Assignment 2: Structured Vector Spaces and Riesz Representation Due Tuesday September 13, 2022

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Problem 1 *Recall the* **Hermitian inner product**

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$$

on $\mathbb{C}^n$ *where* $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ *and* $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. *Given a complex vector* $\mathbf{n} \in \mathbb{C}^2$, *describe the orthogonal space*

$$Z = \{\mathbf{z} \in \mathbb{C}^2 : \langle \mathbf{z}, \mathbf{n} \rangle = 0\}.$$

For example, the corresponding answer for a vector $\mathbf{n} \in \mathbb{R}^2$ *with respect to the dot product would be "a two-dimensional plane through the origin orthogonal to the vector* $\mathbf{n}$*," but you should try to avoid resorting to an explanation in terms of (only) complex dimensions.*

Problem 2 *(norms and norm-like functions; Kreyszig Problems 2.2.3-4; see also Kreyszig Problems 4.3.1-2) Let X be a normed space.*

(a) *If $q : X \to [0, \infty)$ satisfies*

- **(i)** $q(-x) = q(x)$ *for all* $x \in X$, *and*
- **(ii)** $q(x + y) \le q(x) + q(y)$ *for all* $x, y \in X$.

then show

$$|q(y) - q(x)| \le q(y - x) \qquad \text{for all } x, y \in X.$$

In particular,

$$|\ \|y\| - \|x\|\ | \le \|y - x\|.$$

(b) *If $q : X \to \mathbb{R}$ satisfies $q(ax) = aq(x)$ for all $a \ge 0$ and $x \in X$, then*

$$q(\mathbf{0}) = 0.$$

In particular, $\|\mathbf{0}\| = 0$, even if we only assume "$\|x\| = 0$ implies $x = \mathbf{0}$" instead of "$\|x\| = 0$ if and only if $x = \mathbf{0}$" in the definition of "norm."

(c) *If $p : X \to \mathbb{R}$ satisfies*

- **(i)** $p(\alpha x) = |\alpha| p(x)$ *for all* $x \in X$ *and* $\alpha \in F$, *and*
- **(ii)** $p(x + y) \le p(x) + p(y)$ *for all* $x, y \in X$.

then show

$$p(x) \ge 0 \qquad \text{for all } x \in X.$$

In particular, the condition "$\|\cdot\| : X \to [0, \infty)$" can be relaxed to "$\|\cdot\| : X \to \mathbb{R}$" in the definition of "norm," and the function p itself is a **seminorm** *satisfying $p : X \to [0, \infty)$ and*

SN1 $p(\mathbf{0}) = 0$,

SN2 $p(\alpha x) = |\alpha| p(x)$ *for all* $x \in X$, *and*

SN3 $p(x + y) \le p(x) + p(y)$.

See also Kreyszig Problem 2.3.12.

Problem 3 *(inner products and an inner product-like function) Let X be an inner product space. If $h : X \times X \to F$ satisfies*

(i) $h(x, x) \in \mathbb{R}$ *for all* $x \in X$,

(ii) $h(\alpha x + \beta y, z) = \alpha h(x, z) + \beta h(y, z)$ *for all* $\alpha, \beta \in F$ *and* $x, y, z \in X$, *and*

(ii) $h(z, \alpha x + \beta y) = \overline{\alpha} h(z, x) + \overline{\beta} h(z, y)$ *for all* $\alpha, \beta \in F$ *and* $x, y, z \in X$,

then show the following:

(a) $h(y, x) = \overline{h(x, y)}$ *for all* $x, y \in X$. *In particular, the conjugate symmetry*

$$\langle y, x \rangle = \overline{\langle x, y \rangle} \qquad \text{for all } x, y \in X$$

in the definition of the inner product can be replaced with conjugate linearity in the second argument:

$$\langle z, \alpha x + \beta y \rangle = \overline{\alpha} \langle x, z \rangle + \overline{\beta} \langle z, y \rangle \qquad \text{for all } x, y, z \in X \text{ and } \alpha, \beta \in F.$$

Hint: Consider $h(x + y, x + y)$ *and* $h(x + iy, x + iy)$.

