Steps to nondimensionalize a model

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Mechanics is the paradise of mathematical science because here we come to the fruits of mathematics.
—Leonardo da Vinci (1452-1519)

(1) Pick a complete set of characteristic quantities $a_1, \ldots, a_k$, and assume a functional relation for a governed quantity:

$$g = g(a_1, \ldots, a_k, d_1, \ldots, d_l)$$

where $d_1, \ldots, d_l$ are other parameters upon which g depends.

2 By first analyzing the dimensions of all quantities and then rescaling using power monomials in the characteristic quantities obtain the nondimensional form:

$$\begin{aligned} \Pi &= \frac{g}{a_1^p \cdots a_k^r} && (1)\\ &= \frac{1}{a_1^p \cdots a_k^r} g(a_1, \ldots, a_k, a_1^{p_1} \cdots a_k^{r_1} \Pi_1, \ldots, a_1^{p_l} \cdots a_k^{r_l} \Pi_l) && (2)\\ &= \mathcal{F}(a_1, \ldots, a_k, \Pi_1, \ldots, \Pi_l) && (3)\\ &= \Phi(\Pi_1, \ldots, \Pi_l). && (4) \end{aligned}$$

(3) Reexpress $g = a_1^p \cdots a_k^r \Pi$ using the original dimensional variables and the derived form of Π given in (4):

$$g = a_1^p \cdots a_k^r \Phi\left(\frac{d_1}{a_1^{p_l} \cdots a_k^{r_l}}, \ldots, \frac{d_1}{a_1^{p_l} \cdots a_k^{r_l}}\right). \quad (5)$$

Notes

1. In step 1, one hopes there are not too many "additional" parameters.

2. In step 2, the first equations (1-3) are just obtained by renaming variables. The last equation (4) is obtained by the following reasoning. The units of $a_1, \ldots, a_k$ provide a collection of fundamental units for the problem. (They may not be the fundamental units agreed upon at the outset, but they still form such a collection.) Changing one of these units of measurement, abstractly, amounts to scaling the numerical value by some factor. Say we change the unit in which a_1 is measured, but leave the other units unchanged. Then a_1 becomes $\tilde{a}_1 = \lambda a_1$ in the argument of $\mathcal{F}$. We assume, however (and this is the key assumption) that the value of g "changes appropriately" with a change of units. To be precise, a change in the units of a_1 so that $\tilde{a}_1 = \lambda a_1$ results in a change of the units of g so that $\tilde{g} = g(\tilde{a}_1, a_2, \ldots, a_k, \tilde{d}_1, \ldots, \tilde{d}_l)$, and this we require to equal $\lambda^p g(a_1, \ldots, a_k, d_1, \ldots, d_l)$. That is,

$$\tilde{g} = g(\lambda a_1, a_2, \ldots, a_k, \lambda^{p_1}\tilde{d}_1, \ldots, \lambda^{p_l}\tilde{d}_l) = \lambda^p g(a_1, \ldots, a_k, d_1, \ldots, d_l).$$

We have then for the value of Π in the new units:

$$\tilde{\Pi} = \frac{\lambda^p g(a_1, \ldots, a_k, d_1, \ldots, d_l)}{\lambda^p a_1^p \cdots a_k^r} = \Pi.$$

That is, Π is unchanged because it is nondimensional. Therefore $\mathcal{F}$ does not really depend on a_1 (or a_2 or a_3,...). So we get (4).

3. The most useful form of (5) to remember is

$$g = a_1^p \cdots a_k^r \Phi(\Pi_1, \ldots, \Pi_l). \tag{6}$$

4. One can write down (6) directly from the dimensional analysis of step 2.

5. In most instances, one is interested in a case where the nondimensional parameters $\Pi_1, \ldots, \Pi_l$ are close to some fixed values $\Pi_1^0, \ldots, \Pi_l^0$ (usually all zero or ∞). If Φ has a finite limit Φ_0 for $\Pi_1, \ldots, \Pi_l \to \Pi_1^0, \ldots, \Pi_l^0$, we often assume the form

$$g = \Phi_0 a_1^p \cdots a_k^r. \tag{7}$$

Example 1 (Fundamental frequence of a drum) *We assume the fundamental frequency μ depends on the diameter d of the drum, the tension τ on the membrane, and an area-mass density ρ of the membrane. For dimensions we find*

$$[\mu] = 1/T,$$
$$[d] = L, \quad [\tau] = ML/T^2, \quad [\rho] = M/L^2.$$

Since d, τ, and ρ make a complete collection of characteristic quantities,

$$\mu = \Phi_0 d^p \tau^q \rho^r$$

where $[d]^p[\tau]^q[\rho]^r = 1/T$. That is,

$$L^p(M^q L^q/T^{2q})M^r/L^{2r} = 1/T.$$

One easily sees that $q = 1/2$, $r = -1/2$, and $p = -3/2$. Therefore,

$$\mu = \Phi_0 \sqrt{\frac{\tau}{\rho}} d^{-3/2}.$$