1.3 The Lemma of DuBois-Reymond

We needed extra regularity to integrate by parts and obtain the Euler-Lagrange equation. The following result shows that, at least sometimes, the extra regularity in such a situation need not be assumed.

Lemma 3 (cf. Lemma 1.8 in BGH)**.** *(The lemma of DuBois-Reymond) If $f \in C^0(a,b)$ and*

$$\int_a^b f(x)\eta'(x)\,dx = 0 \quad \text{for every } \eta \in C_c^\infty(a,b). \tag{1.16}$$

then $f \equiv c$ (constant).

Proof: Let $\zeta \in C_c^\infty(a,b)$ be arbitrary and take $\mu \in C_c^\infty(a,b)$ with $\int \mu = 1$. Consider

$$\phi = \zeta - \left(\int \zeta\right)\mu = \zeta - c\mu$$

where $c = \int \zeta$. Note that $\phi \in C_c^\infty(a,b)$. Also,

$$\eta(x) = \int_a^x \phi(\xi)\,d\xi$$

has $\eta'(x) = \phi(x)$ and (for $\epsilon > 0$ small)

$$\eta(a+\epsilon) = \int_a^{a+\epsilon} \phi(\xi)\,d\xi = 0 \quad \text{and} \quad \eta(b-\epsilon) = \int_a^b (\zeta - c\mu)\,dx = c - c\int_a^b \mu\,dx = 0.$$

Thus, $\eta \in C_c^\infty(a,b)$ with $\phi = \eta'$. According to (1.16) we have

$$0 = \int f\phi = \int f(\zeta - c\mu) = \int f\zeta - c\int f\mu = \int f\zeta - \left(\int \zeta\right)c_1$$

where

$$c_1 = \int f\mu.$$

Therefore,

$$0 = \int (f - c_1)\zeta \quad \text{for every } \zeta \in C_c^\infty(a,b).$$

The fundamental lemma implies $f \equiv c_1$. □

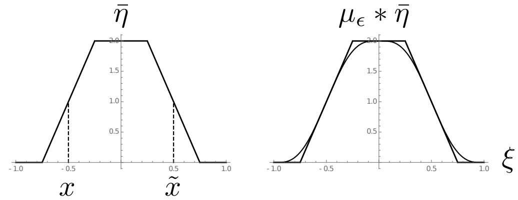

Figure 1.5: For the lemma of DuBois-Reymond, we mollify a piecewise smooth function.

Another proof of the lemma of DuBois-Raymond

Again the authors of BGH give a different argument and a more general result.

Lemma 4 (Lemma 1.8 in BGH)**.** *If $f \in L^1_{\text{loc}}(a,b)$ and*

$$\int_{(a,b)} f\eta' = 0 \quad \text{for every } \eta \in C_c^\infty(a,b). \tag{1.17}$$

then $f \equiv c$ (constant), i.e., there is some constant c such that $f(x) = c$ for almost every x.

Proof: Let x and $\tilde{x}$ be Lebesgue points for $f \in [f_0]$. We might as well assume $a < x < \tilde{x} < b$. As suggested in BGH, let us also take $\delta > 0$ small and fixed so that

$$\bar{\eta}(\xi) = 2\chi_I(\xi) + \left[1 + \frac{1}{\delta}(\xi - x)\right]\chi_T(\xi) - \left[1 - \frac{1}{\delta}(\xi - \tilde{x})\right]\chi_{\tilde{T}}(\xi)$$

with $I = (x + \delta, \tilde{x} - \delta)$, $T = (x - \delta, x + \delta)$ and $\tilde{T} = (\tilde{x} - \delta, \tilde{x} + \delta)$ gives the function with graph indicated in Figure 1.5. The reasoning of Exercise 8 shows $\mu_\epsilon * \bar{\eta} \in C_c^\infty(a,b)$ with

$$\mu_\epsilon * \bar{\eta}(\xi) = \begin{cases} 0, & \xi \in (a,b)\backslash(x - \delta - \epsilon, \tilde{x} + \delta + \epsilon) \\ 2, & \xi \in [x + \delta + \epsilon, \tilde{x} - \delta - \epsilon] \end{cases}$$

where μ_ϵ is a standard mollifier with $\epsilon < \delta$. Also,

$$\frac{d}{dx}(\mu_\epsilon * \bar{\eta}) = \mu_\epsilon * \bar{\eta}' = 0$$

on the interiors

$$(a,b)\backslash[x-\delta-\epsilon,\tilde{x}+\delta+\epsilon] \quad \text{and} \quad (x+\delta+\epsilon,\tilde{x}-\delta-\epsilon)$$

with

$$-\frac{1}{\delta} \leq \frac{d}{dx}(\mu_\epsilon * \bar{\eta}) \leq \frac{1}{\delta} \quad \text{for all } x \in (a,b).$$

