Chapter 6

The Hanging Chain

If a chain or cable has its ends fixed at two different points and hangs under the influence of gravity, it takes the shape of a hyperbolic cosine curve. We now describe this shape precisely and explain how it arises as a minimizer of potential energy among many possible shapes.

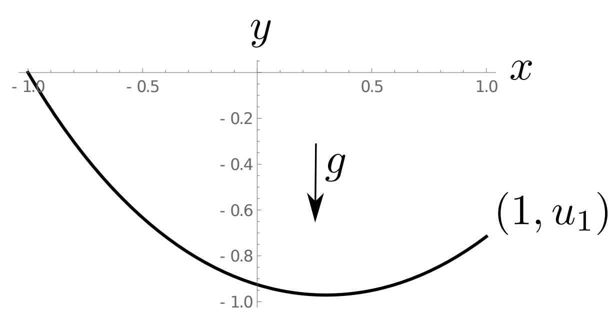

Figure 6.1: the shape of a chain hanging from its endpoints in gravity

6.1 Analysis

6.1.1 Model

Let $\ell > 0$ be the length of the chain and let ρ denote the linear density of mass along the length of chain. Choose x, y-coordinates with the left end of the chain fixed at $(-1, 0)$ and the right end at $(1, u_1)$. We have made a choice of units here so that the horizontal distance between the fixed endpoints is

2 units. This is equivalent to scaling the system given in some particular initial units. We could also assume u_1 has a specific sign, say $u_1 > 0$, but this is not necessary.

Given the length constraint on the chain, we must have

$$1 + u_1^2 < \ell^2. \tag{6.1}$$

There are many curves of length ℓ connecting $(-1, 0)$ to $(1, u_1)$. Among these consider C^1 curves given by the graph of a function $u : [0, 1] \to \mathbb{R}$. The

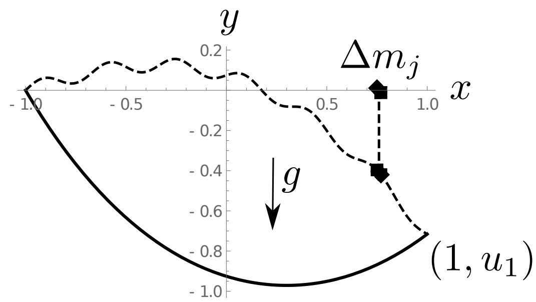

Figure 6.2: an alternative chain shape and the associated potential energy

length constraint may then be written as

$$\int_{-1}^{1} \sqrt{1 + [u'(x)]^2}\, dx = \ell.$$

Assuming a constant gravitational field $\vec{G} = -g(0, 1)$ and zero potential at $y = 0$, we may integrate to approximate the potential energy of a portion of the chain having mass $\Delta m_j = \rho\sqrt{1 + [u'(x_j^*)]^2}\, \Delta x_j$:

$$\text{approximate potential energy } V_j \; = \int_0^{u(x_j^*)} \rho g \sqrt{1 + u'(x_j^*)^2}\, \Delta x_j \, dy.$$

The potential energy associated with a point mass is given by the work required to move the mass from a position of zero potential to another position, that is, $-\int_\gamma F \cdot T$ where F is the force field, γ is a path connecting a position of zero potential to the position of the mass, and T is the unit tangent vector along the path. In this case the force $F = \Delta m_j \vec{G} = -\Delta m_j g(0, 1)$ is assumed constant, and the integral amounts to the force multiplied by the vertical distance to equilibrium:

$$V_j = \rho g u(x_j^*)\sqrt{1+u'(x_j^*)^2}\,\Delta x_j.$$

Summing over all model portions of chain and taking the limit as the mazimum portion length tends to 0, we find an expression for the total potential energy as a function of the chain shape determined by u:

$$\text{potential energy } V \;= \lim \sum \rho g u(x_j^*)\sqrt{1+u'(x_j^*)^2}\,\Delta x_j = \int_{-1}^{1} \rho g u(x)\sqrt{1+u'(x)^2}\,dx.$$

