Assignment 7: Inner Product Spaces etc.
Due Tuesday November 1, 2022

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Problem 1 *(seminorms; my notes Chapter 7) Let X be a normed space, and $\sigma : X \to [0,\infty)$ a seminorm on X. Show that if there is some $M > 0$ for which*

$$\|x\| \le M \qquad \text{for every } x \in X \text{ with } \sigma(x) \le 1,$$

then σ is a norm. In this case, are σ and $\| \cdot \|$ equivalent?

Problem 2 *(convexity) Let V be a vector space.*

(a) *State precisely the definition of convexity as it applies to a subset of V.*

(b) *Show every subspace of a vector space V is convex.*

Problem 3 *(orthogonal space) Let X be an inner product space and let $S \subset X$ be any nonempty subset.*

(a) *Show*

$$S^\perp = \{x \in X : \langle x, v\rangle = 0 \text{ for all } v \in S\}$$

is a subspace of X.

(b) *Show $S^\perp$ is closed. (Kreyszig Problem 3.3.8)*

(c) *Show $S^{\perp\perp} \supset S$.*

Problem 4 *(Kreyszig's proof of Theorem 3.3-2) Let W be a subspace of an inner product space X, and let $q \in W$ is the nearest point projection of a point p so that*

$$\text{dist}(p, W) = \|q - p\|.$$

Assume there is some $w \in W$ with $\langle p - q, w \rangle \neq 0$, and compute

$$\|p - q - \alpha w\|^2 \qquad \text{for } \alpha \in F.$$

Take

$$\alpha = \frac{\langle p - q, w \rangle}{\|w\|^2}$$

and obtain a contradiction. What other result is proved using a very similar technique?

Problem 5 *Let S be any nonempty subset of a finite dimensional inner product space X. Show $X = S^{\perp\perp} \oplus S^{\perp}$. (See Problem 3 above.)*

Problem 6 *Let $\mathcal{H}$ be a Hilbert space and let W be a proper closed subspace in $\mathcal{H}$. Then $\mathcal{H} = W \oplus W^{\perp}$, and there is a well-defined function/projection map $\pi_W : \mathcal{H} \to W$ given by by*

$$\pi_W(x) = q \quad \text{where } x = q + z \text{ with } q \in W \text{ and } z \in W^{\perp}.$$

Show the following

(a) $\pi_W \in \beth(\mathcal{H} \to W)$, *i.e.,* π_W *is linear and continuous.*

(b) π_W *is surjective (onto W).*

(c) $\pi_W\big|_W : W \to W$ *is surjective, in fact,*

$$\pi_W\big|_W = \mathrm{id}_W.$$

(d) $\pi_W\big|_{W^\perp} : W^\perp \to \{\mathbf{0}\}$, *i.e.,*

$$\pi_W\big|_{W^\perp} \equiv \mathbf{0}.$$

(e) $\pi_W^2 : \mathcal{H} \to W$ *by* $\pi_W^2(x) = \pi_W \circ \pi_W(x)$ *satisfies*

$$\pi_W^2 = \pi_W.$$

Such a function is said to be **idemponent of order 2**.

Problem 7 *(Problem 2.10.13 of Kreyszig) If S is any nonempty subset of a normed space X, the* **annihilator** $\mathsf{a}(S)$ *of the set S is defined to be*

$$\mathsf{a}(S) = \left\{ \phi \in \beth(X \to F) : \phi\big|_S \equiv 0 \right\}.$$

(a) *Show $\mathsf{a}(S)$ is a subspace of $\beth(X \to F)$.*

(b) *Compute $\mathsf{a}(X)$ and $\mathsf{a}(\{\mathbf{0}\})$.*

(c) *The* **algebraic annihilator** $\mathsf{a}^{alg}(S)$ *of a nonempty subset $S \subset V$ in any vector space V is*

$$\mathsf{a}^{alg}(S) = \left\{ \phi \in \mathcal{L}(V \to F) : \phi\big|_S \equiv 0 \right\}.$$

Show $\mathsf{a}^{alg}(S)$ is also a subspace of the algebraic dual $V^{alg\ dual} = \mathcal{L}(V \to F)$.

(d) *If X is a normed space, the (continuous) dual space $X' = \beth(X \to F)$ is a subspace of $X^{alg\ dual}$ and the (continuous) annihilator $\mathsf{a}(S)$ of a subset S is a subspace of the algebraic annihilator $\mathsf{a}^{alg}(S)$.*

(e) *If X is an inner product space, find an injection $\psi : S^\perp \to \mathsf{a}(S)$.*

(f) *If $\mathcal{H}$ is a Hilbert space, find a bijection $\psi : S^\perp \to \mathsf{a}(S)$.*

When we have discussed (more fully) the norm on the dual space $X' = \beth(X \to F)$ of a normed space, then it will make sense to address topological questions concerning the annihilator. We will show, for example, that if X is a normed space and $S \subset X$, then $\mathsf{a}(S)$ is closed. Also, if $\mathcal{H}$ is a Hilbert space and $S \subset \mathcal{H}$, then $S^\perp$ and $\mathsf{a}(S)$ are isomorphic as Hilbert spaces.

Problem 8 *(Theorem 3.3-4 of Kreyszig—generalization) If W is a complete subspace of an inner product space X (which may not be complete) and $p \in X$, then there exists a unique $q \in W$ for which*

$$\operatorname{dist}(p, W) = \|q - p\|.$$

Furthermore, $q - p \in W^\perp$ and $X = W \oplus W^\perp$

Problem 9 *(Riesz representation) If $\phi \in \beth(X \to F)$ where X is an inner product space and $\mathcal{N}(\phi)$ is complete, then there exists a unique $w \in X$ for which $\phi(x) = \langle x, w \rangle$ for all $x \in X$.*

Problem 10 *(Kreyszig Problems 3.2.5-6) Show the following:*

(a) *Let* $\{z_n = a_n + ib_n\}_{n=1}^{\infty} \subset \mathbb{C}$ *with* $a_n, b_n \in \mathbb{R}$ *for* $n = 1, 2, 3, \ldots$ *and* $z = a + ib \in \mathbb{C}$ *with* $a, b \in \mathbb{R}$ *be complex numbers for which the following hold:*

$$\lim_{n\to\infty} |z_n| = |z| \qquad \text{and} \qquad \lim_{n\to\infty} z_n \overline{z} = \|z\|^2.$$

Show

$$\lim_{n\to\infty} z_n = z.$$

(b) *Let* X *be a normed space. Consider* $\{x_n\}_{n=1}^{\infty} \subset X$ *and* $x \in X$ *for which the following hold:*

$$\lim_{n\to\infty} \|x_n\| = \|z\| \qquad \text{and} \qquad \lim_{n\to\infty} \langle x_n, x\rangle = \|x\|^2.$$

Show

$$\lim_{n\to\infty} x_n = x.$$