Let us compute what happens in the portions

$$(x-\delta+\epsilon, x+\delta-\epsilon) \quad \text{and} \quad (\tilde{x}-\delta+\epsilon,\tilde{x}+\delta-\epsilon)$$

of the transition intervals T and $\tilde{T}$. For ξ in the first interval

$$\begin{aligned}
\mu_\epsilon * \bar{\eta}(\xi) &= \int_t \mu_\epsilon(t)\bar{\eta}(\xi - t) \\
&= \int_t \mu_\epsilon(t)\left[1 + \frac{1}{\delta}(\xi - t - x)\right] \\
&= 1 + \frac{1}{\delta}(\xi - x) - \frac{1}{\delta}\int_t t\mu_\epsilon(t) \\
&= \bar{\eta}(\xi) - m(\epsilon)
\end{aligned}$$

where

$$m(\epsilon) = \frac{1}{\delta}\int_t t\mu_\epsilon(t) = \frac{1}{\delta}\int_{-\epsilon}^{\epsilon} t\mu_\epsilon(t)\,dt \quad \text{satisfies} \quad |m(\epsilon)| < \frac{\epsilon}{\delta}.$$

Similarly for ξ in the interval around $\tilde{x}$ we have $\mu_\epsilon * \bar{\eta}(\xi) = \bar{\eta}(\xi) + m(\epsilon)$.

The hypothesis (1.17) clearly applies to give

$$\int f(\mu_\epsilon * \bar{\eta})' = 0.$$

We wish to take a limit and conclude

$$\int f\bar{\eta}' = 0. \tag{1.18}$$

To this end, let us estimate

$$\begin{aligned}
\left|\int f(\mu_\epsilon * \bar{\eta})' - \int f\bar{\eta}'\right| &= \left|\int f\left[(\mu_\epsilon * \bar{\eta})' - \bar{\eta}'\right]\right| \\
&\leq \int |f||(\mu_\epsilon * \bar{\eta})' - \bar{\eta}'| \\
&= \int_{(x-\delta-\epsilon,\tilde{x}+\delta+\epsilon)} |f||(\mu_\epsilon * \bar{\eta})' - \bar{\eta}'|.
\end{aligned} \tag{1.19}$$

In this case, we do not know $f \in L^2$, so the Cauchy-Schwarz inequality does not help us. We do know, howevever, that for ϵ small the interval $J = (x-\delta-\epsilon, \tilde{x}+\delta+\epsilon) \subset\subset (a,b)$ so $f \in L^1(J)$. Furthermore, we have shown the function $|(\mu_\epsilon * \bar{\eta})'(\xi) - \bar{\eta}'(\xi)|$ is bounded on all of (a,b) and satisfies

$$\lim_{\epsilon \to 0} |(\mu_\epsilon * \bar{\eta})'(\xi) - \bar{\eta}'(\xi)| = 0$$

for every $\xi \in (a,b)$. It follows that the integrand in (1.19) is bounded independent of ϵ in $L^1(J)$ and limits pointwise to zero almost everywhere. By the Lebesgue dominated convergence theorem, the expression in (1.19) tends to zero, and we have established (1.18).

Since $\bar{\eta}'(\xi) = (1/\delta)\chi_T(\xi) + (1/\delta)\chi_{\tilde{T}}(\xi)$ almost everywhere (i.e., except at the four corner points), we can rewrite (1.18) as

$$\frac{1}{\delta}\int_{(x-\delta,x+\delta)} f - \frac{1}{\delta}\int_{(x-\delta,x+\delta)} f = 0.$$

Taking the limit as $\delta \searrow 0$ and recalling that x and $\tilde{x}$ were Lebesgue points, we get $2f(x) - 2f(\tilde{x}) = 0$. That is, $f(x) = f(\tilde{x})$, and it follows that f is constant, taking a single value on its Lebesgue points. □

Exercises

Exercise 11. *Show that if the mollifier μ_ϵ is chosen to be even, then the quantity*

$$m(\epsilon) = \int_t t\mu_\epsilon(t)$$

vanishes. Show why this quantity need not be zero when μ_ϵ is not even.

