Assignment 10: Final Exam
Due Tuesday December 6, 2022

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Problem 1 (distribution derivative) Find the second distribution derivative of the Heaviside functional $H : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ by

$$H(\phi) = \int_0^\infty \phi(x)\, dx.$$

Problem 2 Given $a, b \in \mathbb{R}$ with $a < b$, consider the spaces $\mathcal{L}^\infty(a, b)$ of essentially bounded functions, the space $L^\infty(a, b)$ of (equivalence classes of) essentially bounded functions, and the space $C^0(a, b)$ of continuous functions.

(a) Define what it means for a measurable function $u : (a, b) \to \mathbb{R}$ to be **essentially bounded**.

(b) State the definition of a **seminorm** σ on a vector space X.

(c) Show $\mathcal{L}^\infty(a, b)$ is a vector space and $\sigma : \mathcal{L}^\infty(a, b) \to [0, \infty)$ by

$$\sigma(u) = \inf \left\{ M > 0 : \int_{(a,b)} \max\{|u| - M, 0\} = 0 \right\}$$

is a seminorm on $\mathcal{L}^\infty(a, b)$ but not a norm.

(d) State the definition of a **norm** on a vector space X.

(e) Show $\| \cdot \| : L^\infty(a, b) \to [0, \infty)$ by $\|[u]\| = \sigma(u)$ is a well-defined norm on $L^\infty(a, b)$ where $[u]$ represents an appropriate equivalence class of functions in $\mathcal{L}^\infty(a, b)$. (You should define the equivalence relation.)

(f) Consider the injections $\phi : C^0[a,b] \to \mathcal{L}^\infty(a,b)$ and $\psi : C^0(a,b) \to L^\infty(a,b)$ by

$$\phi(f) = f \qquad \text{and} \qquad \psi(f) = [f]$$

respectively. Show these maps are both linear and injective.

(g) Show the seminorm σ from part **(c)** is a norm on the subspace $\phi(C^0[a,b])$.

(h) Define what it means for a normed space to be **complete**, i.e., a Banach space.

(i) Show $\phi(C^0[a,b])$, $\psi(C^0[a,b])$, and $L^\infty(a,b)$ are Banach spaces.

(j) Is $\psi(C^0[a,b])$ dense in $L^\infty(a,b)$?

Problem 3 (topologies on a vector space or seminormed space) Let $\{\mathcal{T}_\alpha\}_{\alpha\in\Gamma}$ be any collection of topologies on a vector space X. Typically, in applications these topologies will be the topologies associated with a family of seminorms also indexed by $\alpha \in \Gamma$.

(a) Define the **seminorm topology** associated with a seminorm $\sigma : X \to [0,\infty)$ and show it is a topology. Hint: Seminorm balls.

(b) A **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

$$\bigcup_{U\in\mathcal{B}} U = X$$

and for each $U_1, U_2 \in \mathcal{B}$,

$$x \in U_1 \cap U_2 \qquad \Longrightarrow \qquad \text{there exists some } U \in \mathcal{B} \text{ with } x \in U \subset U_1 \cap U_2.$$

Show the collection

$$\left\{ \bigcup_{U\in\mathcal{V}} U : \mathcal{V} \subset \mathcal{B} \right\}$$

of arbitrary unions of sets in a basis $\mathcal{B}$ is a topology, and is the smallest topology containing $\mathcal{B}$.

(c) Given any collection $\mathcal{C}$ of subsets of X satisfying

$$\bigcup_{U\in\mathcal{C}} U = X,$$

show the collection

$$\mathcal{B} = \left\{ \bigcap_{j=1}^{k} U_k : U_1, U_2, \ldots, U_k \in \mathcal{C} \right\}$$

is a basis for a topology on X.

