint_\Sigma (\boldsymbol{\nu}\cdot\mathbf{i})\cos\gamma\,ds = 0$$

$$\oint_\Sigma (\boldsymbol{\nu}\cdot\mathbf{j})\cos\gamma\,ds = \kappa\mathscr{V}\sin\psi \tag{8.28a, b, c}$$

$$\oint_\Sigma \sin\gamma\,ds = -\kappa\mathscr{V}\cos\psi + \lambda\Omega.$$

Suppose the contact angle γ were constant. Combining (8.28a, b), we obtain

$$(\cos\gamma)\oint_\Sigma \boldsymbol{\nu}\,ds = (\kappa\mathscr{V}\sin\psi)\,\mathbf{j} = 0$$

since

$$\oint_\Sigma \boldsymbol{\nu}\,ds = \int_\Omega \nabla(1)\,d\Omega = 0,$$

and thus $\sin\psi = 0$. We have proved:

Theorem 8.3. *If a drop makes constant contact angle γ with a plane Π inclined at angle ψ to a uniform gravity field, then $\psi = 0$ or π.*

That is, the drop must either be a sessile drop as in Chapter 3 or a pendent drop as in Chapter 4. We observe that Theorem 8.3 holds regardless of the shape of the wetted area Ω.

8.10. The Zero-Gravity Solution

We introduce dimensionless variables, replacing x, y, z by ax, ay, az, a being the radius of wetted disk. Let D_0 be the wetted area in the new coordinates, and $\mathscr{V}_0 = \mathscr{V}/a^3$ the dimensionless volume. We introduce the

Bond number

$$B=\kappa a^2. \tag{8.29}$$

Then (8.22) becomes, with $\mu=a\lambda$,

$$2H_0\equiv 2aH=B(z\cos\psi+y\sin\psi)-\mu. \tag{8.30}$$

When $B=0$, a particular solution is a spherical cap wetting a unit disk, and according to Wente's theorem [186] that is the only solution. We calculate

$$V_0=\frac{\pi(2+\cos\gamma_0)(1-\cos\gamma_0)^2}{3\sin^3\gamma_0}, \tag{8.31}$$

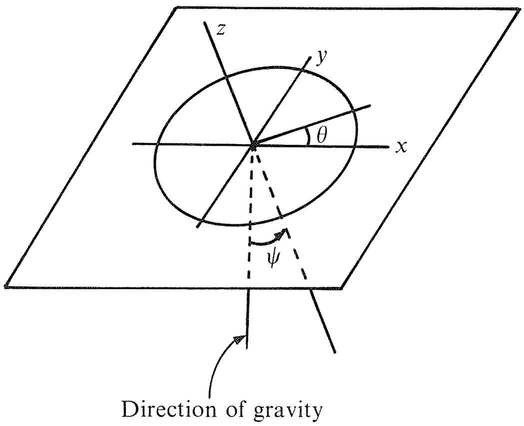
which determines γ_0 and thus the corresponding φ_0 from (8.3). We may assume that this initial solution is the "classical" one, so that $\varphi_0=0$, $\beta=\cos\gamma_0$. Thus, if we allow B to increase, we will have

$$\cos\gamma=\cos\gamma_0-\frac{1}{\sigma}\varphi. \tag{8.32}$$

8.11. Postulated Form for φ

Symmetry considerations suggest that we seek φ as a function only of $y=\sin\psi\sin\theta$, where $\theta=$azimuthal angle (see Fig. 8.4). The first three terms of a formal Taylor expansion thus yield

$$-\frac{1}{\sigma}\varphi=\varepsilon(B;\gamma_0;\psi)+\alpha(B;\gamma_0)\sin\psi\sin\theta+\delta_0(B;\gamma_0)\sin^2\psi\cos 2\theta. \tag{8.33}$$



F gure 8.4. The coordinate system.

The second term in this representation can be determined *a priori* by inserting (8.33) into (8.32) and (8.32) into (8.28b). We obtain

$$\alpha = \frac{1}{\pi} B \mathscr{V}_0 \tag{8.34}$$

and hence, using (8.32),

$$\begin{aligned} \cos\gamma = \cos\gamma_0 + \varepsilon(B;\gamma_0;\psi) + \frac{1}{\pi} B \mathscr{V}_0 \sin\psi \sin\theta \\ + \delta_0(B;\gamma_0)\sin^2\psi\cos 2\theta. \end{aligned} \tag{8.35}$$

This is our projected form for the boundary angle distribution. Although the constants ε and δ_0 are still undetermined, they can be found for any particular configuration from two particular calculations (or laboratory measurements). In general, ε and δ_0 can be expected to be small relative to α_0; the correction is however significant for the determination of geometrically imposed instability (§§8.14, 8.15), as this can occur at (relatively) large values of B.