Exercise 12. *Show that when g has compact support in (a,b), then for ϵ small enough $\mu_\epsilon * g$ also has compact support in (a,b) and may therefore be defined on all of $\mathbb{R}$.*

Exercise 13. *Show directly that*

$$\frac{d}{dx}(\mu_\epsilon * g)(x) = \mu_\epsilon * g'(x) \quad \textit{for all } x \in \mathbb{R}$$

when $g \in \sqsubset_c^1(a,b)$ is any piecewise C^1 function with compact support. Use this calculation to give a (new and different) direct proof that

$$\int f\bar{\eta}' = 0$$

where f satisfies the hypothesis (1.17) or (1.16) and $\bar{\eta}$ is the function defined in the proof of the DuBois-Reymond lemma from BGH.

1.4 The Euler-Lagrange Equation (revisited)

Theorem 4 (Corollary 1.10 in BGH)**.** *(The Euler-Lagrange Equation for weak extremals) If*

$$u \in C^1(a,b)$$

is a weak extremal for the functional

$$\mathcal{F}[u] = \int_a^b F(x, u(x), u'(x))\, dx$$

with Lagrangian $F \in C^1((a,b) \times \mathbb{R} \times \mathbb{R})$, then

$$\frac{d}{dx} F_p(x, u, u') - F_z(x, u, u') = 0 \quad \text{on} \quad (a,b). \tag{1.20}$$

Proof: This result follows, essentially, from integrating by parts in the condition for weak extremals (1.1) in the reverse direction: By the fundamental theorem of calculus

$$\psi(x) = \int_a^x F_z(t, u(t), u'(t))\, dt$$

is a C^1 function with derivative $F_z(x, u(x), u'(x))$. Thus,

$$\int_a^b F_z(x, u, u')\phi\, dx = \psi\phi\Big|_a^b - \int_a^b \psi\phi'\, dx = -\int_a^b \psi\phi'\, dx.$$

Combining this expression with the other integral from (1.1), we get

$$\int_a^b \left[F_p(x, u, u') - \psi\right] \phi'\, dx = 0 \quad \text{for all } \phi \in C_c^\infty(a,b).$$

By the lemma of DuBois-Raymond, there is some constant c such that

$$F_p(x, u, u') - \psi = c.$$

That is,

$$F_p(x, u, u') = \int_a^x F_z(t, u(t), u'(t))\, dt + c. \tag{1.21}$$

While it may not be true that F_p has any higher partial derivatives and it may not be true that u' has any higher partial derivatives, we have shown that the composition $F_p(x, u, u')$ *does* have a derivative:

$$\frac{d}{dx} F_p(x, u, u') = F_z(x, u, u'). \qquad \square$$

It is important to realize that the Euler-Lagrange equation, under these hypotheses, may not allow expansion of the left side by the chain rule. The equation (1.21) is called the *DuBois-Raymond equation* or the Euler-Lagrange equation in integrated form.

The following example (Example 4 on page 14 of BGH) shows the weaker regularity allowed by Theorem 4 is sometimes needed. Consider the functional

$$\mathcal{F}[u] = \int_{-1}^1 u^2 (2x - u')^2\, dx$$

on

$$\mathcal{A} = \{u \in C^1[-1, 1] : u(-1) = 0,\ u(1) = 1\}.$$

Notice that $\mathcal{F}$ is non-negative and $\mathcal{F}[u_0] = 0$ where $u_0 \in C^1[-1, 1] \backslash C^2(-1, 1)$ is given by

$$u_0(x) = \begin{cases} 0, & -1 \le x \le 0 \\ x^2, & 0 \le x \le 1. \end{cases}$$

We wish to show u_0 is the unique minimizer in $\mathcal{A}$. If $u \in \mathcal{A}$ is any minimizer, then we must have

$$\mathcal{F}[u] = \int_{-1}^1 u^2 (2x - u')^2\, dx = 0.$$

This means that on any interval where $u \neq 0$, we must have $u' = 2x$ and $u(x) = x^2 + c$ for some constant c. In particular, integrating from $x = 1$, we must have

$$u(x) = 1 + \int_1^x (2\xi)\, d\xi = x^2 \qquad \text{for } 0 \le x \le 1.$$

If we assume there is some x_0 with $-1 < x_0 < 0$ for which $u(x_0) \neq 0$, then there is a maximal interval (a, b) with $-1 \leq a < x_0 < b \leq 0$ such that

$$u(x) = u(x_0) + x^2 - x_0^2 \neq 0 \quad \text{for } a < x < b, \quad \text{but} \quad u(a) = u(b) = 0.$$

Evaluating $u(x)$ at $x = a$ and $x = b$ we conclude $a^2 = b^2 = x_0^2 - u(x_0)$. This contradicts the fact that $a < b \leq 0$. Consequently, there is no such point x_0, we have $u(x) \equiv 0$ for $-1 \leq x \leq 0$, and $u \equiv u_0$.

