Assignment 4: Examples
Due Tuesday October 4, 2022

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October 5, 2022

Problem 1 *(ℓ^p norms) Prove that for any p_1 and p_2 with $1 \le p_1 < p_2 \le \infty$ there holds*

$$\ell^{p_1} \subset \ell^{p_2},$$

and

$$\left\| \{a_j\}_{j=1}^{\infty} \right\|_{\ell^{p_2}} \le \left\| \{a_j\}_{j=1}^{\infty} \right\|_{\ell^{p_1}}$$

for every sequence $\{a_j\}_{j=1}^{\infty}$.

Problem 2 *(Kreyszig Problem 1.5.2) Show directly that F^n is complete under the ℓ^p norm for $1 \le p \le \infty$.*

Problem 3 *(Kreyszig Problem 1.5.2) Show that if a normed space X is complete, i.e., is a Banach space, under one norm, then X is also a Banach space under any equivalent norm. Use this to give another proof that F^n is complete under the ℓ^p norm. Hint: Recall Problem 3 of Assignment 3.*

Problem 4 *(Kreyszig Example 1.5-4) Show ℓ^p is complete for $1 \le p < \infty$.*

Problem 5 *Consider the ℓ^p norm on F^n. For which p and n is this norm induced by an inner product?*

Problem 6 *Show the dual space of ℓ^2 is Hilbert space isomorphic to ℓ^2. Hint: Use the Riesz representation theorem. You also need to define an inner product on $(\ell^2)' = \beth(\ell^2 \to \mathbb{R})$ and show it is well-defined.*

Problem 7 *(closures; Kreyszig Theorem 2.7-11, Exercise 84 in my notes) If W is a subspace of a normed space X and $L \in \beth(W \to Y)$ where Y is a Banach space, then show the following:*

(a) *There exists a unique extension $M \in \beth(\overline{W} \to Y)$ satisfying*

$$M\big|_W \equiv L.$$

Recall Exercise 7(b) *of* Assignment 3 *in which it was shown $\overline{W}$ is a (closed) vector space.*

(b) *Show the minimal modulus of continuity/operator norm of the extension, i.e.,*

$$\|M\|_{\beth} = \min\{c \geq 0 : \|Mx\| \leq c\|x\| \text{ for all } x \in \overline{W}\}$$

is equal to the operator norm of L.

Problem 8 *(linear isometries) Consider a linear function $\phi : X \to Y$ where X and Y are normed spaces. We say ϕ is a* **linear isometry** *if ϕ is linear and*

$$\|\phi(x)\| = \|x\| \qquad \text{for all } x \in X. \tag{1}$$

If X and Y are inner product spaces, we also require (or require instead)

$$\langle \phi(x), \phi(y) \rangle = \langle x, y \rangle \qquad \text{for all } x, y \in X. \tag{2}$$

(a) *What is the operator norm of an isometry?*

(b) *Show* (2) *implies* (1) *with respect to the induced norm.*

(c) *Show that if $\| \cdot \| : X \to [0, \infty)$ and $\| \cdot \| : Y \to [0, \infty)$ are the induced norms on the inner product spaces X and Y, then* (1) *implies* (2).

(d) *Show every linear isometry is injective.*

Problem 9 *Give an example of a closed proper subspace W of an inner product space X such that W is not complete. Find the completion of your example W.*

Problem 10 *(Kreyszig Problem 3.2.3) Let $\mathcal{P} = \mathcal{P}[t]$ denote the real polynomials in t considered as a subspace of the inner product space $L^2(a,b)$.*

(a) *Show $X = \operatorname{span}\{1, t, t^2\}$ is complete.*

(b) *Use Gram-Schmidt orthonormalization to find an orthonormal basis $\{q_0, q_1, q_2\}$ for X with $q_2(t) = t^2/c$ with*

$$c = \sqrt{\frac{b^5 - a^5}{5}}.$$

(c) *Is $W = \{p \in X : p(a) = 0\}$ a subspace of X?*

(d) *Is $Z = \{p \in X : \deg(p) = 2\}$ a subspace of X?*