By the Leibniz'/Maupertuis' principle of virtual work, or Hamilton's action principle, the observable shape u should be a critical point for

$$V[u] = \int_{-1}^{1} \rho g u(x)\sqrt{1+u'(x)^2}\,dx$$

subject to the constraint

$$L[u] = \int_{-1}^{1} \sqrt{1+u'(x)^2}\,dx = \ell.$$

Under the assumption that ρ and g are positive constants, we may replace the expression for V above with

$$V[u] = \int_{-1}^{1} u(x)\sqrt{1+u'(x)^2}\,dx$$

Introducing a Lagrange multiplier λ associated with the constraint and assuming the existence of the model shape within the admissible class

$$\mathcal{A} = \{u \in C^2[-1,1] : u(-1) = 0,\, u(1) = u_1\},$$

we set $\mathcal{F} = V + \lambda L$ and obtain the necessary condition

$$\delta\mathcal{F}_u[\phi] = \frac{d}{d\epsilon}\int_{-1}^{1}(u+\epsilon\phi+\lambda)\sqrt{1+(u'+\epsilon\phi')^2}\,dx\Big|_{\epsilon=0} = 0$$

for all $\phi \in C_c^\infty(-1,1)$. Differentiating under the integral and evaluating, we find

$$\int_{-1}^{1}\left[\phi\sqrt{1+u'^2} + (u+\lambda)\frac{u'\phi'}{\sqrt{1+u'^2}}\right]\,dx = 0.$$

We may integrate by parts in the second term to obtain

$$\int_{-1}^{1}\left[-\left(\frac{(u+\lambda)u'}{\sqrt{1+u'^2}}\right)' + \sqrt{1+u'^2}\right]\phi = 0 \quad \text{for all } \phi \in C_c^\infty(-1,1).$$

Finally, we may apply the fundamental lemma of the calculus of variations to obtain a two point boundary value problem for a second order nonlinear ordinary differential equation for the observed shape u:

$$\left(\frac{(u+\lambda)u'}{\sqrt{1+u'^2}}\right)' = \sqrt{1+u'^2}, \qquad u(-1)=0,\ u(1)=u_1.$$

We know this equation is satisfied even under the assumption $u \in C^1[-1,1]$.

6.1.2 Extremal graphs

Using the assumed regularity of the observed shape u, we can also write

$$(u+\lambda)\frac{u''}{(1+u'^2)^{3/2}} + \frac{u'^2}{\sqrt{1+u'^2}} = \sqrt{1+u'^2}$$

or

$$(u+\lambda)u'' = 1+u'^2.$$

Under the assumption $u''(-1) > 0$, which (based on observation of the shape of actual physical hanging chains) seems rather reasonable, we can solve for the Lagrange multiplier and find

$$\lambda = \frac{1+u'(-1)^2}{u''(-1)} > 0.$$

More generally, whenever $u+\lambda \neq 0$, we can write

$$\frac{u'}{1+u'^2}\,u'' = \frac{1}{u+\lambda}\,u'.$$

In particular, integrating from $x=-1$ to x,

$$\int_{u'(-1)}^{u'} \frac{t}{1+t^2}\,dt = \int_{u(-1)}^{u} \frac{1}{t+\lambda}\,dt$$

or

$$\frac{1}{2}\left[\ln(1+u'^2) - \ln(1+u'(-1)^2)\right] = \ln(u+\lambda) - \ln\lambda.$$

It follows that

$$\frac{1+u'^2}{1+u'(-1)^2} = \left(\frac{u}{\lambda}+1\right)^2. \tag{6.2}$$

Let us pause at this point to consider the first integral equation

$$u'F_p(u,u') - F(u,u') = -c \tag{6.3}$$

where c is some constant and $F(z,p) = (z+\lambda)\sqrt{1+p^2}$ is the Lagrangian associated with $\mathcal{F}$. We have used $-c$ instead of c here to simplify things later. After a computation, we find

$$\frac{u'^2}{\sqrt{1+u'^2}} - \sqrt{1+u'^2} = -\frac{c}{u+\lambda}.$$

That is,

$$\sqrt{1+u'^2} = \frac{1}{c}(u+\lambda).$$

Taking the contant $c = \lambda/\sqrt{1+u'(-1)^2}$, which it must be, we see several things. First of all, any solution of the first integral equation with $c \neq 0$ will give a solution of (6.2). It is possible to get a solution of (6.3) with the choice $c = 0$, but in this case, we must take $u \equiv -\lambda = 0$, and we must therefore have $u_1 = 0$. This is, indeed, not a solution of the Euler-Lagrange equation for $\mathcal{F} = V + \lambda L$, but this possibility represents the exceptional case of Theorem 9 (Proposition 1.17 in BGH) in which the constraint is degenerate. In this case, the solution $u \equiv 0$ gives the shortest path between $(-1,0)$ and $(1,u_1) = (1,0)$ and is, therefore, a critical point for the length functional L providing the constraint. When $c \neq 0$, we obtain from the first integral equation a global justification for our assumption

$$u + \lambda \neq 0.$$

This is because every solution of the Euler-Lagrange equation must be a solution of the first integral equation. Only the solution $u \equiv 0$ in the case $u_1 = 0$ and $\ell = 2$ is exceptional.