We introduce at this point a heuristic observation on the structure of $\varepsilon(B;\gamma_0;\psi)$. Expanding in powers of B, we find

$$\varepsilon = \varepsilon_0(\gamma_0;\psi) + \varepsilon_1(\gamma_0;\psi)B + \varepsilon_2(\gamma_0;\psi)B^2 + \cdots. \tag{8.36}$$

We note by (8.32) that $\varepsilon_0 = 0$. Further, if $\psi = \pi/2$ there is no component of gravitational force orthogonal to Π, and thus one expects that to first order in B the mean value of $\cos\gamma$ on Σ will remain $\cos\gamma_0$. Thus, it seems reasonable to expect a representation in the form

$$\varepsilon = B\varepsilon_1(\gamma_0)\cos\psi + B^2\varepsilon_2(\gamma_0;\psi). \tag{8.37}$$

In [72] the equation (8.22) is solved by formal asymptotic expansion in B, to order B^2. The results verify (8.35) with ε in the form (8.37). In the following section we outline the procedure in the particular case of the vertical plane, $\psi = \pi/2$.

8.12. Formal Analytical Solution

We suppose $0 < \gamma < \pi/2$ and seek a solution of (8.22) as a graph $u(x,y)$ over Π for small B, with $\psi = \pi/2$. The relation (8.22) then takes the form

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = By - \mu. \tag{8.38}$$

We seek a solution of (8.38) in the unit disk D_0, under the condition

$$u = 0 \quad \text{on } \partial D_0 \tag{8.39}$$

and the constraint

$$\int_{D_0} u\,dx = V_0. \tag{8.40}$$

When $B=0$, the explicit solution is

$$u_0 = \frac{-\cos\gamma_0 + \sqrt{1-r^2\sin^2\gamma_0}}{\sin\gamma_0} \tag{8.41}$$

with

$$\mu_0 = 2\sin\gamma_0 \tag{8.42}$$

and γ_0 the (unique) solution of (8.31). We choose u_0 as starting point to construct a perturbation for small B.

8.13. The Expansion; Leading Terms

We write $u=u_0+Bu_1+B^2u_2$, $\mu=\mu_0+B\mu_1+B^2\mu_2$ in (8.38) and neglect all higher order terms. Setting

$$\mathsf{T}_0 v \equiv \frac{\nabla v}{(1+|\nabla u_0|^2)^{1/2}} - \frac{\nabla u_0 \cdot \nabla v}{(1+|\nabla u_0|^2)^{3/2}} \nabla u_0 \tag{8.43}$$

$$R_2 = \nabla \cdot \left[\frac{2(\nabla u_0 \cdot \nabla u_1)\nabla u_1 + |\nabla u_0|^2 \nabla u_0}{2(1+|\nabla u_0|^2)^{3/2}} - \frac{3(\nabla u_0 \cdot \nabla u_1)^2 \nabla u_0}{2(1+|\nabla u_0|^2)^{5/2}} \right] \tag{8.44}$$

we obtain the two problems

$$\begin{aligned} \nabla \cdot \mathsf{T}_0 u_1 &= y - \mu_1 && \text{in } D_0 \\ u_1 &= 0 && \text{on } \partial D_0 \\ \int_{D_0} u_1\,dx &= 0 \end{aligned} \tag{8.45a, b, c}$$

$$\begin{aligned} \nabla \cdot \mathsf{T}_0 u_2 &= R_2 - \mu_2 && \text{in } D_0 \\ u_2 &= 0 && \text{on } \partial D_0 \\ \int_{D_0} u_2\,dx &= 0 \end{aligned} \tag{8.46a, b, c}$$

for (u_1, μ_1), (u_2, μ_2).