(b) *(Kreyszig Problem 3.1.1; Exercise 57 in my notes)* $h(x+y, x+y) + h(x-y, x-y) = 2[h(x, x) + h(y, y)]$ *for all* $x, y \in X$. *In particular, the* **induced norm** *on an inner product space satisfies the* **parallelogram identity***:*

$$\|x + y\|^2 + \|x - y\|^2 = 2[\|x\|^2 + \|y\|^2] \qquad \text{for all } x, y \in X.$$

Problem 4 *(an inner product space; Kreyszig (sub)section 1.2-3) Let* $\ell^2 = \ell^2(F)$ *denote the collection of all sequences* $\{a_j\}_{j=1}^\infty \subset F$ *of scalars having the property that*

$$\sum_{j=1}^\infty |a_j|^2 < \infty.$$

Such sequences are said to be (absolutely) square summable.

(a) *Show* ℓ^2 *is a vector space by completing the following steps:*

1. *Show* ℓ^2 *is closed under scaling where the scaling* $\alpha\{a_j\}_{j=1}^\infty$ *is defined by*

$$\alpha\{a_j\}_{j=1}^\infty = \{\alpha a_j\}_{j=1}^\infty.$$

2. *Use the triangle inequality for the norm in* F^n *to conclude*

$$\sum_{j=1}^n |a_j + b_j|^2 \le \left[\sqrt{\sum_{j=1}^n |a_j|^2} + \sqrt{\sum_{j=1}^n |b_j|^2}\right]^2.$$

3. *Conclude* ℓ^2 *is closed under addition where the sum* $\{a_j\}_{j=1}^\infty + \{b_j\}_{j=1}^\infty$ *is defined by*

$$\{a_j + b_j\}_{j=1}^\infty.$$

(b) *Show that if* $\{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty \in \ell^2$, *then*

$$\left\langle \{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty \right\rangle = \sum_{j=1}^\infty a_j \overline{b_j}$$

is a well-defined inner product with value in F*. Hint: Use the Cauchy-Schwarz inequality in* F^n *like we used the triangle inequality in* F^n *in the previous part.*

Problem 5 *Give a detailed proof of the triangle inequality for the (induced) norm on* F^n*.*

Problem 6 *(another inner product space) Consider $C([a,b] \to F)$ the collection of all continuous scalar valued functions on the closed interval $[a,b]$. Note that one can define the integral of a continuous scalar valued function by*

$$\int_a^b f(x)\,dx = \int_a^b \mathrm{Re}[f(x)]\,dx + i\int_a^b \mathrm{Im}[f(x)]\,dx$$

where $\mathrm{Re}[f]$ and $\mathrm{Im}[f]$ are the real and imaginary parts of the value of f. Show

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

defines an inner product on $C([a,b] \to F)$.

Problem 7 *(a normed space; exercises 58-59 in my notes) Show that $C^0[a,b]$ with the sup norm*

$$\|f\|_\infty = \max_{a\le x\le b} |f(x)|$$

is a normed space with a norm that is **not** *induced by an inner product. Hint: Consider positive functions $f \equiv 1$ and $g(x) = x + c$; check the paralellogram identity.*

Problem 8 *Recall that a function $f : X \to Y$ where X and Y are topological spaces is continuous at the point $p \in X$ if for every open set V in Y with $f(p) \in V$, there exists an open set U in X with $p \in U$ and*

$$f(U) = \{f(x) : x \in U\} \subset V.$$

Also, the function $f : X \to Y$ is **continuous** *if f is continuous at each point $p \in X$.*

Show this definition of continuity is equivalent to the condition:

The inverse image of every open set is open.

That is, for each open set V in Y, the set

$$f^{-1}(V) = \{x \in X : f(x) \in V\} \qquad \text{is open in } X.$$

Problem 9 *(Riesz representation) Let X be any inner product space.*

(a) *Show that given any vector $w \in X$, the function defined by*

$$\ell(x) = \langle x, w \rangle$$

is a continuous linear functional.