Finally, the first integral equation tells us something about the sign of $u + \lambda$ because

$$\sqrt{1+u'^2} = \frac{\sqrt{1+u'(-1)^2}}{\lambda}(u+\lambda).$$

It follows that $u+\lambda$ and λ must share the same sign, and under our, seemingly justified, assumpetion $u''(-1) > 0$, that sign is positive. Thus, we may proceed to solve either the Euler-Lagrange equation or the first integral equation under this assumption. Making the substituion $v = (u+\lambda)\sqrt{1+u'(-1)^2}/\lambda$, we find

$$u' = \pm\sqrt{v^2-1} \qquad \text{or} \qquad \frac{\lambda}{\sqrt{1+u'(-1)^2}}v' = \pm\sqrt{v^2-1}.$$

It follows that

$$\cosh^{-1} v - \cosh^{-1} v(-1) = \pm\frac{\sqrt{1+u'(-1)^2}}{\lambda}(x+1),$$

$$v = \frac{\sqrt{1+u'(-1)^2}}{\lambda}(u+\lambda) = \cosh\left[\pm\frac{\sqrt{1+u'(-1)^2}}{\lambda}(x+1) + \cosh^{-1} v(-1)\right],$$

or

$$u = -\lambda + \frac{\lambda}{\sqrt{1+u'(-1)^2}}\cosh\left[\frac{\sqrt{1+u'(-1)^2}}{\lambda}(x+1) \pm \cosh^{-1}\sqrt{1+u'(-1)^2}\right].$$

This looks rather complicated, but it does tell us that the extremals have the form of hyperbolic cosine curves This also confirms that the constant c from the first integral equation should be positive with $c < 0$ extremals corresponding to maximizers of the energy. Substituting the value of c from the first integral equation and differentiating, we also see

$$u' = \sinh\left((x+1)/c \pm \cosh^{-1}\sqrt{1+u'(-1)^2}\right).$$

This allows us to nominally locate the vertex or lowest point on the hyperbolic cosine curve which occurs for

$$x = \mu = -1 \mp c\cosh^{-1}(\lambda/c).$$

In terms of this parameter, the extremals may be written as

$$u = -\lambda + c\cosh\left(\frac{x-\mu}{c}\right).$$

There are now three unknown parameters λ, μ, and c, but the initial condition $u(-1) = 0$ implies

$$\lambda = c \cosh\left(\frac{1+\mu}{c}\right)$$

and

$$u = c\cosh\left(\frac{x-\mu}{c}\right) - c\cosh\left(\frac{1+\mu}{c}\right).$$

The other endpoint condition takes the symmetric form

$$c\cosh\left(\frac{1-\mu}{c}\right) - c\cosh\left(\frac{1+\mu}{c}\right) = u_1.$$

Another equation we can use to determine the parameters c and μ is given by the length constraint $L[u] = \ell$.

$$u' = \sinh\left(\frac{x-\mu}{c}\right) \qquad \text{and} \qquad 1 + u'^2 = \cosh^2\left(\frac{x-\mu}{c}\right).$$

Therefore,

$$L[u] = \int_{-1}^{1} \sqrt{1+u'^2}\, dx = \int_{-1}^{1} \cosh\left(\frac{x-\mu}{c}\right) dx,$$

and writing down $L[u] = \ell$ we are led to the fundamental symmetric system:

$$c\cosh\left(\frac{1-\mu}{c}\right) - c\cosh\left(\frac{1+\mu}{c}\right) = u_1. \tag{6.4}$$

and

$$c\sinh\left(\frac{1-\mu}{c}\right) + c\sinh\left(\frac{1+\mu}{c}\right) = \ell. \tag{6.5}$$

In this symmetric form, it is possible to eliminate μ as follows: Square both equations and subtract the first from the second, noting $\ell^2 - u_1^2 \geq 4$. We get

$$c^2\left[-2 + 2\cosh\left(\frac{1-\mu}{c}\right)\cosh\left(\frac{1+\mu}{c}\right) + 2\sinh\left(\frac{1-\mu}{c}\right)\sinh\left(\frac{1+\mu}{c}\right)\right] = \ell^2 - u_1^2.$$

That is,

$$-1 + \cosh\left(\frac{2}{c}\right) = 1 + \cosh^2\left(\frac{1}{c}\right) + \sinh^2\left(\frac{1}{c}\right) = \frac{\ell^2 - u_1^2}{2c^2}.$$

That is,

$$c \sinh\left(\frac{1}{c}\right) = \frac{1}{2}\sqrt{\ell^2 - u_1^2} > 1. \tag{6.6}$$