Exercise 14. **(a)** *Find the Euler-Lagrange equation for*

$$\mathcal{F}[u] = \int_{-1}^{1} u^2 (2x - u')^2 \, dx,$$

and show that u_0 given above is a solution of the equation.

(b) *Assume $u \in C^2[-1, 1]$ is a classical extremal for $\mathcal{F}$, and use the chain rule (product rule etc.) to write the Euler-Lagrange equation as a second order quasilinear ODE. Is u_0 also a solution of this equation?*

(c) *What can you say about $C^2[-1, 1]$ classical extremals for this functional?*

1.5 Examples

We now return to some examples from the introduction and write down the associated Euler-Lagrange equations. We also make some elementary observations about those examples and introduce some additional examples.

1.5.1 Dirichlet energy

Recall that $\mathcal{D}[u] = 0$ if $u \equiv c$ (constant), and these are absolute minimizers in $C^1[a, b]$, but they may not be admissible.

For the Dirichlet energy, the Lagrangian is $F(p) = p^2$ and the Euler-Lagrange equation is

$$u'' = 0. \tag{1.22}$$

Notice the argument of DuBois-Raymond now *implies* added regularity: Any C^1 (weak) extremal must be a C^2 (classical) extremal. Given the admissible class $\mathcal{A}_0 = \{u \in C^1[a,b] : u(a) = u_a,\ u(b) = u_b\}$, it is easy to integrate (1.22) to obtain the unique admissible extremal:

$$u_0(x) = \frac{u_b - a_a}{b - a}(x - a) + u_a.$$

Let's try to show u_0 is the minimizer. Say u is some other admissible competitor. Then

$$\begin{aligned}
\mathcal{D}[u] - \mathcal{D}[u_0] &= \int_a^b \left\{[u_0' + (u' - u_0')]^2 - {u_0'}^2\right\} dx \\
&= \int_a^b \left\{(u' - u_0')^2 + 2u_0'(u' - u_0')\right\} dx \\
&\geq 2\int_a^b u_0'(u' - u_0')\, dx \\
&= 2u_0' \int_a^b (u' - u_0')\, dx \\
&= 2u_0'[u_b - u_b - (u_a - u_a)] \\
&= 0.
\end{aligned}$$

This shows $\mathcal{D}[u] \geq \mathcal{D}[u_0]$ with equality if and only if $u' \equiv u_0'$, which means $u \equiv u_0$. That is, u_0 is the unique minimizer.

This approach suggests the following simple rephrasing of the condition for minimizers:

Lemma 5 (Lemma 2.1 in Troutman)**.** *If the universal set $\mathcal{U}$ is a linear space (e.g., $C^1[a,b]$ or $\sqsubset^1[a,b]$, etc.) and $u_0 \in \mathcal{A}$ satisfies*

$$\mathcal{F}[u_0 + v] - \mathcal{F}[u_0] \geq 0 \quad \textit{whenver } v \in \mathcal{U} \textit{ and } u_0 + v \in \mathcal{A}, \tag{1.23}$$

then u_0 is a minimizer. Conversely, if $u_0 \in \mathcal{A}$ is a minimizer, then (1.23) holds.

Furthermore, if equality holds in (1.23) only for $v = 0$, then u_0 is the unique minimizer (and conversely, if u_0 is the unique minimzer, then equality in (1.23) can only hold when $v = 0$.)

1.5.2 Poisson's functional

Here we consider

$$\mathcal{F}[u] = \int_0^1 (u + u'^2)\, dx$$

on

$$\mathcal{A}_0 = \{u \in C^1[a,b] : u(0) = 0 = u(1)\}.$$

Here the function $u \equiv 0$ is admissible and gives $\mathcal{F}[u] = 0$, but this is not the minimizer. The Lagrangian is $F(z,p) = z + p^2$ and the Euler-Lagrange equation is

$$2u'' - 1 = 0.$$

Again, we have extra regularity for extremals, and the unique extremal is

$$u_0 = \frac{1}{4}x(x-1).$$

Computing $\mathcal{F}[u_0]$ we get

$$\int_0^1 \left[\frac{1}{4}x^2 - \frac{1}{4}x + \frac{1}{4}\left(x - \frac{1}{2}\right)^2\right] dx = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{24} + \frac{1}{24}\right) = -\frac{1}{24}.$$

Thus, $\mathcal{F}[u_0] < 0$. See Exercise 16.