(d) Show/conclude

$$\mathcal{T} = \left\{ \bigcup_{\beta \in \mathcal{V}} \left(\bigcap_{j=1}^{k_\beta} U_j^\beta \right) : U_1^\beta, U_2^\beta, \ldots, U_{k_\beta}^\beta \in \bigcup_{\alpha \in \Gamma} \mathcal{T}_\alpha \quad \text{for} \quad \beta \in \mathcal{V} \right\} \tag{1}$$

is a topology on X and is the smallest topology containing $\mathcal{T}_\alpha$ for every $\alpha \in \Gamma$. Here the symbol $\mathcal{V}$ represents any indexing set for which the set conditional holds, and the topology $\mathcal{T}$ is called **the topology generated by the family** $\{\mathcal{T}_\alpha\}_{\alpha \in \Gamma}$.

(e) Show a sequence $\{x_j\}_{j=1}^\infty \subset X$ converges (to an element $x \in X$) with respect to the topology $\mathcal{T}$ if and only if $\{x_j\}_{j=1}^\infty$ converges (to x) with respect to every topology $\mathcal{T}_\alpha$ with $\alpha \in \Gamma$.

Note: In general, the topology generated by a family of topologies need not be Hausdorff; in particular, it may be that a given sequence converges to two distinct points, i.e., limits are not unique.

Problem 4 ($C_c^\infty(U)$ topology) Given an open set $U \subset \mathbb{R}$, a natural number $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, and an open set $V \subset\subset U$, let $\sigma_{k,V}$ denote the seminorm $\sigma : C_c^\infty(U) \to [0, \infty)$ by

$$\sigma(u) = \sum_{j=0}^{k} \left\| \frac{d^j u}{dx^j} \right\|_{C^0(\overline{V})} = \sum_{j=0}^{k} \max_{x \in \overline{V}} \left| \frac{d^j u}{dx^j}(x) \right|.$$

As in the previous problem, let $\mathcal{T}$ denote the topology generated by this family of seminorms.

(a) Let $S = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ be any finite collection of the seminorms $\sigma_{k,V}$. Explain how to construct a sequence $\{u_j\}_{j=1}^\infty \subset C_c^\infty(U)$ which converges to two distinct functions $v_1 \neq v_2$ with respect to every seminorm in Σ.

(b) Show that a sequence $\{u_j\}_{j=1}^\infty \subset C_c^\infty(U)$ converges to a function $u \in C_c^\infty(U)$ if and only if $\{u_j\}_{j=1}^\infty$ converges to u in $C^k(\overline{V})$ for every $k \in \mathbb{N}_1$ and every $V \subset\subset U$.

(c) Conclude $C_c^\infty(\Omega)$ is a Hausdorff space with respect to the topology $\mathcal{T}$.

The topology in this case is called the C^∞ topology on $C_c^\infty(U)$.

Problem 5 (distributions) Recall that a **distribution** is by definition a continuous linear functional $F : C_c^\infty(\mathbb{R}) \to \mathbb{R}$.

(a) Define the **distribution derivative** of a distribution F.

(b) If $c \in \mathbb{R}$, consider the distribution $C : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ by

$$C(\phi) = \int_{\mathbb{R}} c\phi.$$

Calculate the distribution derivative of C.

(c) Show every distribution F determines uniquely a set of distribution **antiderivitives**

$$\{G = G_0 + C : \text{the distribution derivative of } G_0 \text{ is } F\}.$$

Note: Technically, once a linear functional $G_0 : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ is identified satisfying $-G_0(\phi') \equiv F(\phi)$, then one needs to (and one should) show the functional $G_0 : C_c^\infty \to \mathbb{R}$ is continuous, i.e., $G_0 \in X'$ where $X = C_c^\infty(\mathbb{R})$. Unfortunately, as explained in the lecture the topology on $Y = C_c^\infty(\mathbb{R})$ described in Problem 4 (and called "the $\mathcal{K}$ topology" in the lecture) is not the correct topology to consider when using the space $X = C_c^\infty(\mathbb{R})$ in the theory of distributions. In particular, even the constant distributions C are discontinuous with respect to the $\mathcal{K}$ topology on $Y = C_c^\infty(\mathbb{R})$ unless $c = 0$. Schwarz suggested the use of a different topology. I called this the **Schwarz topology** $\mathcal{S}$ on $X = C_c^\infty(\mathbb{R})$ and G_0 should be continuous with respect to this topology. For this exercise (at the moment) perhaps it is enough to write down the formula for G_0 and verify the required properties (linearity and $-G_0(\phi') = F(\phi)$) other than continuity. I will try to describe/construct the Schwarz topology at the end of this assignment, and you should then (with some work) be able to verify G_0 (and every integral functional and evaluation functional) is continuous. Incidentally, as noted in the lecture, the evaluation functionals $E_x : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ by $E_x(\phi) = \phi(x)$ do satisfy $E_x \in Y'$ for all fixed $x \in \mathbb{R}$, as you can verify.

