Assignment 5: Examples
Due Tuesday October 11, 2022

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Problem 1 *(ℓ^p and other norms on $\mathbb{R}^n$; Kreyszig Problems 2.2.6,8,10) Plot, i.e., draw pictures of, the unit spheres*

$$\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$$

when $n = 2$ and 3 for the following norms:

$$\|\mathbf{x}\|_1 = \sum_{j=1}^{n} |x_j|.$$

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}, \qquad 1 < p < 2.$$

$$\|\mathbf{x}\|_2 = \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2}.$$

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}, \qquad 2 < p < \infty.$$

$$\|\mathbf{x}\|_\infty = \max_{1 \le j \le n} |x_j|.$$

Problem 2 *(Kreyszig Problem 1.5.9) Show the uniform limit of a sequence of continuous functions (on a closed interval of $\mathbb{R}$) is continuous. More precisely, assume $\{f_j\}_{j=1}^{\infty} \subset C^0[a,b]$ where $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ satisfy*

$$\lim_{j \to \infty} \sup_{a \le x \le b} |f_j(x) - f(x)| = 0,$$

and show $f \in C^0[a,b]$.

Problem 3 *(L^1 norm) Show* $\|\cdot\|_{L^1} : C^0[a,b] \times C^0[a,b] \to \mathbb{R}$ *by*

$$\|f\|_{L^1} = \int_a^b |f(x)|\, dx$$

defines a norm on $C^0[a,b]$*, but that norm is not induced by an inner product.*

Problem 4 *Give a detailed proof of the fact that the* L^p *norm is positive definite on* $C^0[a,b]$*. More precisely, given* $a, b \in \mathbb{R}$ *with* $a < b$ *and a function* $f \in C^0[a,b]$ *for which*

$$\int_a^b |f(x)|^p\, dx = 0,$$

show $f(x) = 0$ *for every* $x \in [a,b]$.

Problem 5 *Prove the fundamental lemma of the calculus of variations: If* $f \in C^0[a,b]$ *and*

$$\int_a^b f(x)\phi(x)\, dx = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b),$$

then $f(x) = 0$ *for every* $x \in [a,b]$.

Problem 6 *Let* $\{f_j\}_{j=1}^\infty \subset C^0[a,b]$ *be a sequence of continuous functions converging with respect to the* L^p *norm to a continuous function* $f \in C^0[a,b]$*. Let*

$$A = \left\{ x \in [a,b] : f(x) = \lim_{j\to\infty} f_j(x) \right\}$$

be the **set of pointwise convergence** *of the sequence, let* $B = [a,b]\backslash A$ *and* $C = (a,b)\backslash A$*. Find examples for which*

(a) $B = \{a\}$.

(b) $C = \{x_0\}$ *for some* $x_0 \in (a,b)$.

(c) $\overline{C} = [a,b]$.

Problem 7 *(absolute convergence in normed spaces; Kreyszig problems 2.3.7–2.3.9)* *Let* $\{x_j\}_{j=1}^{\infty} \subset X$ *be a sequence in a Banach space* X*. Assume*

$$\sum_{j=1}^{\infty} \|x_j\| < \infty.$$

Show there is some $x \in X$ *for which*

$$\lim_{k\to\infty} \sum_{j=1}^{k} x_j = x.$$

Problem 8 *We gave as examples of closed proper subspaces of* $C^0[a,b]$ *with respect to the* C^0 *norm (or any* L^p *norm) the finite dimensional subspaces spanned by a finite collection of functions. We also know the polynomials* $\mathcal{P}$ *are an infinite dimensional proper subspace, but it turns out* $\mathcal{P}$ *is not closed. Give an example of an infinite dimensional proper subspace of* $C^0[a,b]$ *which is closed (and hence complete) with respect to the* C^0 *(or* L^∞*) norm.*

Problem 9 *(measure; my notes section 1.3.5 subsection on the Lebesgue spaces* L^p*)* *Given a* σ*-algebra* $\mathcal{A}$ *of subsets of a set* X*, a function* $\mu : \mathcal{A} \to [0,\infty]$ *is a* **general measure** *if*

(i) $\mu(\phi) = 0$,

(ii) *If* $A_1, A_2, A_3, \ldots$ *is a sequence of pairwise disjoint sets in* $\mathcal{A}$*, then*

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j).$$

Notice that this definition allows μ *to take the value* $+\infty$ *in the extended real numbers and does not require* μ *to be* σ*-finite.*

(a) *State precisely the definition of* σ**-algebra** *and what it means for a measure to be* σ**-finite**.

(b) *(Royden Chapter 11 Problem 20) Show that* **cardinality** $\# : 2^X \to [0,\infty]$ *with* $\#(A)$ *giving the number of elements in a set* A *is a general measure on any set* X*. This measure is called* **counting measure**.