To solve (8.45) we observe that the right side of (8.45a) has the form $r\sin\theta - \mu_1$, and we seek a solution in the form of a Fourier series $u_1(r,\theta) = \Sigma[v_k(r)\cos k\theta + w_k(r)\sin k\theta]$. Substituting into (8.45), we find that v_k

satisfies

$$r(1+u_0'^2)\frac{d}{dr}\left[\frac{rv'}{(1+u_0'^2)^{3/2}}\right]=k^2 v, \qquad 0<r<1, \tag{8.47}$$

$$\limsup_{r\to 0}|v(r)|<\infty, \qquad v(1)=0$$

if $k\geq 1$, and

$$\frac{1}{r}\frac{d}{dr}\left[\frac{rv'}{(1+u_0'^2)^{3/2}}\right]=-\mu_1, \qquad 0<r<1$$

$$\limsup_{r\to 0}|v(r)|<\infty, \qquad v(1)=0 \qquad \text{(8.48a, b, c)}$$

$$\int_0^1 r\,v(r)\,dr=0$$

when $k=0$.

The relations (8.47) *admit only the trivial solution when* $k>0$. In fact, since (8.47a) has a regular singular point at $r=0$, any solution can be represented near $r=0$ in the form $v=\sum_0^\infty a_n r^{n+\alpha}$, and the indicial equation yields $\alpha=\pm k$. The root $-k$ is excluded by (8.47b), and thus $v=O(r^k)$ at $r=0$.

We now multiply (8.47a) by $v/r\sqrt{1+u_0'^2}$ and integrate, obtaining

$$-\int_0^1 \frac{rv'^2}{(1+u_0'^2)^{3/2}}\,dr=k^2\int_0^1 \frac{v^2}{r(1+u_0'^2)^{1/2}}\,dr,$$

from which we conclude $v\equiv 0$ on $(0,1]$.

The relations (8.48) can be solved explicitly, yielding again $v\equiv 0$.

In an analogous way, we find $w_k\equiv 0$ when $k\neq 1$, while setting $t=r\sin\gamma_0$ we obtain

$$w_1(r)=\frac{1}{6\sin^3\gamma_0}\cdot\left\{\frac{2-t^2}{t\sqrt{1-t^2}}-\frac{1}{t}-\frac{t}{\sqrt{1-t^2}}\left(\frac{1-\cos\gamma_0}{1+\cos\gamma_0}+\log\frac{1+\sqrt{1-t^2}}{1+\cos\gamma_0}\right)\right\}. \tag{8.49}$$

Finally, (8.46) can be solved, again by the same procedure; the calculations are elementary but lengthy, see [72]. They lead to a formal expression for $u(r;\theta)$ to order B^2, from which the contact angle γ can be obtained, again to order B^2, from the relation $\tan\gamma(\theta)=-u'(1;\theta)$. We state here only the result (for $\psi=\pi/2$):

$$\cos\gamma\cong\cos\gamma_0+\left(\frac{V_0}{\pi}\sin\theta\right)B+(l\cos 2\theta-m)B^2 \tag{8.50}$$

in formal agreement with the postulated form (8.35). Here

$$l=\cos^2\gamma_0\left[c\sin\gamma_0+w'^2(1)\frac{3\cos^2\gamma_0-2}{4}\right] \tag{8.51}$$

with

$$c=\frac{1}{\cos^2\gamma_0(1-\cos\gamma_0)^2(2+\cos\gamma_0)}\int_0^{\sin\gamma_0}\frac{C(t)}{t}\left[\frac{2}{\sqrt{1-t^2}}-(2+t^2)\right]dt \tag{8.52}$$

and

$$C(t)=\sin^3\gamma_0\left\{\frac{3}{4t}\frac{d}{dt}\left[\frac{t^2(1-t^2)^2}{\sin^2\gamma_0}w_1'^2(t)-(1-t^2)w_1^2(t)\right]-w_1^2(t)\right\}; \tag{8.53}$$

$$m=\cos^2\gamma_0\left[-a\sin\gamma_0+w'^2(1)\frac{3\cos^2\gamma_0-2}{4}\cos\gamma_0\right] \tag{8.54}$$

with

$$a=\frac{1}{\sin^2\gamma_0\cos^2\gamma_0(1-\cos\gamma_0)}\int_0^{\sin\gamma_0}tA(t)\left[\frac{1+\cos\gamma_0}{\sqrt{1-t^2}}-2\right]dt \tag{8.55}$$

and

$$A(t)=-\frac{3}{4t}\sin^3\gamma_0\frac{d}{dt}\left[\frac{t^2(1-t^2)^2}{\sin^2\gamma_0}w_1'^2(t)+\frac{1-t^2}{3}w_1^2(t)\right]. \tag{8.56}$$