Problem 6 (weak topology) Let X be a normed space and denote by X' the dual space $\beth(X \to F)$ of continuous linear functionals on X. The **weak topology** on X is the topology $\mathcal{T}_w$ generated by

$$\mathcal{B} = \left\{ \bigcap_{j=1}^{k} \omega_j^{-1}(V_1) : \omega_1, \ldots, \omega_k \in X' \text{ and } V_1, \ldots, V_k \text{ are open sets in } F. \right\}.$$

A sequence $\{x_j\}_{j=1}^{\infty} \subset X$ is said to **converge weakly** to an element $x \in X$ if

$$\lim_{j\to\infty} \omega(x_j) = \omega(x) \qquad \text{for every } \omega \in X'.$$

(a) Show every (strongly) convergent sequence in X is weakly convergent.

(b) Show that a sequence in X is weakly convergent if and only if it is convergent with respect to the weak topology.

(c) Which topology has more open sets, the norm topology $\mathcal{T}$ or the weak topology $\mathcal{T}_w$?

Problem 7 (topology) Let $\mathcal{T}$ and $\mathcal{T}_w$ be two topologies on a space X satisfying $\mathcal{T}_w \subsetneq \mathcal{T}$. For the following assertions, choose $\mathcal{S}_1, \mathcal{S}_2 \in \{\mathcal{T}, \mathcal{T}_w\}$ with $\mathcal{S}_1 \neq \mathcal{S}_2$ to make the assertion true, and then prove the assertion.

(a) If the function $f : X \to Y$ from X into the topological space Y is continuous with respect to $\mathcal{S}_1$, then f is continuous with respect to $\mathcal{S}_2$.

(b) If $\{x_j\}_{j=1}^{\infty} \subset X$ converges to $x \in X$ with respect to $\mathcal{S}_1$, then $\{x_j\}_{j=1}^{\infty}$ converges to x with respect to $\mathcal{S}_2$.

(c) There exists a continuous function $f : X \to Y$ which is continuous with respect to $\mathcal{S}_2$ but not with respect to $\mathcal{S}_1$.

(d) There exists a sequence $\{x_j\}_{j=1}^{\infty} \subset X$ which is convergent to an element $x \in X$ with respect to $\mathcal{S}_2$ but $\{x_j\}_{j=1}^{\infty}$ does not converge to x with respect to $\mathcal{S}_1$.

Problem 8 (Shur's result(s)) Let X be a normed space and let $X' = \beth(X \to F)$ denote the dual space of X. Let $\{\omega_1, \ldots, \omega_k\}$ be a finite subset of X' and let $V_1, \ldots, V_k$ be open sets in $\mathbb{R}$. Assume

$$\omega_\ell(\mathbf{0}) = 0 \in V_\ell \qquad \text{for } \ell = 1, \ldots, k.$$

Consider $\Phi : X \to \mathbb{R}^k$ by $\Phi(x) = (\omega_1(x), \ldots, \omega_k(x))$.

(a) Show $\Phi \in \beth(X \to \mathbb{R}^k)$.

(b) Show

$$\mathcal{N}(\omega_\ell) = \{x \in X : \omega_\ell(x) = 0\} \subset \omega^{-1}(V_\ell).$$

(c) Show

$$\mathcal{N}(\Phi) = \{x \in X : \Phi(x) = \mathbf{0} \in \mathbb{R}^k\} = \bigcap_{\ell=1}^{k} \mathcal{N}(\omega_\ell).$$

(d) Show that if $\dim X > k$, then Φ is not injective.