In this way, we obtain a single transcendental equation for c. One can show $c \sinh(1/c)$ is monotone decreasing in c for $c > 0$ and takes every value greater than 1. Let us verify the equivalent assertions for the function $f(z) = \sinh z/z$. First of all if $z \searrow 0$, we have by L'Hopital's rule

$$\lim_{z\searrow 0} \frac{\sinh z}{z} = \lim_{z\searrow 0} \cosh z = 1 \quad \text{and} \quad \lim_{z\nearrow\infty} \frac{\sinh z}{z} = \lim_{z\nearrow\infty} \cosh z = \infty.$$

Also,

$$f'(z) = \frac{z\cosh z - \sinh z}{z^2}.$$

Setting $f_1(z) = z\cosh z - \sinh z$ we see $f_1(0) = 0$ and $f_1'(z) = z\sinh z > 0$ for $z > 0$. In particular, $f_1(z) > 0$ for $z > 0$, so $f'(z) > 0$ for $z > 0$. Also,

$$\lim_{z\searrow 0} f'(z) = \lim_{z\searrow 0} \frac{f_1'(z)}{2z} = 0.$$

We have shown that f takes every value on $[1, \infty)$ uniquely and has a well-defined inverse on that interval. Thus, we have a unique solution

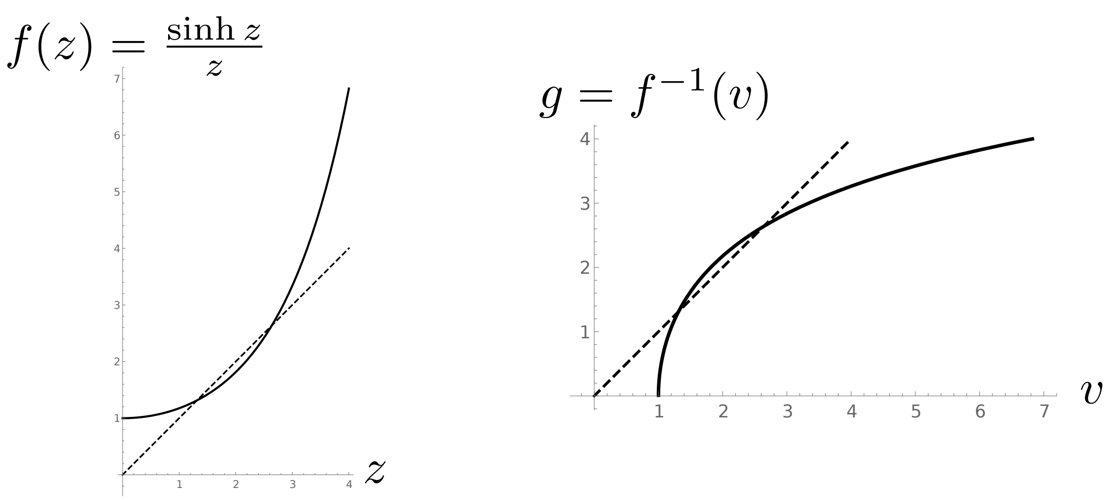

Figure 6.3: $\sinh z/z$ and its inverse

$$c = \frac{1}{f^{-1}\left(\frac{1}{2}\sqrt{\ell^2 - u_1^2}\right)}.$$

Once we know $c > 0$, we can expand (6.4) to see

$$-2c \sinh\left(\frac{1}{c}\right) \sinh\left(\frac{\mu}{c}\right) = u_1.$$

Therefore, substituting from (6.6),

$$\mu = -c \sinh^{-1}\left(\frac{u_1}{\sqrt{\ell^2 - u_1{}^2}}\right)$$

Setting $g = f^{-1}$, we also know

$$\lim_{v \searrow 1} g'(v) = +\infty.$$

In order to get an accurate approximation for $g = f^{-1}$ near $v = 1$, we compute the next derivative of f:

$$f''(z) = \frac{z^2 f_1' - 2z f_1}{z^4} = \frac{z^2 \sinh z - 2z \cosh z + 2 \sinh z}{z^3}.$$

Again, using L'Hopital's rule

$$\lim_{z \searrow 0} f''(z) = \lim_{z \searrow 0} \frac{\cosh z}{3} = \frac{1}{3}.$$

Thus, we have to leading order $v \sim 1 + g(v)^2/6$ or $g(v) \sim g_0(v) = \sqrt{6(v-1)}$.