8.14. Computer Calculations

The above section completes the formal analysis: the postulated representation for the resistance forces led to the representation (8.35) for the contact angle distribution, to order B^2. A formal asymptotic integration of the strict equations for the surface interface led independently to the same representation, and additionally to the determination of the coefficients that appear. Presumably the results should be asymptotically correct for small B, however that has not yet been proved. Nevertheless, a strict solution of the basic equations (8.38)–(8.40) does exist for small B, as follows from a result of Wente [185]. In order to obtain direct comparison with such a solution, the equations were integrated numerically by Milinazzo [131]. In Figs. 8.5(a)–(c) are shown comparisons between the computer calculations (shown as individual points) and the predictions from (8.50) (shown as curves) for the cases $\gamma_0=30°$, $75°$, $105°$, with $\psi=\pi/2$, for varying values of B.

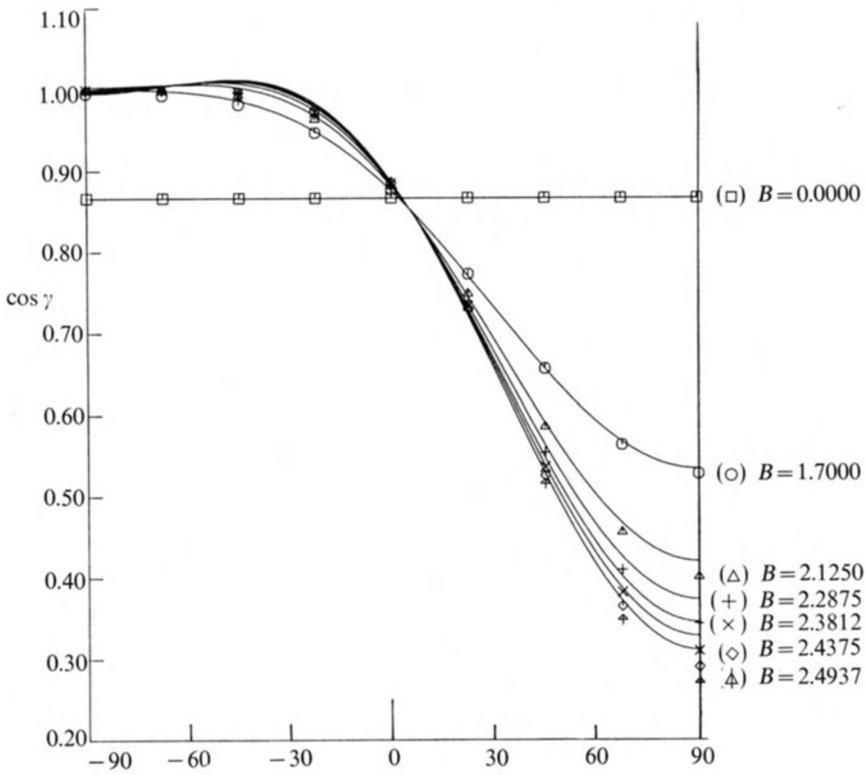


Figure 8.5 (a). Contact angle distributions. $\gamma_0 = 30°$.

8.15. Discussion

We note that when $\gamma_0 = 30°$, some of the (large B) curves yield values $\cos\gamma > 1$ for values of θ near $\pi/2$. Thus, the representation (8.50) is not valid in this range, and the validity of these curves in the remaining θ interval must also be held in some question. The value $B_M = \max\{B: \max_\theta|\cos\gamma| \leq 1\}$ can be taken as an upper stability bound for the given γ_0, since for $B > B_M$ the formal solution would presumably penetrate the supporting plane, see the following section.

Note that in the figures for $\gamma_0 = 75°$ and $105°$ the range of B shown for the computer curves is not sufficient to achieve the value $|\cos\gamma| = 1$. That is because the computer calculations ceased to converge for values of B significantly larger than those shown. Two possibilities suggest themselves to account for the difficulty. One is that the chosen coordinate system — spherical coordinates based at the center of the wetted disk — is not suited to the configurations that appear. If the drop profile deve-

