Assignment 8: Examples etc.
Due Tuesday November 8, 2022

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October 23, 2022

Problem 1 *(sequence spaces; Kreyszig Example 1.5-3 and Problem 2.3.1) Let* c *denote the collection of all* **convergent sequences** *in* $\mathbb{R}$ *(or alternatively* $c(F)$ *the collection of all convergent sequences in the field* F *with* $F = \mathbb{R}$ *or* $F = \mathbb{C}$*.)*

(a) *Show* c *is a subspace of* ℓ^∞*.*

(b) *Show* c *is a Banach space with the* ℓ^∞ *norm.*

Problem 2 *(sequence spaces; Kreyszig Problems 2.3.1-2) Let* c_0 *denote the collection of all sequences in* $\mathbb{R}$ *that* **converge to** 0 *(or alternatively* $c_0(F)$ *the collection of all sequences in the field* F *with* $F = \mathbb{R}$ *or* $F = \mathbb{C}$ *that converge to* $0 \in F$*.)*

(a) *Show* c_0 *is a subspace of* $c \subset \ell^\infty$*.*

(b) *Show* c_0 *is a Banach space with the* ℓ^∞ *norm.*

Problem 3 *(sequence spaces) Recall that* ℓ^p *for* $1 \leq p < \infty$ *denotes the sequence space of* p*-summable sequences, usually considered with respect to the norm*

$$\left\| \{x_j\}_{j=1}^{\infty} \right\| = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}. \tag{1}$$

Though this norm is essentially used to define ℓ^p*, because* p*-summable means precisely that the quantity in* (1) *is funite, we can also think of* ℓ^p *as a vector space with respect to other norms. In the following, we consider* ℓ^p *as a normed vector space with respect to the* ℓ^∞ *norm.*

(a) ℓ^p *is a subspace of* $c_0 \subset c \subset \ell^\infty$ *for* $1 \leq p < \infty$*.*

(b) $\ell^p \neq c_0$ *for* $1 \leq p < \infty$*.*

(c) ℓ^p *is not complete with respect to the* ℓ^∞ *norm for* $1 \leq p < \infty$*.*

(d) *What is the closure of* $\ell^p \subset \ell^\infty$ *with respect to the* ℓ^∞ *norm? (Hint: If you have trouble with this, try the next problem.)*

Problem 4 *(sequence spaces; Kreyszig Problems 2.3.3) Let* c_{00} *denote the collection of all sequences in* $\mathbb{R}$ *that are* **eventually** 0*, i.e,* $\{x_j\}_{j=1}^{\infty} \in c_{00}$ *if there exists some* $N > 0$ *such that* $x_j = 0$ *for* $j \geq N$*, (or alternatively* $c_{00}(F)$ *the collection of all sequences in the field* F *with* $F = \mathbb{R}$ *or* $F = \mathbb{C}$ *that are eventually* $0 \in F$*.)*

(a) *Show* c_{00} *is a subspace of* $\ell^p \subset c_0 \subset c \subset \ell^\infty$ *for* $1 \leq p < \infty$*.*

(b) *Show* c_{00} *is not complete with respect to the* ℓ^∞ *norm.*

(c) *What is the closure of* c_{00} *with respect to the* ℓ^∞ *norm?*

Problem 5 *(Schauder basis; Kreyszig Problem 2.3.10) In this problem let* ℓ^p *denote the* p*-summable sequences with respect to the* ℓ^p *norm given in* (1) *for* $1 \leq p < \infty$ *as usual.*

(a) *State precisely the definition of* **Schauder basis***.*

(b) *Show* $\{\mathbf{e}_j\}_{j=1}^{\infty}$ *where* $\mathbf{e}_j = \{\delta_{jk}\}_{k=1}^{\infty}$ *and*

$$\delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

is a Schauder basis for ℓ^p*.*

Problem 6 *(Schauder basis) Let us return, once again to consideration of the ℓ^∞ norm including for the vector spaces ℓ^p for $1 \le p < \infty$ as in Problem 3 and Problem 4 above.*

(a) *What is the relation of*

$$W = \mathrm{span}\{\mathbf{e}_j\}_{j=1}^{\infty}$$

where $\{\mathbf{e}_j\}_{j=1}^{\infty}$ is defined in Problem 5 above to the ("five") sequence spaces $c_{00} \subset \ell^p \subset c_0 \subset c \subset \ell^\infty$? Note: This question is purely algebraic; it has nothing to do with which norm is used.

