The Wave Equation in One Spatial Dimension D'Alembert's Solution and Comments on Regularity

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April 20, 2021

Recall that we have derived D'Alembert's solution $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ for the wave equation $u_{tt} = u_{xx}$ with Cauchy data $u(x, 0) = u_0(x)$ and $u_t(x, 0) = v_0(x)$ using the **method of characteristics** as follows: First we write the 1-D wave equation as

$$(u_t - u_x)_t + (u_t - u_x)_x = 0$$

and set $w = u_t - u_x$. Then w satisfies the first order linear PDE $w_t + w_x = 0$ with Cauchy data $w(x, 0) = v_0(x) - u_0'(x)$. Along the characteristic $\gamma(t) = (x_0 + t, t)$ we have

$$\frac{d}{dt} w(x_0 + t, t) = w_x + w_t = 0,$$

so choosing x_0 with $\gamma(t) = (x_0 + t, t) = (x, t)$, we have

$$w(x, t) \equiv w(x_0, 0) = v_0(x - t) - u_0'(x - t).$$

Having determined w, we consider the non-homogeneous first order linear PDE $u_t - u_x = w$ with Cauchy data $u(x, 0) = u_0(x)$. On the characteristic $\gamma(t) = (x_0 - t, t)$ we have

$$\frac{d}{dt} u(x_0 - t, t) = -u_x + u_t = w(x_0 - t, t),$$

so choosing x_0 with $\gamma(t) = (x_0 - t, t) = (x, t)$, we have

$$\begin{aligned}
u(x_0 - t, t) &= u(x_0, 0) + \int_0^t w(x_0 - \tau, \tau)\, d\tau \\
&= u_0(x_0) + \int_0^t [v_0(x_0 - 2\tau) - u_0'(x_0 - 2\tau)]\, d\tau \\
&= u_0(x + t) + \int_0^t [v_0(x + t - 2\tau) - u_0'(x + t - 2\tau)]\, d\tau \\
&= u_0(x + t) + \frac{1}{2} u_0(x + t - 2\tau)\Big|_{\tau=0}^{t} + \int_0^t v_0(x + t - 2\tau)\, d\tau \\
&= u_0(x + t) + \frac{1}{2}[u_0(x - t) - u_0(x + t)] + \int_0^t v_0(x + t - 2\tau)\, d\tau \\
&= \frac{1}{2}[u_0(x - t) + u_0(x + t)] + \frac{1}{2}\int_{x-t}^{x+t} v_0(\xi)\, d\xi.
\end{aligned}$$

In the last equality we have used the change of variables $\xi = x + t - 2\tau$. Thus, the d'Alembert formula is

$$u(x, t) = \frac{1}{2}[u_0(x - t) + u_0(x + t)] + \frac{1}{2}\int_{x-t}^{x+t} v_0(\xi)\, d\xi.$$

It follows from this formula that if $u_0 \in C^2(\mathbb{R})$ and $v_0 \in C^1(\mathbb{R})$, then the solution u of the wave equation satisfies $u \in C^2(\mathbb{R} \times (0, \infty))$.

Exercise 1 *Prove the solution given by d'Alembert's formula when $u_0 \in C^2(\mathbb{R})$ and $v_0 \in C^1(\mathbb{R})$ is the unique solution of the problem*

$$\begin{cases} u_{tt} = u_{xx} & \text{on } \mathbb{R} \times (0, \infty) \\ u(x,0) = u_0(x) \text{ and } u_t(x,0) = v_0(x). \end{cases}$$

Regularity

Recall that every solution of Laplace's equation $\Delta u = 0$ on an open domain $\mathcal{U}$ satisfies $u \in C^\infty(\mathcal{U})$. Recall also that every solution $u \in C^2(\mathcal{U} \times (0,T))$ of the heat equation $u_t = \Delta u$ satisfies $u \in C^\infty(\mathcal{U} \times (0,T))$. We wish to show that no similar higher regularity of solutions can be expected for solutions of the wave equation. We first observe that the function

$$u_0(x) = \begin{cases} x^2 + x^3/6, & x \le 0 \\ x^2, & x \ge 0 \end{cases}$$

satisfies $u \in C^2(\mathbb{R}) \backslash C^3(\mathbb{R})$. In particular, $u'''(0^-) = 1$ while $u'''(0^+) = 0$. Thus, taking as Cauchy data $u(x,0) = u_0(x)$ and $u_t(x,0) \equiv 0$, we obtain a solution