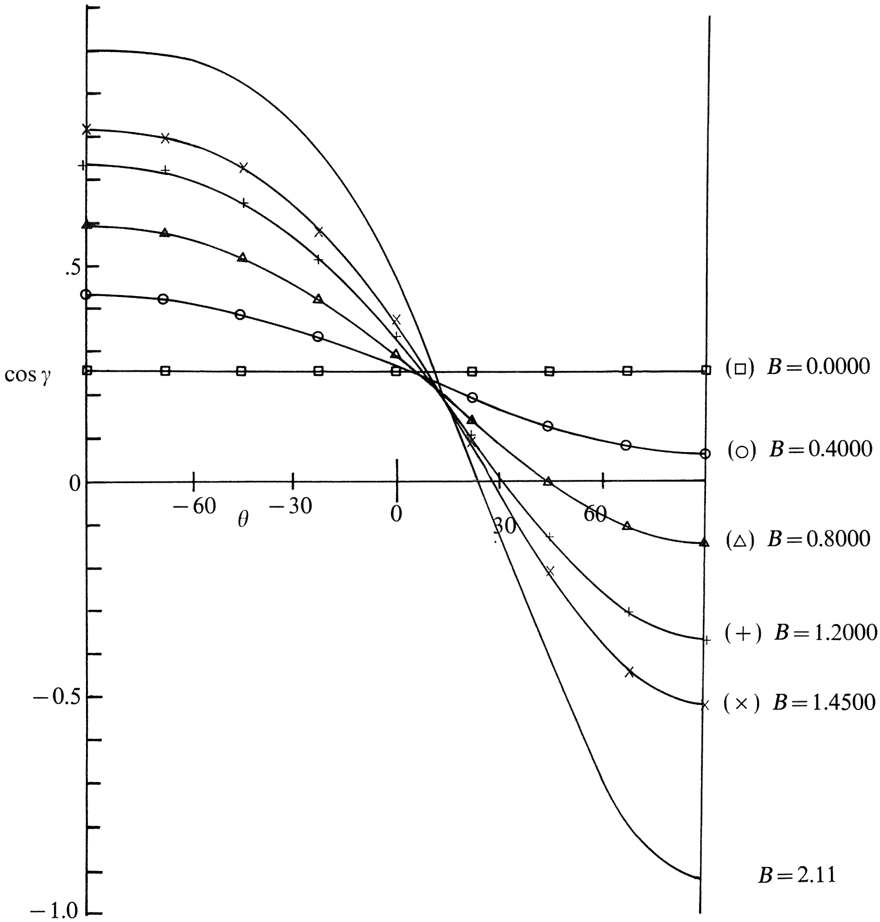


Figure 8.5(b). Contact angle distributions. $\gamma_0=75°$.

lops an overhang, the representation could fail at points of the surface interface. Computer calculations of the drop profile show however no evidence of such overhang.

It is conceivable that a bifurcation is encountered. This possibility is suggested by the existence of multiple pendent drop solutions with identical volumes and contact angles; these solutions nevertheless do not themselves bifurcate. In this connection see Note 7 to Chapter 4.

8.16. Further Discussion

It is natural—especially in view of the stability bound introduced above —to look for extrema of $\cos\gamma$ on the contact circumference. From the