(e) Conclude that if $\dim X > k$, then no nonempty set of the form

$$W = \bigcap_{\ell=1}^{k} \omega_\ell^{-1}(V_\ell)$$

is contained in **any** open ball $B_r(\mathbf{0}) \subset X$.

(f) Conclude that if $\dim X > k$, then no nonempty open set in the weak topology on X is a subset of any bounded set in X.

Problem 9 (topological vector spaces; cf. Yôsida Chapter 1, section 6, Lemma 1) Let X be a vector space and let $\mathcal{T}$ be a topology of open subsets of X.

(a) (first lemma of topological vector spaces) Assume the addition mapping $p : X \times X \to X$ given by $p(x, y) = x + y$ is continuous. Show

$$\{x + v : x \in U\} \qquad \text{is open in } X \tag{2}$$

for every (fixed) $v \in X$ and every (fixed) open set $U \subset X$. The topology on a vector space X is said to be **translation invariant** if the assertion of (2) holds.

(b) (second lemma of topological vector spaces) Assume X and Y are vector spaces over the same field, and each has a translation invariant topology. Recall the equivalent definitions of continuity for a linear function $L : X \to Y$.

(i) (continuity at each point) For each $x_0 \in X$ and each open set $V \subset Y$ with $Lx_0 \in V$, there is some open set $U \subset X$ with $x_0 \in U$ and

$$LU = \{Lx : x \in U\} \subset V.$$

(ii) (topological continuity) For each open set $V \subset Y$, the set

$$L^{-1}V = \{x \in X : Lx \in V\} \qquad \text{is open in } X.$$

Note: There is no (obvious) characterization of continuity in terms of an operator norm/Lipschitz continuity modulus in this case.

Show the linear function $L : X \to Y$ is continuous if and only if L is continuous at $\mathbf{0} \in X$.

Problem 10 (convergence of distributions) Let $g : \mathbb{R} \to \mathbb{R}$ denote the absolute value function given by $g(x) = |x|$, let $h : \mathbb{R} \to \mathbb{R}$ denote the Heaviside function given by

$$h(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

and let μ_δ denote the standard mollifier for $\delta > 0$; recall your work on the mollifications $\mu_\delta * g$ and $\mu_\delta * h$ (perhaps from Problems 6 and 7 of Assignment 9 and/or from section 1.5.2 of my notes).

We say a family of distributions $\{M_\delta\}_{\delta>0} \subset [C_c^\infty(\mathbb{R})]'$ **converges strongly** to a distribution $M \in [C_c^\infty(\mathbb{R})]'$ as $\delta \searrow 0$ if the following holds:

For any $\epsilon > 0$, any $N > 0$, and any open set $V \subset [-N, N] \subset \mathbb{R}$, there exists some $\delta_0 > 0$ such that $\delta < \delta_0$ implies

$$|M_\delta(\phi) - M(\phi)| < \epsilon \qquad \text{uniformly for } \phi \in \{\psi \in C_c^\infty(V) : \|\psi\|_{L^1} < N\}.$$

(a) Let $M_\delta \in [C_c^\infty(\mathbb{R})]'$ denote the integral functional associated with the (classical) derivative $(\mu_\delta * g)'$, and show M_δ converges strongly to $M : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ by

$$M\phi = \int_{\mathbb{R}} (-1 + 2h)\phi$$

as $\delta \searrow 0$.

(b) Now let $M_\delta \in [C_c^\infty(\mathbb{R})]'$ denote the integral functional associated with the (classical) derivative $(\mu_\delta * h)'$, and show M_δ **does not** converge strongly to $E : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ by

$$E\phi = \phi(0)$$

as $\delta \searrow 0$.

(c) Show the distributional derivative of h is the evaluation function E mentioned in part **(b)** above.