$$\begin{aligned} u(x,t) &= \frac{1}{2}[u_0(x-t) + u_0(x+t)] \\ &= \begin{cases} [(x-t)^2 + (x-t)^3/6 + (x+t)^2 + (x+t)^3/6]/2, & x \le -t \\ [(x-t)^2 + (x-t)^3/6 + (x+t)^2]/2, & -t \le x \le t \\ [(x-t)^2 + (x+t)^2]/2, & x \le -t \end{cases} \end{aligned}$$

of the wave equation. For positive times t and $x = t$, we see

$$u_{xxx}(t^-, t) = 1 \qquad \text{and} \qquad u_{xxx}(t^+, t) = 0.$$

Thus, $u \notin C^3(\mathbb{R} \times (0, \infty))$.

There are similar solution formulas for u satisfying $u_{tt} = \Delta u$ in all spatial dimensions,[1] but we will not discuss those formulas this semester.

Finally, we wish to consider the **domain of dependence** and the **finite propogation speed** associated with solutions of the wave equation. Since there is nothing particularly special about having one space dimension in this discussion, we will consider the general case of $u_{tt} = \Delta u$ on $\mathbb{R} \times (0,T)$. The computation we will make uses an identity which is a special case of what is called a **kinematical identity**. Kinematical identities, in general, give a formula for the derivative of an integral with respect to a parameter associated with change in the domain of integration. That is, if $\mathcal{V}_t$ is a domain which changes with respect to a parameter t, say within a larger domain $\mathcal{W}$ so that $\mathcal{V}_t \subset \mathcal{W}$ for all t, and $f : \mathcal{W} \to \mathbb{R}$, then a kinematical identity tells you how to compute

$$\frac{d}{dt} \int_{\mathcal{V}_t} f.$$

In our case, we will consider an integral of a function over a ball of varying radius and fixed center:

$$\int_{B_r(\mathbf{p})} f.$$

[1]A good reference on this topic and all of the PDE material covered in these notes is Chapter 2 of Craig Evans' book *Partial Differential Equations* **AMS**.

More precisely, say $f : \mathbb{R}^n \to \mathbb{R}$ with $f \in C^0(\mathbb{R}^n)$ and we wish to compute

$$\frac{d}{dr}\int_{B_r(\mathbf{p})} f.$$

By the generalized polar coordinates/fubini theorem we can write

$$\frac{d}{dr}\int_{B_r(\mathbf{p})} f = \frac{d}{dr}\int_0^r \left(\int_{\partial B_t(\mathbf{p})} f\right) dt.$$

Then the simple 1-D fundamental theorem of calculus gives us the remarkable formula:

$$\frac{d}{dr}\int_{B_r(\mathbf{p})} f = \int_{\partial B_r(\mathbf{p})} f.$$

We will now use this in conjunction with the chain rule in a manner which is a little more complicated. As usual, we denote by Du the spatial gradient of u. For $r > 0$ and $0 < t \le r$,

$$\frac{d}{dt}\left(\frac{1}{2}\int_{B_{r-t}(\mathbf{p})} (u_t^2 + |Du|^2)\right) = \int_{B_{r-t}(\mathbf{p})} (u_t u_{tt} + Du \cdot Du_t) + \frac{1}{2}\int_{\partial B_{r-t}(\mathbf{p})} (u_t^2 + |Du|^2)\, \frac{d}{dt}(r-t).$$

Notice that we have t dependence in both the domain (ball) of integration and in the integrand. The first term differentiates the dependence in the integrand while the second term uses the kinematical identity and the chain rule to differentiate the dependence in $B_{r-t}(\mathbf{p})$. Simplifying and using the PDE etc., we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\int_{B_{r-t}(\mathbf{p})} (u_t^2 + |Du|^2)\right) &= \int_{B_{r-t}(\mathbf{p})} (u_t \Delta u + Du \cdot Du_t) - \frac{1}{2}\int_{\partial B_{r-t}(\mathbf{p})} (u_t^2 + |Du|^2) \\
&= \int_{B_{r-t}(\mathbf{p})} \mathrm{div}(u_t Du) - \frac{1}{2}\int_{\partial B_{r-t}(\mathbf{p})} (u_t^2 + |Du|^2) \\
&= \int_{\partial B_{r-t}(\mathbf{p})} \left[u_t Du \cdot \mathbf{n} - \frac{1}{2}(u_t^2 + |Du|^2)\right] \\
&\le \int_{\partial B_{r-t}(\mathbf{p})} \left[|u_t Du \cdot \mathbf{n}| - \frac{1}{2}(u_t^2 + |Du|^2)\right].
\end{aligned}$$