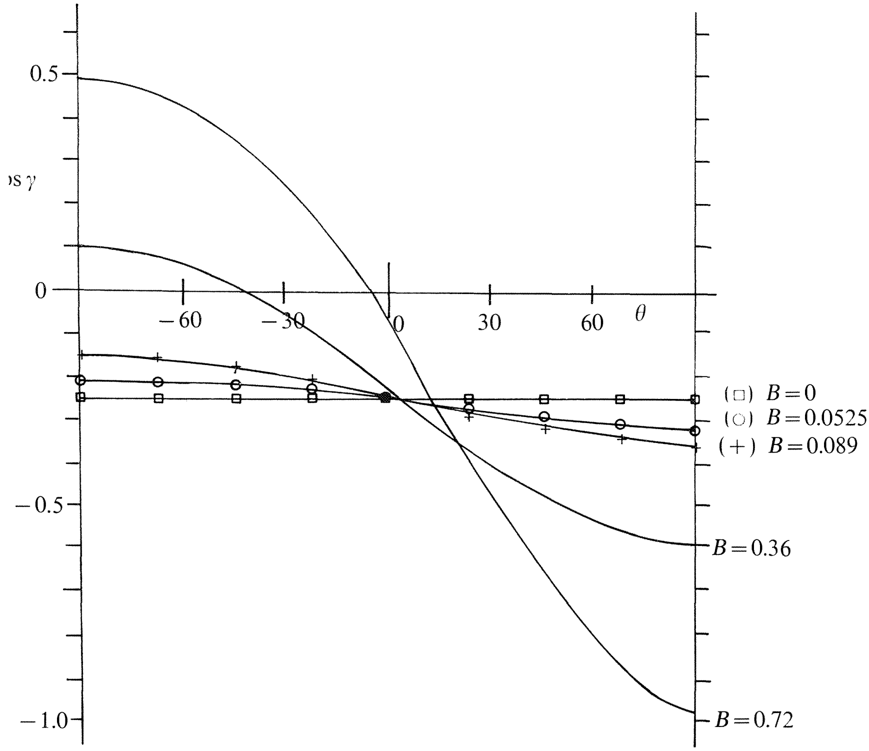


Figure 8.5 (c). Contact angle distributions. $\gamma_0 = 105°$.

relation (8.50) we find that if $B < |n/4l|$, then there is a single maximum at $\pi/2$ and a single minimum at $-\pi/2$; however, if $B > |n/4l|$, then $\pi/2$ becomes a minimum, $-\pi/2$ a maximum, and new extrema appear where $|\sin\theta| = |n/4lB|$. Let B_1 and B_{-1} be the values of B for which respectively $\cos\gamma(\pi/2) = 1$, $\cos\gamma(-\pi/2) = -1$. We have from (8.50)

$$B_1 = \frac{n - \sqrt{n^2 - 4(l+m)(1-\cos\gamma_0)}}{2(l+m)}$$
$$B_{-1} = \frac{-n + \sqrt{n^2 + 4(l+m)(1+\cos\gamma_0)}}{2(l+m)}. \tag{8.57}$$

Table 8.1 shows the calculated values of B_1, B_{-1} for varying γ_0, and also the corresponding values of $|n/4l|$. It is seen that in every case the value $|\cos\gamma| = 1$ is attained at $\pm\pi/2$, for a value $B < |n/4l|$. It thus appears that in the range of physical interest ($|\cos\gamma| \le 1$) the only maxima and minima for $\cos\gamma$ occur at $\theta = \pm\pi/2$.

Figure 8.6 shows the stability bound $B_M = \min\{B_1, B_{-1}\}$ as function

Table 8.1. Stability Bounds for B

γ_0	B_1	B_{-1}	$\lvert n/4l \rvert$
15	0.955	6.87	1.012
30	1.684	5.65	1.826
45	2.131	4.405	2.308
60	2.271	3.233	2.423
75	2.116	2.201	2.237
90	1.725	1.360	1.886
105	1.206	0.736	1.540
120	0.690	0.3284	1.564
135	0.295	0.1068	0.971
150	0.0760	0.0191	0.0407
165	0.0060	0.0008	0.0010

of γ_0, with $\psi=\pi/2$. The result is compared with that of the computer calculations, to the extent comparison is possible, see the remarks in §8.14. The dotted portion of the computer curve represents the upper limit of attainable B, when the value $|\cos\gamma|=1$ could not be achieved.

We note that for γ_0 below a critical value (slightly over 75°) the drop becomes parallel to the support plane ($\gamma=0$) at the top ($\theta=\pi/2$) before

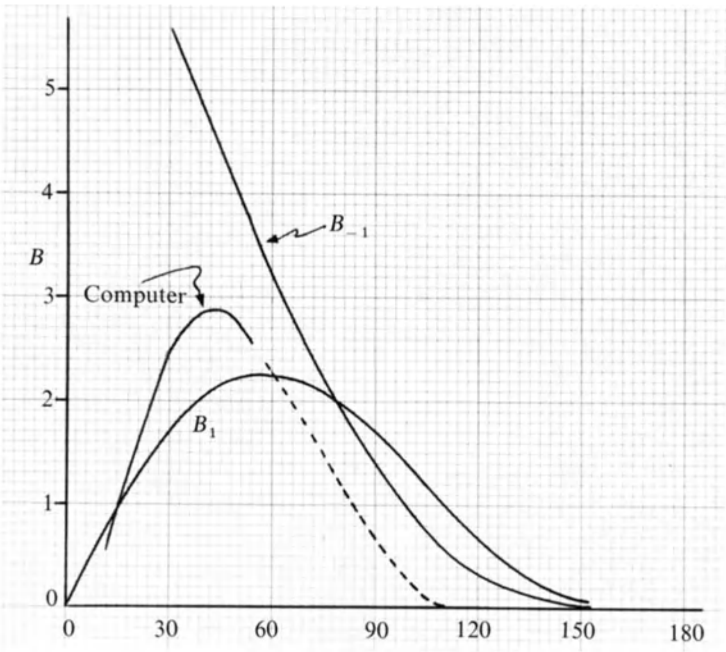


Figure 8.6. Stability bounds.

