Assignment 3: Inner Product Spaces
Kreyszig Chapter 3
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Problem 1 *(product norm; Kreyszig Problems 2.1.13 and 2.3.15) Let X and Y be normed spaces over the same field. In most instances it is easy to see which norm is being used by the context, but if we want to emphasize or clarify a particular usage we can use $\| \cdot \|_X$ for the norm on X and $\| \cdot \|_Y$ for the norm on Y.*

(a) *Show $X \times Y = \{(x, y) : x \in X, y \in Y\}$ becomes a vector space with the operations of addition given by*

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and scaling

$$\alpha(x, y) = (\alpha x, \alpha y).$$

(b) *Show the vector space $X \times Y$ becomes a normed space with norm*

$$\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\}. \tag{1}$$

Problem 2 *(equivalent norms; Kreyszig section 2.4) Let X be a normed space with two norms $\| \cdot \|_1$ and $\| \cdot \|_2$. These two norms are said to be* **equivalent norms** *if there exist $m_1, m_2 > 0$ such that*

$$\|x\|_2 \leq m_1 \|x\|_1 \qquad \text{and} \qquad \|x\|_1 \leq m_2 \|x\|_2 \qquad \textit{for all } x \in X.$$

(a) *(Kreyszig Problem 2.4.4) Show equivalent norms determine the same metric topology on a space X.*

(b) *Show*

$$\|(x, y)\| = \sqrt{\|x\|_X^2 + \|y\|_Y^2} \qquad \textit{for } x \in X \textit{ and } y \in Y \tag{2}$$

determines a norm on the Cartesian product $X \times Y$ of normed spaces X and Y.

(c) *Show the generalized Euclidean norm defined in* (2) *is equivalent to the max norm defined in* (1).

Problem 3 *(Kreyszig Theorem 2.4-5) Show that any two norms on a finite dimensional vector space X are equivalent. Hint: Use Kreyszig's linear combinations lemma (Lemma 2.4-1 in Kreyszig or Lemma 7 and Exercise 18 in my notes). This was used to show every linear operator on a finite dimensional space is continuous. Suggestion: Try to give the proof without looking at Kreyszig's proof of Theorem 2.4-5.*

Problem 4 *(inner product on a Cartesian product) Given two inner product spaces X and Y, define an inner product on the Cartesian product $X \times Y$. What is the induced norm?*

Problem 5 *(Pythagorean identity; Kreyszig Problem 3.1.2-3) Let X be an inner product space.*

(a) *If X is a real inner product space then show two vectors $x, y \in X$ satisfy $x \perp y$ if and only if*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

(b) *If X is a complex inner product space then show two vectors $x, y \in X$ satisfy $x \perp y$ only if*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2. \tag{3}$$

(c) *Under what circumstances can it be the case for two vectors $x, y \in X$ that* (3) *holds but x is not orthogonal to y?*

Problem 6 *(Closures, Kreyszig section 1.3) Recall that a set $A \subset X$ in a topological space X is* **closed** *if the complement $A^c = X \backslash A$ is open. The intersection of all closed sets containing a given set A is called the* **closure** *of A and is denoted*

$$\overline{A} = \bigcap_{C^{\text{closed}} \supset A} C.$$

(a) *Show that in any normed space X the closure of an open ball $B_r(p) = \{x \in X : \|x - p\| < r\}$ satisfies*

$$\overline{B_r(p)} = \{x \in X : \|x - p\| \leq r\}.$$

(b) *Given any set X with at least two elements, show the function $d : X \times X \to \{0, 1\}$ by*

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

defines a metric distance on X for which every open ball of radius $r = 1$ satisfies

$$\{p\} = B_1(p) = \overline{B_1(p)} \subsetneq \{x \in X : d(x, p) \leq 1\}.$$

(c) *Consider $X = [0, 1] \cup \{2\}$ as a metric subspace of $\mathbb{R}$ (with the usual Euclidean metric distance). Show that in this space*

$$B_1(1) \subsetneq \overline{B_1(1)} \subsetneq \{x \in X : |x - 1| \leq 1\}.$$

Problem 7 *(closures) If W is a subspace of a normed space X, then show the following:*

(a) *(Kreyszig Theorem 2.4-3) If W is finite dimensional, then W is closed.*

(b) *Even if W is infinite dimensional the closure $\overline{W}$ is a vector space, i.e., a subspace of X.*

Problem 8 *(the completion theorem for inner product spaces; an equivalence relation among Cauchy sequences) Given any set S, a* **relation** *is any subset of $S \times S$. A subset R of $S \times S$ is said to be an* **equivalence relation** *if the following hold:*

(i) $(x, x) \in R$ *for every* $x \in S$,

(ii) $(x, y) \in R$ *implies* $(y, x) \in R$, *and*

(iii) *If* $(x, y) \in R$ *and* $(y, z) \in R$, *then* $(x, z) \in R$.