(c) *Let* $\Sigma = \Sigma^+$ *denote the collection of all* **non-negative simple functions** $\phi : X \to [0, \infty]$ *given by*

$$\phi(x) = \sum_{j=1}^{k} c_j \chi_{A_j}(x)$$

where χ_A *is the characteristic function of the set* A*, the values* $c_1, c_2, \ldots, c_k \in [0, \infty]$ *and* $A_1, A_2, \ldots, A_k$ *are pairwise disjoint subsets in* X*. Let us define integration* $I_\Sigma : [0, \infty]^X \to \mathbb{R}$ *and* $I : [0, \infty]^X \to [0, \infty]$ *on the collection* $[0, \infty]^X$ *of all real valued function on* X *by*

$$I_\Sigma[\phi] = \sum_{j=1}^{k} c_j \#(A_j) \qquad \text{and} \qquad I[f] = \sup\{I_\Sigma[\phi] : \phi \in \Sigma \text{ with } \phi \le f\}$$

respectively. Show that if the integral $\int f = I[f]$ *satisfies*

$$\int f < \infty$$

for some nonnegative real valued function $f \in [0, \infty]^X$*, then*

$$\{x \in X : f(x) > 0\} \qquad \textit{is } \sigma\textit{-finite.}$$

Note: *For parts* **(b)** *and* **(c)** *above we have used cardinality* $\# : 2^X \to [0, \infty]$ *in a "course" manner, so that the value of* $\#$ *is either finite or infinite. Of course, we can think of cardinality in the more usual precise manner saying* $\#(A) = \#(B)$ *if there exists a bijection* $\nu : A \to B$ *and then ordering infinite sets by cardinality. We use this distinction among infinite sets in part* **(d)**.

(d) *(Royden's bizzare example, chapter 11 section 1) Show*

$$\mathfrak{C} = \{A \subset X : \#(A) \le \#(\mathbb{N}) \text{ or } \#(X \backslash A) \le \#(\mathbb{N})\}$$

is a σ*-algebra on any* **uncountable** *set* X*, i.e., a set with* $\#(X) > \#(\mathbb{N})$*, and* $\beta : \mathfrak{C} \to \{0, 1\}$ *by*

$$\beta(A) = \begin{cases} 0, & \#(A) \le \#(\mathbb{N}) \\ 1, & \#(X \backslash A) \le \#(\mathbb{N}) \end{cases}$$

is a general measure.

(e) *(Dirac measure) Given* $a, b \in \mathbb{R}$ *with* $a < b$ *and a point* $x_0 \in [a, b]$, *the* **evaluation functional** $\mathcal{E} \in \daleth(C^0[a, b] \to \mathbb{R})$ *is given by* $\mathcal{E}[f] = f(x_0)$. *This is also sometimes called the* **Dirac functional**; *there is no such thing as a Dirac* δ *function. There is, however, also a* **Dirac measure**. *If* $p \in X$ *is an element of any set* X, *then* $\delta : 2^X \to \{0, 1\}$ *by*

$$\delta(A) = \begin{cases} 1, & p \in A \\ 0, & p \in X \backslash A. \end{cases}$$

This is also sometimes called an **atomic measure** *and the singleton set* $\{p\} \subset X$ *is called an* **atom**. *Show* δ *is a measure, and define integration with respect to this measure. What is the relation between the evaluation functional and the Dirac measure?*

Problem 10 *(simple functions and measurable functions) For this problem, let* $a, b \in \mathbb{R}$ *with* $a < b$, *and consider simple functions* $\phi : [a, b] \to \mathbb{R}$ *given by*

$$\phi(x) = \sum_{j=1}^{k} c_j \chi_{A_j}(x) \tag{1}$$

where χ_A *is the characteristic function of the set* A, *the values* $c_1, c_2, \ldots, c_k \in \mathbb{R}$, *and* $A_1, A_2, \ldots, A_k \in \mathfrak{M}$ *are pairwise disjoint Lebesgue measurable sets in* $[a, b]$. *Let us denote the collection of these simple functions by* Σ.

(a) *Show* Σ *is a real vector space.*

(b) *Recall that a function* $f : [a, b] \to \mathbb{R}$ *is* **Lebesgue measurable** *if*

$$\{x \in [a, b] : f(x) \in U\} \in \mathfrak{M} \qquad \text{for every open set } U \subset \mathbb{R}.$$

Show that if $f : [a, b] \to \mathbb{R}$ *satisfies*

$$\{x \in [a, b] : f(x) > t\} \in \mathfrak{M} \qquad \text{for every } t \in \mathbb{R},$$

then f *is Lebesgue measurable. Hint(s): Recall that* $\mathfrak{M}$ *is a* σ*-algebra and every open set in* $\mathbb{R}$ *is a countable union of open intervals.*

(c) *Show every simple function in* Σ *is Lebesgue measurable and every characteristic function* χ_N *where* N *is a non-measurable set, i.e.,* $N \subset [a, b]$ *by* $N \notin \mathfrak{M}$, *is not measurable.*

(d) *Show the integral of $\phi \in \Sigma$ is well-defined by*

$$\int_{[a,b]} \phi = \sum_{j=1}^{k} c_j \mu(A_j)$$

where μ is Lebesgue measure. The point is that the sets $A_1, A_2, \ldots, A_k$ in the representation (1) are not uniquely determined.

(e) *Show*

$$\int_{[a,b]} \phi = \sum_{j=1}^{k} c_j \mu(A_j)$$

whenever

$$\phi(x) = \sum_{j=1}^{k} c_j \chi_{A_j}(x) \qquad \text{for all } x \in [a,b]$$

for measurable sets $A_1, A_2, \ldots, A_k \in \mathfrak{M}$. The point is that the sets $A_1, A_2, \ldots, A_k$ in this representation are not assumed to be disjoint.