that occurs (with $\gamma=\pi$) at the bottom ($\theta=-\pi/2$). For larger γ_0 the stability bound is achieved first at the bottom. The reason B_1 tends to zero with increasing γ_0 derives from the chosen normalization: the Bond number B is based on the radius of the wetted disk, and thus the drop volume for given B becomes infinite as $\gamma_0 \to \pi$.

Notes to Chapter 8

1. §8.2. If the support surface Π is curved, then the identity giving rise to (8.2) must be changed to

$$\int_\Omega \nabla\varphi \, d\Omega + 2\int_\Omega \varphi \mathbf{H}_\Pi \, d\Omega = \oint_\Sigma \varphi \nu \, ds.$$

In this case we observe that a constant areal distribution φ, which yields a vanishing gradient in Π, can nevertheless provide a non-null net force on the drop. This is analogous to the ability of a surface tension, which acts within a curved surface, to account for pressure changes across the surface.

2. §8.11. It should be noted at this point that the same formal results could have been obtained by assuming φ to be a function only of hydrostatic pressure, which in turn depends only on y. However, it was shown in §8.4 that φ cannot in general be determined by pressure alone. In this respect an interesting further example might be a drop in a wedge formed by a horizontal and a vertical wall. In such a case, the pressure would be constant on the horizontal contact line; the distribution of contact angle on that line could shed some light as to the factors determining φ.

The expression (8.35) could also have been written without introducing φ, simply by expanding $\beta=\cos\gamma$ in a formal Fourier series, subject to symmetry considerations. The interpretation that has been made is however suggestive, particularly in view of the formal relation (8.2). Its scientific correctness and (or) usefulness will have to be determined by further study.

3. §8.13. A calculation of equilibrium configurations for a drop on an inclined plane appears also in Brown, Orr, and Scriven [20]. This paper suffers from a number of errors and ambiguities.

4. The actual behavior of fluid along a triple interface can be much more complicated than is envisaged in any of the current theories. Hardy [95] observed that for drops of liquids with large vapor pressure on a steel plate, a microscopically thin film of the liquid surrounds the drop. Later investigators [7, 79, 153] have confirmed this "precursor film" in varying situations. Hardy wrote "There can be no manner of doubt that primary spreading on solid surfaces occurs through the intervention of the vapor." However, Ghiradella, Radigan, and Frisch [79] in an ingenious experiment found that for a contact line advancing with speed v, the film width increased with v, thus suggesting that other factors must be taken into account.

Although these authors were able to determine a nonzero limiting film width as $v \to 0$, the corresponding measurements for $v = 0$ were inconclusive (H.L. Frisch, oral communication).

Chapter 9

Identities and Isoperimetric Relations

A number of integral identities for capillary surfaces have already been employed in the text, e.g., the Laplace volume formula in §1.9, and the relations of §8.9. Further integral relations involving mean curvature on surfaces can be found, e.g., in Minkowski [133] and Hsiung [102]. The following result of [30] seems not to be generally known. Let V denote a domain in 3-space (or its volume), let $\partial V = S$ have mean curvature H^S. If S is star with respect to the origin, we may extend H^S to be constant on radial segments to the origin, to obtain a function continuous in $V - \{0\}$ with a bounded singularity at $\{0\}$. We define

$$\bar{H}^S = \frac{1}{V}\int_V H^S(x)\,dx. \tag{9.1}$$

Theorem 9.1. *There holds*

$$3\bar{H}^S = \frac{S}{V}. \tag{9.2}$$

Proof. In a spherical coordinate system (r, ω) we may describe S by an equation $r = f(\omega)$. We set $F(r, \omega) = r/f(\omega)$; we then have on S that $F = 1$ and

$$H^S = \frac{1}{2}\operatorname{div}\frac{\nabla F}{|\nabla F|}\bigg|_{F=1}. \tag{9.3}$$

Since F is defined for all r, ω we may integrate over V to obtain

$$\int_V \operatorname{div}\frac{\nabla F}{|\nabla F|}\,dx = \oint_S \frac{\nabla F}{|\nabla F|}\cdot \nu\, d\sigma = S \tag{9.4}$$

since ∇F is orthogonal to S on that surface.