It is less obvious how to obtain a simple approximation for $g(v)$ when v is large. Let us begin with an intermediate approximation obtained from the Taylor expansion of f at $z = 1$. We have

$$f(1) = \sinh(1), \qquad f'(1) = \cosh(1) - \sinh(1), \qquad \text{and} \qquad f''(1) = 3\sinh(1) - 2\cosh(1).$$

Thus, $v \sim \sinh(1) + (\cosh(1) - \sinh(1))(g-1) + (3\sinh(1) - 2\cosh(1))(g-1)^2/2$, and we have an approximation

$$g(v) \sim g_1(v) = 1 + \frac{\sinh(1) - \cosh(1) + \sqrt{\cosh^2(1) + \sinh(2) - 5\sinh^2(1) + 2(3\sinh(1) - 2\cosh(1))v}}{3\sinh(1) - 2\cosh(1)}.$$

This approximation is relatively accurate on a rather small interval about $\sinh(1)$. The average $(g_1(v) + g_2(v))/2$ is accurate for somewhat larger v.

For $v > 1$, we also have recourse to a recursive approximation scheme. Picking an initial value g_0, for example, we could take $g_0 = \sinh^{-1}(v)$ which will be smaller than $g(v)$ as long as $v > \sinh(1)$. We see the actual value $g(v)$ satisfies

$$\frac{\sinh g(v)}{g(v)} = v \qquad \text{or} \qquad g(v) = \sinh^{-1}(vg(v)).$$

This suggests setting $g_1 = \sinh^{-1}(vg_0)$ and $g_{j+1} = \sinh^{-1}(vg_j)$ in general for $j = 0, 1, 2, \ldots$.

Conjecture 1. *The sequence g_j tends (upward) to $g(v)$ with the estimate*

$$g(v) - g_{j+1} \le g_{j+1} - g_j.$$

For example, if we take $v = 33.6189 \approx \sinh[6]/6$, then

$$\begin{aligned} g_0 &= \sinh^{-1}(v) \approx 4.20846 \\ g_1 &= \sinh^{-1}(g_0 v) \approx 5.64534 \\ g_2 &= \sinh^{-1}(g_1 v) \approx 5.93907 \\ g_3 &= \sinh^{-1}(g_2 v) \approx 5.98979. \end{aligned}$$

6.1.3 Minimality of extremals

We have established the existence of a unique catenary extremal given by the graph of a function $u \in C^\infty[-1, 1]$ and satisfying

$$u(-1) = 0, \quad u(1) = u_1, \qquad \text{and} \qquad \int_{-1}^{1} \sqrt{1 + u'^2}\, dx = \ell.$$

The function u satisfies

$$u(x) = c \cosh\left(\frac{x - \mu}{c}\right) - c \cosh\left(\frac{1 + \mu}{c}\right) \tag{6.7}$$

where $c > 0$ is the unique solution of $c \sinh(1/c) = \sqrt{\ell^2 - u_1^2} \Big/ 2 > 0$, and

$$\mu = -c \sinh^{-1}\left(\frac{u_1}{\sqrt{\ell^2 - {u_1}^2}}\right).$$

We now wish to establish the following result.

Theorem 16. *The function u given in (6.7) is the unique minimizer of*

$$V[u] = \int_{-1}^{1} u\sqrt{1+u'^2}\,dx$$

on

$$\mathcal{A} = \{u \in C^1[-1,1] : u(-1) = 0,\ u(1) = u_1\}$$

subject to

$$L[u] = \int_{-1}^{1} \sqrt{1+u'^2}\,dx = \ell.$$

A fundamental difficulty in establishing this result is that the Lagrangian $F(z,p) = (z+\lambda)\sqrt{1+p^2}$ associated with the augmented functional $\mathcal{F} = V + \lambda L$ where

$$\lambda = c\cosh\left(\frac{1+\mu}{c}\right) > 0$$

is not (always) convex. Showing this is Problem 30 of Chapter 3 in Troutman. Following Troutman, we take the special case $u_1 = 0$. In this case $\mu = 0$, and the extremal is given by

$$u(x) = c\cosh\left(\frac{x}{c}\right) - \lambda \qquad \text{with} \qquad \lambda = c\cosh\left(\frac{1}{c}\right).$$

On the other hand, the function $u_0 \equiv 0$ satisfies $u_0 \in \mathcal{A}$, and $\delta\mathcal{F}_u[v] \equiv 0$. Taking $v = -u$, we have $u + v = u_0$ and showing $\mathcal{F}$ is **not** convex amounts to showing

$$\mathcal{F}[u_0] - \mathcal{F}[u] < 0$$

(under some circumstances). In fact,

$$\begin{aligned}
\mathcal{F}[u_0] - \mathcal{F}[u] &= \int_{-1}^{1} \lambda\,dx - \int_{-1}^{1} (u+\lambda)\sqrt{1+u'^2}\,dx \\
&= 2c\cosh\left(\frac{1}{c}\right) - c\int_{-1}^{1} \cosh^2\left(\frac{x}{c}\right)\,dx \\
&= 2c\cosh\left(\frac{1}{c}\right) - \frac{c}{2}\int_{-1}^{1} \left[\cosh\left(\frac{2x}{c}\right) + 1\right]\,dx \\
&= 2c\cosh\left(\frac{1}{c}\right) - \frac{c^2}{2}\sinh\left(\frac{2}{c}\right) - c \\
&c\left[2\cosh\left(\frac{1}{c}\right) - c\sinh\left(\frac{1}{c}\right)\cosh\left(\frac{1}{c}\right) - 1\right].
\end{aligned}$$