Different notions of **weak convergence** for distributions can be obtained in the following way: Let $\sigma : C_c^\infty(\mathbb{R}) \to [0, \infty)$ be a seminorm. We say a family of distributions $\{M_\delta\}_{\delta>0} \subset [C_c^\infty(\mathbb{R})]'$ **converges σ-weakly** to a distribution $M \in [C_c^\infty(\mathbb{R})]'$ as $\delta \searrow 0$ if the following holds:

For any $\epsilon > 0$, any $N > 0$, and any open set $V \subset [-N, N] \subset \mathbb{R}$, there exists some $\delta_0 > 0$ such that $\delta < \delta_0$ implies

$$|M_\delta(\phi) - M(\phi)| < \epsilon \qquad \text{uniformly for } \phi \in \{\psi \in C_c^\infty(V) : \sigma(\psi) < N\}.$$

Note that taking the $L^1(V)$ norm of (the restriction of) a function $\phi \in C_c^\infty(\mathbb{R})$ gives a seminorm on $C_c^\infty(\mathbb{R})$, and it is this choice of norm that gives the definition of strong convergence of distributions.

(d) Determine a "natural" choice of seminorm $\sigma : C_c^\infty(\mathbb{R}) \to [0, \infty)$ for which the family of distributions $\{M_\delta\}_{\delta>0}$ of part **(b)** above does converge to the evaluation functional σ-weakly. If you have trouble with this, you may want to consider part **(e)** below first.

(e) Let $\sigma_0 : C_c^\infty(\mathbb{R}) \to [0, \infty)$ be given by the C^0/sup norm. Show that strong convergence of any family $\{M_\delta\}_{\delta>0}$ of distributions to a distribution $M \in [C_c^\infty(\mathbb{R})]'$ as $\delta \searrow 0$ implies σ_0-weak convergence of M_δ to M. Just to be clear, σ_0-weak convergence means the following:

> For any $\epsilon > 0$, any $N > 0$, and any open set $V \subset [-N, N] \subset \mathbb{R}$, there exists some $\delta_0 > 0$ such that $\delta < \delta_0$ implies
>
> $$|M_\delta(\phi) - M(\phi)| < \epsilon \qquad \text{uniformly for } \phi \in \{\psi \in C_c^\infty(V) : \|\psi\|_{C^0(\overline{V})} < N\}.$$

What is being clarified here is the precise use/restriction of the seminorm. Of course the C^0 norm is actually a norm on $C_c^\infty(\mathbb{R})$. Can you explain why one would not (want to) use the much simpler condition

> For any $\epsilon > 0$ and any $N > 0$, there exists some $\delta_0 > 0$ such that $\delta < \delta_0$ implies
>
> $$|M_\delta(\phi) - M(\phi)| < \epsilon \qquad \text{uniformly for } \phi \in \{\psi \in C_c^\infty(\mathbb{R}) : \|\psi\|_{C^0(\mathbb{R})} < N\}$$

as a definition of C^0-weak convergence of distributions?

Problem 11 (Extra/Bonus: The third lemma of topological vector spaces, aka Yôsida Theorem 1, Chapter 1, Section 6) Let X and Y be topological vector spaces (over the same field) in which the operation mappings

$$\begin{aligned} p_X : X \times X \to X \qquad & \text{by } p_X(x,y) = x + y, \\ p_Y : Y \times Y \to Y \qquad & \text{by } p_Y(x,y) = x + y, \\ \tau_X : F \times X \to X \qquad & \text{by } \tau_X(\alpha,x) = \alpha x, \text{ and} \\ \tau_Y : F \times Y \to Y \qquad & \text{by } \tau_Y(\alpha,y) = \alpha y \end{aligned}$$

are continuous. Assume further that the topology on X is the topology generated by a/the family of topologies $\{\mathcal{T}_\sigma\}_{\sigma \in A}$ where each topology $\mathcal{T}_\sigma$ is the "seminorm ball" topology associated with a seminorm $\sigma : X \to [0,\infty)$. Recall that this means the topology on X is the smallest topology containing all the open sets in $\cup_\sigma \mathcal{T}_\sigma$. Similarly, assume the topology on Y is the topology generated by a/the family of topologies $\{\mathcal{S}_\nu\}_{\nu \in B}$ where each topology $\mathcal{S}_\nu$ is the "seminorm ball" topology associated with a seminorm $\nu : Y \to [0,\infty)$.

