Calculus of Variations
Homework 3

January 30, 2012

1. If $\Omega \subset \mathbb{R}^n$ is open and $f \in C^1(\Omega)$, show $f \in C^0(\Omega)$.

2. Sketch a proof of the following version of the fundamental lemma:

   If $f \in L^1[a,b]$ and
   $$\int_{[a,b]} f\eta = 0$$
   for every $\eta \in C_c^\infty[a,b]$, then $f = 0$ almost everywhere.

3. Formulate and prove a similar general version of the Lemma of DuBois-Reymond.

4. We say that u is a *Lipschitz* (weak) *extremal* if there is some constant c such that $|u(x_1) - u(x_2)| \leq c|x_1 - x_2|$ for all $x_1, x_2 \in [a,b]$ and
   $$\int_a^b [D_z F(x,u,u') \cdot \phi + D_p F(x,u,u') \cdot \phi'] = 0$$
   for every $\phi \in C_c^\infty[a,b]$.

   Prove that a Lipschitz weak extremal satisfies
   $$F_p(x, u(x), u'(x)) = c + \int_a^x F_z(\xi, u(\xi), u'(\xi))\, d\xi$$
   for almost every $x \in [a,b]$. You may use the fact that u' is defined almost everywhere and is in $L^1[a,b]$.