1.5.3 Arclength functional

Here is an instance where C^2 regularity is not immediate from the Euler-Lagrange equation. The Lagrangian is $\sqrt{1+p^2}$, and the Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right) = 0.$$

With additional regularity, the differentiation may be carried out so that the left side is immediately recognizable as the curvature of the graph of u:

$$k = \frac{u''}{(1+u'^2)^{3/2}}.$$

Of course, if the curvature vanishes, the graph is a straight line. See Exercise 17.

1.5.4 Potential equation

An ordinary differential equation (ODE) central to the study of many physical phenomena is the second order linear potential equation

$$-u'' + c(x)u = 0$$

associated with the variational integral

$$\mathcal{F}[u] = \int_a^b \left[u'^2 + c(x)u^2\right] \, dx.$$

The function $c = c(x)$ is called the *potential*. See Exercise 19. This is Example 2 on page 14 of BGH.

1.6 Exercises

Exercise 15. *Compute the Dirichlet energy of the sequence of functions* $u_j \in \mathcal{A}_0 = \{u \in C^1[0,1] : u(0) = 0,\ u(1) = 1\}$ *which converge in* $L^1(0,1)$ *to the constant zero function.*

Exercise 16. *Show* $u_0 = x(x-1)/4$ *is the unique minimizer of Poisson's functional*

$$\mathcal{F}[u] = \int_0^1 (u + u'^2)\, dx$$

on

$$\mathcal{A}_0 = \{u \in C^1[0,0] : u(0) = 0 = u(1)\}.$$

Exercise 17. *Explicitly integrate the Euler-Lagrange equation for the arclength functional to verify all extremal solutions must be affine, having the form* $u(x) = \alpha x + \beta$ *for some constants* α *and* β.

Exercise 18. *Find the extremals for the total variation functional.*

Exercise 19. *Show the potential equation is the Euler-Lagrange equation of the functional*

$$\mathcal{F}[u] = \int_a^b \left[u'^2 + c(x)u^2\right] \, dx.$$

What can you say about the regularity of solutions in $C^1[a,b]$*?*

Exercise 20 (project exercise)**.** *Consider a cylindrical container (modeled by)*

$$\{(x,y,z) : x^2 + y^2 = 1,\ z \geq 0\} \cup \{(x,y,0) : x^2 + y^2 \leq 1\}.$$

Assume the container contains a volume $V = \pi$ *of liquid in a gravity field so that the volume initially occupies the space*

$$\{(x,y,z) : x^2 + y^2 < 1,\ 0 < z < 1\}.$$

(Ignore surface tension and wetting energy.) Assume, more generally, that the liquid is bounded by the graph of an even function $z = u(r)$ for $-1 < r < 1$ (in polar coordinates) in the admissible class

$$\mathcal{A} = \left\{ u \in C^1[0,1] : u'(0) = 0,\ 2\pi \int_0^1 r u(r)\, dr = \pi \right\}.$$

If the container (along with the liquid it contains) is rotating at a constant angular velocity ω, calculate the physical energy of the system two ways:

(a) *The physical energy is the kinetic energy of rotation.*

(b) *With respect to a rotating frame fixed to the container, the physical energy is the potential energy with respect to the apparent field.*

Find (and solve) the Euler-Lagrange equation for the interface. What happens if the container has some other shape (but still rotates with a given volume of liquid around some axis)?

Solutions

Solution 1 (Exercise 8)**.** *Let*

$$f(x) = \chi_{(-\delta,\delta)}\left(\frac{x - x_0}{\delta}\right)$$

and consider the convolution integral

$$\mu_\epsilon * f(x) = \int_{\xi \in \mathbb{R}} \mu(\xi) f(\xi - x).$$

Show the following:

(a) $\mu_\epsilon * f = f * \mu_\epsilon$.

(b) $\mu_\epsilon * f \in C_c^\infty(\mathbb{R})$.

(c) $0 \le \mu_\epsilon * f(x) \le 1$ *for all* $x \in \mathbb{R}$.

(d) *When* $\epsilon < \delta$

$$\mu_\epsilon * f(x) = \begin{cases} 0, & |x| \ge \delta + \epsilon \\ 1, & |x| \le \delta - \epsilon. \end{cases}$$