Definition: Given any inner product space X, let $\mathcal{R}(X \to F)$ denote the **represented linear functionals** on X. That is,

$$\mathcal{R}(X \to F) = \{\phi \in \mathcal{L}(X \to F) : \text{there exists some } w \in X \text{ such that} \\ \phi(x) = \langle x, w \rangle \text{ for all } x \in X\}.$$

Part **(a)** *above asserts $\mathcal{R}(X \to F)$ is a subset of the (continuous) dual space $\beth(X \to F)$ of bounded linear functionals. The Riesz representation theorem says that if X is a Hilbert space, then $\mathcal{R}(X \to F) = \beth(X \to F)$. Part (a) also shows the assumption of continuity cannot be left out of the Riesz representation theorem. Parts* **(b)** *and* **(c)** *below give an explicit example.*

(b) *Let ℓ^2 denote the inner product space of square summable real sequences; see Problem 4 above. Let $\mathbf{e}_k = \{\delta_{kj}\}_{j=1}^{\infty} \in \ell^2$ be the sequence with all zeros except for 1 in the k-th entry. Notice that*

$$\{a_j\}_{j=1}^{\infty} = \sum_{j=1}^{\infty} a_j \mathbf{e}_j$$

is a convergent series in ℓ^2 for every $\{a_j\}_{j=1}^{\infty} \in \ell^2$. Consider the inner product space $W = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots\}$. Note that W is a proper subspace of ℓ^2. Show $\ell : W \to \mathbb{R}$ by

$$\ell\left(\{a_j\}_{j=1}^{\infty}\right) = \sum_{j=1}^{\infty} a_j$$

is a well-defined linear functional.

(c) *Show the linear functional ℓ from part* **(b)** *does not admit Riesz representation, that is, there is no sequence $w \in W$ (or even $w \in \ell^2$) for which $\ell(v) = \langle v, w \rangle$ for all $v \in W$. Hint: Show ℓ is discontinuous and use the assertion of part* **(a)** *above.*

Problem 10 *(Riesz representation) Our proof of the Riesz representation theorem, i.e., Riesz' proof, relied very strongly on the fact that we were able to take a nonzero vector in the orthogonal complement of a certain (proper) subspace. More specifically, we needed to find a nonzero vector in in the null space $\mathcal{N}(\ell)^\perp$ where ℓ was a continuous linear functional on a Hilbert space.*

Recall that given any subspace W of an inner product space X, the **orthogonal complement** *of W is defined by*

$$W^\perp = \{x \in X : \langle x, w \rangle = 0 \textit{ for all } w \in W\}.$$

(a) *Show that $W^\perp$ is a subspace.*

(b) *Take it as given that $L^2(a, b)$, the collection of all square integrable functions $f : (a, b) \to \mathbb{R}$ satisfying*

$$\int_{(a,b)} f^2 < \infty,$$

is an inner product space with inner product

$$\langle f, g \rangle = \int_{(a,b)} fg.$$

(We will prove this in some detail later.) If you believe this, then it is clear that $W = C^0[a, b]$ is a subspace of $X = L^2(a, b)$. Show $W = C^0[a, b]$ is a **proper** *subspace of $X = L^2(a, b)$, i.e., $W \subsetneq X$, but $W^\perp = \{\mathbf{0}\}$. (It may be a little difficult for you to give all the details correctly here, but you should be able to see the main idea.)*

(c) *It was also important in our proof that $\mathcal{N}(\ell)^\perp$ was one-dimensional that $W^{\perp\perp} = W$ for a certain subspace, namely for $W = \operatorname{span}\{w\}$ where w was the representing vector. Show that in general*

$$W \subset W^{\perp\perp} \qquad \textit{for any subspace } W.$$

(d) *Give an example in which $W \subsetneq W^{\perp\perp}$. Hint: Look at part* **(b)** *above.*