Interior to $V - \{0\}$ we have $0 < r < f(\omega)$ and

$$\frac{1}{2}\operatorname{div}\frac{\nabla F}{|\nabla F|} = \frac{f}{r}H^S. \tag{9.5}$$

This last result is evident geometrically, since the left side of (9.4) is the mean curvature of the similar surface obtained by contracting S with respect to the origin in the ratio r/f.

Placing (9.5) into (9.4) and integrating with respect to r, we obtain

$$\oint_S H^S r^3 d\omega = \tfrac{1}{3}\int_V H^S dx = S \tag{9.6}$$

which completes the proof.

We have also

Corollary 9.1 (a). *If V lies interior to a ball B_R of radius R, then*

$$\bar{H}^S \geq \frac{1}{R}. \tag{9.7}$$

Proof. Let B_{R_0} be a ball with volume V. By the general isoperimetric inequality

$$\frac{S}{V} \geq \left|\frac{\partial B_{R_0}}{B_{R_0}}\right| = \frac{3}{R_0}. \tag{9.8}$$

Clearly $R_0 \leq R$, and the result thus follows from (9.2).

The above results can be applied to the symmetric capillary tube, or to the sessile drop as considered in Chapter 3. The surface is no longer closed, but if the point of projection is taken as the center of the wetted disk Ω in the support plane, then $(\nabla F/|\nabla F|)\cdot v \equiv -\cos\gamma$ on Ω. Repeating the proof of Theorem 9.1, we obtain

Corollary 9.1 (b). *For a symmetric drop (or capillary surface) with center of projection as above, there holds*

$$3\bar{H}^S = \frac{S - \Omega\cos\gamma}{V}. \tag{9.9}$$

Various other applications and generalizations are clearly possible.

We consider various questions relating to possible isoperimetrical relations for the capillary tube of general section without volume constraint (cf. Chapter 2). In all questions, it is assumed that γ is a given constant, $0 \leq \gamma < \pi/2$, and that $\kappa > 0$.

i) For given boundary length $|\Sigma|$, which tube raises the maximum (or minimum) volume of fluid?
Answer: According to the formula (1.55) of Laplace, every tube of given $|\Sigma|$ raises the same volume above reference level.

ii) For given sectional area $|\Omega|$, which tube raises the maximum volume of fluid?
Answer: Since $|\Sigma|$ can be arbitrarily large, the result of i) shows that the problem has no solution.

iii) For given $|\Omega|$, which tube raises the minimum volume?
Answer: We have

$$|V| = \frac{1}{\kappa}|\Sigma|\cos\gamma \geq \frac{2\sqrt{\pi}}{\kappa}|\Omega|^{1/2}\cos\gamma$$

by the isoperimetric inequality, with equality holding if and only if Ω is a disk. Thus, the circular section raises the smallest volume.

iv) For given $|\Sigma|$ (or $|\Omega|$) which tube achieves a maximum fluid height?
Answer: Since by Theorem 5.5 an infinite height can be achieved, the problem has no solution.

v) For given $|\Sigma|$ (or $|\Omega|$) which tube section achieves a minimum fluid height?
Conjecture: The disk.

vi) For given $|\Sigma|$ (or $|\Omega|$) for which tube section is the minimum height a maximum?
Answer: The examples of appropriately chosen rectangular sections show that the problem has no solution.

vii) For given $|\Sigma|$ (or $|\Omega|$) for which tube section is the maximum height a minimum?
Conjecture: The disk.

viii) Is the maximum height always achieved at points of maximum curvature of Σ?
Answer: By Theorem 5.4, at a corner for which $\alpha+\gamma \geq \pi/2$ the height is bounded although the curvature at the vertex is infinite. But if $\alpha+\gamma<\pi/2$, the height is unbounded. An approximation by smooth arcs thus leads to a negative answer (cf. §7.10).

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276. Liggett: Interacting Particle Systems
277. Fulton/Lang: Riemann-Roch Algebra
278. Barr/Wells: Toposes, Triples, and Theories
279. Bishop/Bridges: Constructive Analysis
280. Neukirch: Class Field Theory
281. Chandrasekharan: Elliptic Functions
282. Lelong/Gruman: Entire Functions of Several Complex Variables
283. Kodaira: Complex Manifolds and Deformation of Complex Structures
284. Finn: Equilibrium Capillary Surfaces