Now consider the value $|u_t Du \cdot \mathbf{n}|$. By the inequality $ab \le (a^2 + b^2)/2$, we can write

$$|u_t Du \cdot \mathbf{n}| \le \frac{1}{2}[u_t^2 + (Du \cdot \mathbf{n})^2] \le \frac{1}{2}[u_t^2 + |Du|^2].$$

The last inequality is a consequence of the Cauchy-Schwarz inequality. Returning to our main calculation we have

$$\frac{d}{dt}\left(\frac{1}{2}\int_{B_{r-t}(\mathbf{p})} (u_t^2 + |Du|^2)\right) \le \frac{1}{2}\int_{\partial B_{r-t}(\mathbf{p})} [(u_t^2 + (Du \cdot \mathbf{n})^2) - (u_t^2 + |Du|^2)] \le 0.$$

This means that given initial Cauchy data $u(\mathbf{x}, 0) = u_0(\mathbf{x})$ and $u_t(\mathbf{x}, 0) = v_0(\mathbf{x})$ satisfying

$$u_0 = v_0 \equiv 0 \qquad \text{on } B_r(\mathbf{p})$$

we must have

$$u(\mathbf{x}, t) \equiv 0 \quad \text{for } 0 \le t \le r \text{ and } \mathbf{x} \in B_{r-t}(\mathbf{p}). \tag{1}$$

In particular, no values of u_0 and v_0 at points $\mathbf{x} \in \mathbb{R}^n \backslash B_{r-t}(\mathbf{p})$, or change in those values, can alter or effect in any way the conclusion (1). In particular, the value $u(\mathbf{p}, t) \equiv 0$ at all times $0 \le t \le r$. Nothing can move the value at $\mathbf{x} = \mathbf{p}$ before time $t = r$.

Here is a streamlined/simpler presentation of the main calculation above:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\int_{B_{r-t}(\mathbf{p})}(u_t^2+|Du|^2)\right) &= \frac{d}{dt}\left(\frac{1}{2}\int_0^{r-t}\left[\int_{\partial B_\tau(\mathbf{p})}(u_t^2+|Du|^2)\right]\,d\tau\right)\\
&= -\frac{1}{2}\int_{\partial B_{r-t}(\mathbf{p})}(u_t^2+|Du|^2)+\int_0^{r-t}\left[\int_{\partial B_\tau(\mathbf{p})}(u_t\Delta u+Du\cdot Du_t)\right]\,d\tau\\
&= -\frac{1}{2}\int_{\partial B_{r-t}(\mathbf{p})}(u_t^2+|Du|^2)+\int_{B_{r-t}(\mathbf{p})}\operatorname{div}(u_t Du)\\
&= \int_{\partial B_{r-t}(\mathbf{p})}\left[u_t Du\cdot\mathbf{n}-\frac{1}{2}(u_t^2+|Du|^2)\right]\\
&\le \int_{\partial B_{r-t}(\mathbf{p})}\left[|u_t Du\cdot\mathbf{n}|-\frac{1}{2}(u_t^2+|Du|^2)\right]\\
&\le \frac{1}{2}\int_{\partial B_{r-t}(\mathbf{p})}(|Du\cdot\mathbf{n}|^2-|Du|^2)\\
&\le 0.
\end{aligned}
$$

Lower Regularity

Notice, finally, that d'Alembert's formula makes sense for any Cauchy data $u_0, v_0 \in L^1_{loc}(\mathbb{R})$.

Exercise 2 *Use mathematical software to plot d'Alembert's formula for various choices of nonsmooth Cauchy data. For example, try* $u_0(x) = h_{x_0}(x) = \chi_{[x_0,\infty)}(x)$ *given by a Heaviside function which turns on at* $x = x_0$*. Another interesting choice might be to take the Heaviside function as the initial velocity* v_0*. Another choice might be to take* u_0 *to be a "tent function" which is something like* $-|x-x_0|+r$ *for* $|x| \le r$ *and zero elsewhere.*

The wave equation is the prototypical example of a **hyperbolic** second order linear partial differential equation. If you have a second order linear partial differential operator

$$Lu = \sum_{i,j} a_{ij} D_i D_j u$$

with leading order (constant, symmetric) coefficient matrix $A = (a_{ij})$ having one negative eigenvalue and the rest positive, you can expect similar properties for the solutions.