Show a linear operator $L : X \to Y$ is continuous if and only if for each seminorm $\nu \in B$, there exists a seminorm $\sigma \in A$ and some $c > 0$ for which

$$\nu(Lv) \le c\,\sigma(v) \qquad \text{for all } v \in X.$$

1 The Schwarz topology on $C_c^\infty(\mathbb{R})$.

The construction of the Schwarz topology on $X = C_c^\infty(\mathbb{R})$ uses the $\mathcal{K}$ topologies (or the topologies of uniform convergence of derivatives on compact subsets). I will briefly attempt to give the details here as I have adapted them from Chapter 6 of Rudin's book *Functional Analysis* (1973, first edition). One of the main differences is that Rudin considers $\mathcal{K}$ seminorms on subspaces called (by Schwarz) $\mathcal{D}_K$ where $\mathcal{D}_K$ as a vector space is $C_c^\infty(K)$ and K is a compact set while I restrict to the situation where $K = \overline{V}$ with V an open set and $\overline{V}$ compact. Also, Rudin considers $K \subset \mathbb{R}^n$, and I restrict attention to to $V \subset\subset \mathbb{R}$. Finally, Rudin considers a general subset $\Omega \subset \mathbb{R}^n$ as the primary domain of functions under consideration, and I restrict attention to the special case where $\Omega = \mathbb{R}$. Some additional background material and discussion may be found in my notes.

The starting point is with the $\mathcal{K}$ topology on all of $C^\infty(\mathbb{R})$. Then the main tool to construct the Schwarz topology is the topology on $C_c^\infty(\overline{V})$ or what Rudin would call $\mathcal{D}_{\overline{V}}$. Here is a quick review:

The $\mathcal{K}$ topology on $C^\infty(\mathbb{R})$ is the topology generated by the C^k norms

$$\|f\|_{C^k(\overline{V})} = \max\left\{\left|\frac{d^j f}{dx^j}(x)\right| : x \in \overline{V} \text{ and } 0 \le j \le k\right\}$$

considered as seminorms on $C^\infty(\mathbb{R})$. These seminorms are actually norms on the subspaces $C^k(\overline{V})$ as V ranges over open sets with $V \subset\subset \mathbb{R}$ and k takes all values in the set $\{0, 1, 2, 3, \ldots\}$ with $k = 0$ corresponding to the uniform C^0 norm. Each of these seminorms determines a topology $\mathcal{T}_{k,\overline{V}}$ on $C^\infty(\mathbb{R})$.

Precisely, the $\mathcal{K}$ topology on $C^\infty(\mathbb{R})$ is the smallest topology $\mathcal{T}$ containing all the topologies $\mathcal{T}_{k,\overline{V}}$ or

$$\bigcup\{\mathcal{T}_{k,\overline{V}} : V \text{ is open in } \mathbb{R} \text{ with } V \subset\subset \mathbb{R} \text{ and } k \in \{0, 1, 2, 3, \ldots\}\}.$$

Recall that $Y = C_c^\infty(\mathbb{R})$ is a subspace of $C^\infty(\mathbb{R})$ with respect to the $\mathcal{K}$ topology.