(b) *For which of the "five" spaces above with respect to the ℓ^∞ norm is $\{\mathbf{e}_j\}_{j=1}^{\infty}$ a Schauder basis? What interesting thing does this tell you about Schauder bases?*

Problem 7 *(Kreyszig Problems 2.3.7-8; see also Assignment 5 Problem 7)*

(a) *Give an example of a normed space* X *and a sequence* $\{x_j\}_{j=1}^{\infty} \subset X$ *for which*

$$\sum_{j=1}^{\infty} \|x_j\| < \infty$$

but the sequence of partial sums $\{s_k\}_{k=1}^{\infty} \subset X$ *where*

$$s_k = \sum_{j=1}^{k} x_j$$

has no convergent subsequence.

(b) *Show that if* X *is a normed space and* $\{x_j\}_{j=1}^{\infty} \subset X$ *satisfies*

$$\sum_{j=1}^{\infty} \|x_j\| < \infty,$$

then the sequence of partial sums $\{s_k\}_{k=1}^{\infty} \subset X$ *where*

$$s_k = \sum_{j=1}^{k} x_j$$

is a Cauchy sequence.

(c) *Show that if* X *is a normed space, but* X *is* **not** *a Banach space, i.e.,* X *is not complete, then there exists a sequence* $\{x_j\}_{j=1}^{\infty} \subset X$ *with*

$$\sum_{j=1}^{\infty} \|x_j\| < \infty$$

and sequence of partial sums $\{s_k\}_{k=1}^{\infty} \subset X$ *where*

$$s_k = \sum_{j=1}^{k} x_j$$

having no convergent subsequence.

Problem 8 *(L^p spaces; Section 1.4 in my notes) Let $\mathfrak{M}$ denote the Lebesgue measurable sets in an interval (a, b), and let $\mu : \mathfrak{M} \to [0, \infty)$ denote Lebesgue measure.*

(a) *State the definition of what it means for a function $f : (a, b) \to \mathbb{R}$ to be* **measurable**.

(b) *Show that if $f : (a, b) \to \mathbb{R}$ is measurable, then*

$$\{x \in (a, b) : f(x) \neq 0\}$$

is a measurable set.

(c) *State the definition of what it means for a measurable function $f : (a, b) \to \mathbb{R}$ to be* **essentially bounded**.

(d) *Let $\mathcal{L}^\infty(a, b)$ denote the collection of essentially bounded measurable functions $f : (a, b) \to \mathbb{R}$. Show $\sigma : \mathcal{L}^\infty(a, b) \to [0, \infty)$ by*

$$\sigma(f) = \inf\{M > 0 : \mu\{x : |f(x)| \geq M\} = 0\}$$

is a seminorm on $\mathcal{L}^\infty(a, b)$.

Problem 9 *(L^p spaces; Section 1.4 in my notes; Assignment 5 Problem 10) Let $\mathfrak{M}$ denote the Lebesgue measurable sets in an open set $U \subset \mathbb{R}^1$, and let $\mu : \mathfrak{M} \to [0, \infty)$ denote Lebesgue measure. A function $\tau : U \to \mathbb{R}$ is said to be a* **measurable simple function** *if there exist finitely many sets $A_1, A_2, \ldots, A_k \in \mathfrak{M}$ and corresponding $c_1, c_2, \ldots, c_k \in \mathbb{R}$ such that*

$$\tau(x) = \sum_{j=1}^{k} c_j \chi_{A_j}(x) \tag{2}$$

where χ_A denotes the **characteristic function** $\chi_A : \mathbb{R} \to \mathbb{R}$ **of the set** *A given by*

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \in \mathbb{R}\backslash A. \end{cases}$$

Let $\Sigma(U)$ denote the collection of all measurable simple functions on U.

(a) *Show $\Sigma(U)$ is a (real) vector space.*

(b) *Given a simple function τ of the form* (2) *show there exist pairwise disjoint measurable sets $E_1, E_2, \dots E_\ell$ and corresponding constants $\gamma_1, \gamma_2, \dots, \gamma_\ell \in \mathbb{R}$ such that*

$$\tau(x) = \sum_{j=1}^{k} c_j \chi_{A_j}(x) = \sum_{j=1}^{\ell} \gamma_j \chi_{E_j}(x). \tag{3}$$

(c) *Is there a necessary inequality ($\ell \leq k$ or $k \leq \ell$) between the natural numbers ℓ and k in* (3)*?*