Since $x \sinh x \to \infty$ as $x \nearrow \infty$, we see that for $c > 0$ small enough

$$c \sinh\left(\frac{1}{c}\right) > 2,$$

and $\mathcal{F}[u_0] - \mathcal{F}[u] < 0$. Recalling that c is determined by

$$c \sinh\left(\frac{1}{c}\right) = \frac{1}{2}\sqrt{\ell^2 - u_1^2} = \frac{\ell}{2},$$

we find nonconvexity for chains of any length $\ell > 4$.

In spite of this nonconvexity, Troutman suggests a rephrasing of the problem which leads to a much stronger result than Theorem 16 above. The function u determines a parametric curve parameterized by arclength. This is given by the function $\mathbf{x} \in C^1([0, \ell] \to \mathbb{R}^2)$ by $\mathbf{x}(s) = (\xi(s), \eta(s))$ where

$$\begin{cases} \xi(s) = \mu + c \sinh^{-1}\left[\frac{s}{c} - \sinh\left(\frac{1+\mu}{c}\right)\right] \\ \eta(s) = u(\xi(s)) = c\cosh\left(\sinh^{-1}\left[\frac{s}{c} - \sinh\left(\frac{1+\mu}{c}\right)\right]\right) - c\cosh\left(\frac{1+\mu}{c}\right). \end{cases} \tag{6.8}$$

This parametric map $\mathbf{x}$ also satisfies

$$|\mathbf{x}'| \equiv 1 \qquad \text{and} \qquad 2 = \int_{-1}^{\ell} \xi'(s)\, ds = \int_0^{\ell} \sqrt{1 - \eta'^2}\, ds.$$

Now if we let $\mathbf{x} = (\xi, \eta) \in C^1([0, \ell] \to \mathbb{R}^2)$ be any parametric curve parameterized by arclength ($|\mathbf{x}'| \equiv 1$) with $\mathbf{x}(0) = (-1, 0)$ and $\mathbf{x}(\ell) = (1, u_1)$, then the potential energy expression

$$V[u] = \int_{-1}^{1} u\sqrt{1 + u'^2}\, dx$$

generalizes to

$$V_1[\mathbf{x}] = \int_0^{\ell} \eta\, ds.$$

To see this, we may again consider a portion of chain of mass $\Delta m_j = \rho \Delta s_j$ located at a point $\mathbf{x}(s_j^*)$. The potential energy of this particular section of chain is approximately

$$\int_0^{\eta} \rho g \Delta s_j\, dy = \rho g \eta \Delta s_j.$$

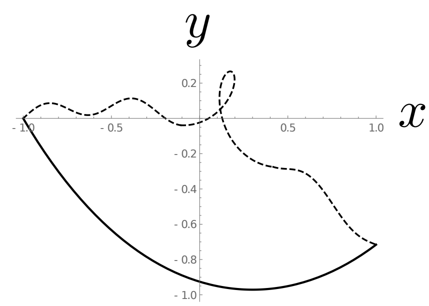

Figure 6.4: an parametric chain shape: These shapes are also not required to satisfy $-1 \le \xi \le 1$ though the one illustrated does. (Actually, this shape has length a little longer than the original catenary chain shape.)

Summing over a partition of such portions and taking the limit as the maximum length Δs_j tends to zero (and dividing out by the constant ρg as usual), we arrive at the expression for V_1 above. The following result treats these general parametric curves of length ℓ connecting $(-1, 0)$ to $(1, u_1)$ and asserts that the catenary graph extremal is the unique minimizer among such curves.