For this construction, we back up and consider

$$C_c^\infty(\overline{V}) = \{f \in C_c^\infty(\mathbb{R}) : \operatorname{supp}(f) \subset \overline{V}\}.$$

as a subspace of $C^\infty(\mathbb{R})$ with respect to the $\mathcal{K}$ topology. Note this $C_c^\infty(V)$ is a proper subspace of the space $C_c^\infty(\overline{V})$ which Rudin (and Schwarz) would call $\mathcal{D}_{\overline{V}}$. Most importantly, denote the $\mathcal{K}$ subspace topology on $C_c^\infty(\overline{V})$ by

$$\mathcal{S}_{\overline{V}} = \{\mathcal{U} \cap C_c^\infty(\overline{V}) : \mathcal{U} \in \mathcal{K}\}.$$

Now we use the spaces $C_c^\infty(\overline{V})$ with the $\mathcal{K}$ subspace topology $\mathcal{S}_{\overline{V}}$ as follows: Denote by X the vector space $C_c^\infty(\mathbb{R})$. We define a **local base $\mathcal{B}_0$ at $\mathbf{0}$** $\in X$, i.e., at the zero function in X, by

$$\mathcal{B}_0 = \{\mathcal{V} \subset X : \mathcal{V} \text{ is convex, centrally symmetric, and } \\ \mathcal{V} \cap \overline{V} \in \mathcal{S}_{\overline{V}} \text{ for every } V \subset\subset \mathbb{R}\}.$$

A base for the Schwarz topology is then obtained by translating the sets in $\mathcal{B}_0$:

$$\mathcal{B} = \{\{\phi + f : f \in \mathcal{V}\} : \phi \in X = C_c^\infty(\mathbb{R}) \text{ and } \mathcal{V} \in \mathcal{B}_0\}.$$

The Schwarz topology $\mathcal{S}$ on $C_c^\infty(\mathbb{R})$ is then the collection of arbitrary unions of these basis elements:

$$\mathcal{S} = \left\{\bigcup_{\alpha\in\Gamma} \mathcal{V}_\alpha : \{\mathcal{V}_\alpha\}_{\alpha\in\Gamma} \subset \mathcal{B}\right\}.$$

Problem 12 *Let $X = C_c^\infty(\mathbb{R})$ denote the topological vector space defined above with respect to the Schwarz topology.*

(a) *Show $\mathcal{S}$ is a topology which is strictly finer than the $\mathcal{K}$ topology on $Y = C_c^\infty(\mathbb{R})$, that is $\mathcal{K} \subsetneq \mathcal{S}$.*

(b) *Show the sequence $\{\phi_j\}_{j=1}^\infty \subset C_c^\infty(\mathbb{R})$ with $\phi_j(x) = \mu_1(x - j)$ where μ_1 is the standard mollifier*

(i) *converges to the zero function in Y, and*

(ii) *does not converge to the zero function in X.*

(c) *Show that a sequence $\{f_j\}_{j=1}^\infty \subset X = C_c^\infty(\mathbb{R})$ converges to a function $f \in X$ if and only if there is a fixed compact set $K \subset \mathbb{R}$ for which*

$$\text{supp}(f_j) \subset K \textit{ for all } j = 1, 2, 3, \ldots \textit{ and } \lim_{j\to\infty} \|f_j - f\|_{C^k} = 0 \textit{ for all } k = 0, 1, 2, 3, \ldots.$$

(d) *The integral functionals associated with $L^1_{loc}(\mathbb{R})$ functions as well as the evaluation functionals are all in X', i.e., they are all continuous with respect to the Schwarz topology on $C_c^\infty(\mathbf{r})$. Hint: Continuity at the zero function $\mathbf{0} \in X = C_c^\infty(\mathbb{R})$ implies continuity for a linear function $G_0 : X \to \mathbb{R}$, even if there is no norm. See part* **(b)** *of Problem 9 above.*

(e) *Your antiderivative functional $G_0 : X \to \mathbb{R}$ from Problem 5 above is continuous with respect to the Schwarz topology.*

(f) *Consider the antiderivative operator $\Phi : X \to X$ by*

$$\Phi(\psi) = \phi \qquad \text{with} \qquad \phi(x) = \int_{-\infty}^x \psi(t)\, dt - \left(\int_{\mathbb{R}} \psi \right) \int_{-\infty}^x \eta(t)\, dt$$

where $\eta \in X = C_c^\infty(\mathbb{R})$ is fixed with $\int_{\mathbb{R}} \eta = 1$. Show Φ is continuous.