(d) *Given a simple function $\tau \in \Sigma(U)\backslash\{\mathbf{0}\}$ of the form* (2) *show there exist* **unique** *pairwise disjoint measurable sets $E_1, E_2, \dots E_\ell$ and corresponding* **distinct nonzero** *constants $\gamma_1, \gamma_2, \dots, \gamma_\ell \in \mathbb{R}$ such that*

$$\tau(x) = \sum_{j=1}^{k} c_j \chi_{A_j}(x) = \sum_{j=1}^{\ell} \gamma_j \chi_{E_j}(x). \tag{4}$$

(e) *Is there a necessary inequality ($\ell \leq k$ or $k \leq \ell$) between the natural numbers ℓ and k in* (4)*?*

(f) *Define the* **integral of a simple function** $\tau \in \Sigma(U)$ *having the form* (2) *by*

$$\int_U \tau = \sum_{j=1}^{k} c_j \mu A_j.$$

Show $\int_U \tau$ is well-defined.

(g) *Show that for each $\tau \in \Sigma(U)$, the function $|\tau| : U \to \mathbb{R}$ by $|\tau|(x) = |\tau(x)|$ satisfies $|\tau| \in \Sigma(U)$.*

(h) *Show $\sigma : \Sigma(U) \to [0, \infty)$ by*

$$\sigma(\tau) = \inf\{M > 0 : \mu\{x : |\tau(x)| \geq M\} = 0\}$$

is a seminorm on $\Sigma(U)$ (but not a norm).

(i) *Show $\sigma_1 : \Sigma(U) \to [0, \infty)$ by*

$$\sigma_1(\tau) = \int_U |\tau|$$

is a seminorm on $\Sigma(U)$ (but not a norm).

Problem 10 *(L^p spaces; Section 1.4 in my notes) Let $\mathfrak{M}$ denote the Lebesgue measurable sets in an open set $U \subset \mathbb{R}^1$, and let $\mu : \mathfrak{M} \to [0, \infty)$ denote Lebesgue measure.*

(a) *(pointwise L^∞ on an open set) Let $\mathcal{L}^\infty(U)$ denote the collection of measurable functions $f : U \to \mathbb{R}$ for which*

$$\sigma(f) = \inf\{M > 0 : \mu\{x : |f| \geq M\} = 0\}. \tag{5}$$

You may wish to verify that $\{x : |f| \geq M\}$ is measurable, so that σ is well-defined. Show that $\mathcal{L}^\infty(U)$ is a vector space and $\sigma : \mathcal{L}^\infty(U) \to [0, \infty)$ given by (5) is a seminorm on $\mathcal{L}^1(U)$ (but not a norm).

(b) *(locally bounded functions) Let $\mathcal{L}^\infty_{loc}$ denote the collection of measurable functions $f : U \to \mathbb{R}$ for which $f \in \mathcal{L}^\infty(V)$ for every open set $V \subset\subset U$, meaning $\overline{V}$ is compact and $\overline{V} \subset U$. Show $C^0(U)$ is a subspace of $\mathcal{L}^\infty_{loc}(U)$ but $C^\infty(U)$ is not a subspace of $\mathcal{L}^\infty(U)$.*

(c) *(pointwise L^1 functions on an open set) Let $\mathcal{L}^1(U)$ denote the collection of measurable functions $f : U \to \mathbb{R}$ for which*

$$\sigma_1(f) = \int_U |f| = \sup_{\tau \leq f} \int_U \tau < \infty \tag{6}$$

where the supremum is taken over all simple functions $\tau : U \to \mathbb{R}$. Show $\sigma_1 : \mathcal{L}^1(a, b) \to [0, \infty)$ defined by (6) is a seminorm on $\mathcal{L}^1(a, b)$ (but not a norm).

(d) *(locally integrable functions) Let $\mathcal{L}^1_{loc}$ denote the collection of measurable functions $f : U \to \mathbb{R}$ for which $f \in \mathcal{L}^1(V)$ for every open set $V \subset\subset U$. Show $\mathcal{L}^\infty_{loc}(U)$ is a subspace of $\mathcal{L}^1_{loc}(U)$ but $C^\infty(U)$ is not a subspace of $\mathcal{L}^1(U)$.*

(e) *Show $\mathcal{L}^\infty(U)$ is a subspace of $\mathcal{L}^1(U)$, and*

$$\left\{(f, g) \in X \times X : \int_U |f - g| = \sigma_1(f - g) = 0\right\}$$

is an equivalence relation on X where $X = \mathcal{L}^\infty(U)$ or $X = \mathcal{L}^1(U)$.