Theorem 17. *The catenary graph satisfying (6.8) is the unique minimizer of*

$$V_1[\mathbf{x}] = \int_0^\ell \eta \, ds$$

on

$$\mathcal{B} = \{\mathbf{x} \in C^1([0, \ell] \to \mathbb{R}^2) : \mathbf{x}(0) = (-1, 0),\ \mathbf{x}(\ell) = (1, u_1),\ |\mathbf{x}'| \equiv 1\}$$

subject to

$$L_1[\mathbf{x}] = \int_0^\ell \sqrt{1 - \eta'^2} \, ds = 2.$$

Finally, we simplify the previous result slightly and prove something even more general. It will be noted that the functionals appearing above only depend on the second coordinate function of $\mathbf{x}$, namely, $\eta \in C^1[0, \ell]$. Thus, it makes sense to extend their domains and rename them:

$$V_1 : C^1[0, \ell] \to \mathbb{R} \qquad \text{by} \qquad V_1[\eta] = \int_0^\ell \eta \, ds$$

and

$$L_1 : \{\eta \in C^1[0,\ell] : |\eta'(s)| \leq 1 \text{ for } 0 \leq s \leq \ell\} \to \mathbb{R} \quad \text{by} \quad L_1[\mathbf{x}] = \int_0^\ell \sqrt{1-\eta'^2}\, ds.$$

We now state the main result.

Theorem 18. *The second component of the parametric map defined in (6.8) is the unique minimizer of*

$$V_1[\eta] = \int_0^\ell \eta\, ds$$

on

$$\mathcal{B} = \{\mathbf{x} \in C^1([0,\ell] \to \mathbb{R}^2) : \eta(0) = 0,\ \eta(\ell) = u_1\}$$

subject to

$$L_1[\eta] = \int_0^\ell \sqrt{1-\eta'^2}\, ds = 2.$$

Notice the absence of the condition $|\mathbf{x}'| \equiv 1$ in the definition of $\mathcal{B}$. Notice, furthermore, that the functional L_1 is not (even) defined on all of $\mathcal{B}$, but only on

$$\mathcal{B}_1 = \{\eta \in \mathcal{B} : |\eta'(s)| \leq 1 \text{ for } 0 \leq s \leq \ell\}.$$

Proof of Theorem 18: We show first that η from (6.8) is the unique minimizer of

$$\mathcal{G}[\eta] = (V_1 - cL_1)[\eta] = \int_0^\ell \left[\eta - c\sqrt{1-\eta'^2}\right] ds$$

on $\mathcal{B}_1$ (without constraint). This follows from two facts

1. The augmented functional $\mathcal{G} = V_1 - cL_1$ is strictly convex on $\mathcal{B}_1$ in the sense of Definition 5.

2. The function η from (6.8) is an extremal for $\mathcal{G}$, that is $\delta\mathcal{G}_\eta[v] = 0$ whenever $\eta + v \in \mathcal{B}_1$.

If we can establish these two assertions, we may apply Theorem 14 on minimizing convex functionals. The strict convexity does not follow from our previous result because the augmented Lagrangian $G(z,p) = z - c\sqrt{1-p^2}$ is not strictly second order convex. We do have

$$D^2G = \begin{pmatrix} 0 & 0 \\ 0 & \frac{c}{(1-p^2)^{3/2}} \end{pmatrix}.$$

Therefore, for each $v \in C^1[0,\ell]$ such that $\eta + v \in \mathcal{B}_1$, we have

$$\begin{aligned} G(\eta+v,\eta'+v') - G(\eta,\eta') &= G_z(\eta,\eta')v + G_p(\eta,\eta')v' + \frac{c}{2(1-p_*^2)^{3/2}}v'^2 \\ &\geq G_z(\eta,\eta')v + G_p(\eta,\eta')v' \end{aligned}$$

with equality only if $v' = 0$ (pointwise). Integrating this inequality

$$\mathcal{G}[\eta+v]-\mathcal{G}[\eta] = \int_0^\ell [G_z(\eta,\eta')v+G_p(\eta,\eta')v']\,ds+\frac{c}{2}\int_0^\ell \frac{v'^2}{(1-p_*^2)^{3/2}}\,ds \geq \delta\mathcal{G}_\eta[v]$$

with equality only if $v' \equiv 0$. But if $\eta + v \in \mathcal{B}_1$, then $v(0) = v(\ell) = 0$, so equality implies $v \equiv 0$. This establishes the strict convexity of $\mathcal{G}$.

On the other hand, the Euler-Lagrange equation for $\mathcal{G}$ is

$$c\left(\frac{\eta'}{\sqrt{1+\eta'^2}}\right)' = 1$$

where the derivatives are with respect to the arclength s. To compute this for the function η from the arclength parameterization of the catenary we observe first that

$$s = \int_0^\xi \sqrt{1+u'^2}\,dx = c\sinh\left(\frac{\xi-\mu}{c}\right) + c\sinh\left(\frac{1+\mu}{c}\right).$$

Therefore,

$$\frac{d\xi}{ds} = \frac{1}{\cosh\left(\frac{\xi-\mu}{c}\right)}.$$

Having made this observation/calculation we have from (6.8)

$$\eta'(s) = \frac{du}{dx}(\xi)\frac{d\xi}{ds} = \frac{\sinh\left(\frac{\xi-\mu}{c}\right)}{\cosh\left(\frac{\xi-\mu}{c}\right)}.$$