Very often the fact that an ordered pair (x, y) is an element of an equivalence relation R is signified by writing $x \sim y$ (or something similar). Thus, the three properties can be written (with their usual names as)

(i) $x \sim x$ *for every $x \in S$ (the relation is* **reflexive***),*

(ii) $x \sim y$ *implies $y \sim x$ (the relation is* **symmetric***), and*

(iii) *If $x \sim y$ and $y \sim z$, then $x \sim z$ (the relation is* **transitive***).*

Given an equivalence relation on a set S and any element $p \in S$, the **equivalence class** *determined by p is*

$$[p] = \{x \in S : x \sim p\}.$$

Show the following:

(a) *If $p, q \in S$, then either $[p] \cap [q] = \phi$ or $[p] = [q]$. Consequently, the equivalence relation* **partitions** *the set S into equivalence classes.*

(b) *The relation*

$$\lim_{j \to \infty} (y_j - x_j) = \mathbf{0} \in X$$

is an equvalence relation among the Cauchy sequences $\{x_j\}_{j=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ in a normed space X.

(c) *Let $\mathcal{S}$ be the collection of all Cauchy sequences in a normed space X. Then*

$$\left[\{x_j\}_{j=1}^{\infty}\right] = \{\{\xi_j\}_{j=1}^{\infty} \subset X : \lim_{j \to \infty} (\xi_j - x_j) = \mathbf{0}\} \subset \mathcal{S}.$$

(Notice the set definition appearing in the middle does not require the sequence(s) $\{\xi_j\}_{j=1}^{\infty}$ to be Cauchy. You should show this.)

Problem 9 *(completion of an inner product space) Let X be any inner product space. Recall that the completion $\mathcal{H}$ of X was constructed as the Hilbert space of equivalence classes*

$$\left[\{x_j\}_{j=1}^{\infty}\right]$$

of Cauchy sequences $\{x_j\}_{j=1}^{\infty} \subset X$. Consider $\phi : X \to \mathcal{H}$ by

$$\phi(v) = \left[\{v\}_{j=1}^{\infty}\right]$$

where $\{v\}_{j=1}^{\infty}$ is the constant sequence with all terms v.

(a) *Show ϕ is linear and injective.*

(b) *(Kreyszig Problem 1.4.4) Show a Cauchy sequence in any metric space is bounded.*

(c) *Use part* **(b)** *to show the inner product $\langle \cdot\, , \,\cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to F$ by*

$$\left\langle \left[\{x_j\}_{j=1}^{\infty}\right], \left[\{y_j\}_{j=1}^{\infty}\right] \right\rangle_{\mathcal{H}} = \lim_{j\to\infty} \langle x_j, y_j \rangle_X$$

is well-defined. (You need to show the limit exists and is independent of the chosen representatives for the elements in $\mathcal{H}$.)

Problem 10 *(closure; extension of a linear operator; Kreyszig Theorem 2.7-11) Recall that a set $A \subset X$ is said to be* **dense** *in X if the closure of A satisfies*

$$\overline{A} = X.$$

(a) *Let X be a topological space. Show a subset $A \subset X$ is dense in X if and only if for each nonempty open set $U \subset X$ there holds*

$$A \cap U \neq \phi.$$

(b) *Let X be a metric space and $A \subset X$ with A dense in X. Show that for each $x \in X$, there exists a sequence $\{x_j\}_{j=1}^{\infty} \subset A$ with*

$$\lim_{j \to \infty} x_j = x.$$

(c) *Let X and Y be normed spaces with X_0 a dense subspace of X and Y a Banach space. If*

$$L_0 : X_0 \to Y \qquad \text{is a bounded linear operator,}$$

then the function $L : X \to Y$ defined by

$$Lx = \lim_{j \to \infty} L_0 x_j \qquad \text{where } \{x_j\}_{j=1}^{\infty} \subset X_0 \text{ and } \lim_{j \to \infty} x_j = x$$

is a well-defined linear operator.

(d) *Show the operator $L : X \to Y$ in part* **(c)** *above is bounded with minimal Lipschitz constant, i.e., operator norm, the same as that of L_0.*