Therefore,

$$\frac{d}{ds}\left(\frac{\eta'}{\sqrt{1-\eta'^2}}\right) = \frac{d}{dx}\left(\sinh\left(\frac{\xi-\mu}{c}\right)\right)\Big|_{x=\xi}\frac{d\xi}{ds} = \frac{1}{c},$$

and η is a C^2 classical extremal for $\mathcal{G}$. In particular, $\delta\mathcal{G}_\eta[v] \equiv 0$, and $\mathcal{G}[\eta + v] - \mathcal{G}[\eta] \geq 0$ whenever $\eta + v \in \mathcal{B}_1$ with equality only if $v \equiv 0$.

The usual argument of Theorem 10 now applies. That is, it happens that

$$L_1[\eta] = \int_0^\ell \sqrt{1-\eta'^2}\, ds = 2,$$

so for any $v \in C^1[0,\ell]$ such that $\eta + v \in \mathcal{B}$ and for which $L_1[\eta + v] = 2$, we have

$$V_1[\eta + v] - cL_1[\eta + v] = \mathcal{G}[\eta + v] \geq \mathcal{G}[\eta] = V_1[\eta] - cL_1[\eta]$$

with equality only if $v \equiv 0$. Since $L_1[\eta + v] = L_1[\eta] = 2$, we have

$$V_1[\eta + v] \geq V_1[\eta] \quad \text{with equality only if } v \equiv 0.$$

This establishes Theorem 18. □

Proof of Theorem 17: If $\tilde{\mathbf{x}} = (\tilde{\xi}, \tilde{\eta}) \in \mathcal{B}$ satisfies

$$\int_0^\ell \sqrt{1-\tilde{\eta}'^2}\, ds = 2$$

and $\mathbf{x}$ is the parametric catenary, then $\tilde{\eta} \in \mathcal{B}_1 \subset \mathcal{B}$ and satisfies $L_1[\tilde{\eta}] = 2$. Thus, by Theorem 18

$$V_1[\tilde{\mathbf{x}}] = V_1[\tilde{\eta}] \geq V_1[\eta] = V_1[\mathbf{x}] \quad \text{with equality only if } \tilde{\eta} \equiv \eta.$$

We have, in particular, $V_1[\tilde{\mathbf{x}}] \geq V_1[\mathbf{x}]$ for all $\tilde{\mathbf{x}} \in \mathcal{B}$ satisfying the constraint

$$L_1[\tilde{\mathbf{x}}] = 2.$$

In the case of equality we have $\tilde{\xi}' = \pm\sqrt{1-\eta'^2}$ and

$$2 = \int_0^\ell \tilde{\xi}'\, ds = \int_0^\ell \sqrt{1-\eta'^2}\, ds.$$

Since $\eta'(s) = 1$ for at most one arclength s, we conclude $\tilde{\xi} = \sqrt{1-\eta'^2}$ and $\tilde{\mathbf{x}} \equiv \mathbf{x}$. □

Finally we prove the initial (and weakest) assertion.
Proof of Theorem 16: If $\tilde{u} \in \mathcal{A}$ and

$$L[\tilde{u}] = \int_{-1}^{1} \sqrt{1 - \tilde{u}'^2}\, dx = \ell,$$

then the graph of $\tilde{u}$ may be parameterized by arclength to give a parameterized curve $\tilde{\mathbf{x}} \in \mathcal{B}$ satisfying the constraint

$$L_1[\tilde{\mathbf{x}}] = \int_0^{\ell} \sqrt{1 - \tilde{\eta}'^2}\, ds = 2.$$

By Theorem 17, we know $V_1[\tilde{\mathbf{x}}] \geq V_1[\mathbf{x}]$ with equality only if $\tilde{\mathbf{x}} = \mathbf{x}$. Changing variables, we find

$$V_1[\tilde{\mathbf{x}}] = \int_0^{\ell} \tilde{\eta}\, ds = \int_{-1}^{1} \tilde{u}\sqrt{1 + \tilde{u}'^2}\, dx = V[\tilde{u}]$$

and

$$V_1[\mathbf{x}] = \int_0^{\ell} \eta\, ds = \int_{-1}^{1} u\sqrt{1 + u'^2}\, dx = V[u].$$

The result evidently follows. □

Relations to physical parameters

If we wished to consider the right endpoint to have a general coordinate (c, d) with $c > 0$ and $c^2 + d^2 < \ell^2$, we could first make a choice of units so that the length c measures one unit in the new system. Equivalently, we consider the problem with right endpoint at $(1, d/c)$. If $d > 0$, we can reverse the endpoints.