Lecture 18 MATH 6702 March 9, 2020

Last time: θ thermal energy density

$\vec{\Phi}$ heat flux field

- volume flux
- mass flux AND MASS flow

spatial velocity field $\mathbb{V} = \mathbb{V}(\mathbb{X}, t)$

flow $\begin{cases} \dot{\mathbb{X}} = \mathbb{V}(\mathbb{X}, t) \\ \mathbb{X}(0) = \mathbb{P} \end{cases}$

Spatial position $\mathbb{X} = \mathbb{P}$ (position)

Material position $\mathbb{X} = \mathbb{X}(t) = \mathbb{X}(t; \mathbb{P})$

spatial mass density $\rho(\mathbb{X}, t)$ (particle)

[redacted]

material mass density

$\rho(\mathbb{X}(t), t)$

spatial derivative: $\frac{\partial \rho}{\partial t}$

material derivative: $\frac{D\rho}{Dt}(\mathbb{X}(t), t) = D\rho \cdot \mathbb{V} + \frac{\partial \rho}{\partial t}$

Heat PDE: θ, $\vec{\phi}$, Q

↑ independent source/sink forcing (rate)

$[Q] = \dfrac{[\text{energy}]}{L^3 \cdot T}$

$$\boxed{\frac{d}{dt}\int_{\mathcal{V}} \theta = -\int_{\partial\mathcal{V}} \vec{\phi}\cdot N + \int_{\mathcal{V}} Q}$$ (accounting of heat energy w.r.t. time.)

$$\boxed{\frac{\partial\theta}{\partial t} = -\operatorname{div}\vec{\phi} + Q}$$ (heat PDE I)

unknowns: θ, $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$

given: $Q = Q(X, t)$

<u>Temperature</u> : $u = u(x, t)$ $\quad$ $[u] = [\text{temp}]$

LAW OF SPECIFIC HEAT : $\theta = \sigma \rho u$

specific heat capacity

mass density

$[\rho] = \frac{M}{L^3}$

$[\sigma] = \frac{[\text{energy}]}{M \cdot [\text{temp}] \cancel{T}}$

$\sigma = \sigma(x, \overset{\text{time}}{u}, \ldots)$; $\rho = \rho(x)$

no explicit time dependence.

$$\frac{\partial \theta}{\partial t} = \rho \frac{\partial}{\partial t}(\sigma u)$$

[redacted]

Fourier's Law

$$\boxed{\vec{\phi} = -K\, Du}$$

high

energy

low

temp gradient

$[Du] = \dfrac{[\text{temp}]}{L}$

thermal conductivity:

$[K] = \dfrac{[\text{energy}]}{L \cdot T\,[\text{temp}]}$

$K = K(x, u, \ldots \rho)$

one unknown u.

given: ρ, σ, K, Q

$-\text{div}(-K\, Du)$

$$\boxed{\rho \frac{\partial}{\partial t}[\sigma u] = \text{div}(K\, Du) + Q}$$

HEAT EQN II

[redacted]

$$\boxed{\rho \frac{\partial}{\partial t}[\sigma u] = \operatorname{div}(K \cdot Du) + Q}$$ Heat PDE III

presumably nonlinear

Usual Assumptions: $\begin{cases} \sigma, \rho & \text{constant} \\ K & \text{constant} \end{cases}$

$$\boxed{u_t = k \Delta u + q}$$ Heat PDE IV

linear

$k = \frac{K}{\sigma\rho}$, $q = \frac{Q}{\sigma\rho}$

constant

$q = q(x,t)$

facing.

scaling in space

$$\boxed{u_t = \Delta u + f}$$ Heat PDE!

[redacted]

1D derivation (interval)

$$\frac{d}{dt}\int_{\underbrace{[0,L]}_{I}} \Theta_1 = -\Big[\phi_1(\underset{b}{L},t) - \phi_1(\underset{a}{0},t)\Big] + \int_{\underbrace{[0,L]}_{I}} Q_1$$

exit rate

$$= -\int_{[0,L] \leftarrow I} \frac{\partial \phi_1}{\partial x} + \int_{[0,L] \leftarrow I} Q_1$$

$$(\Theta_1 = \sigma \rho u , \ \phi_1 = -K u_x)$$

$$\boxed{u_t = k u_{xx} + q}$$

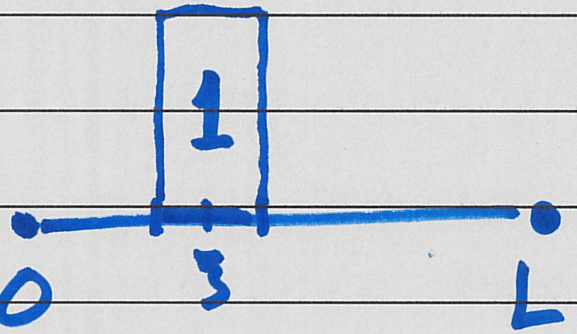
$$k = 1$$

$$\boxed{u_t = u_{xx} + f}$$

$$\boxed{u_t = u_{xx} + f}$$ 1-D heat eqn.

Steady state $u = u(x)$, $u_t = 0$.

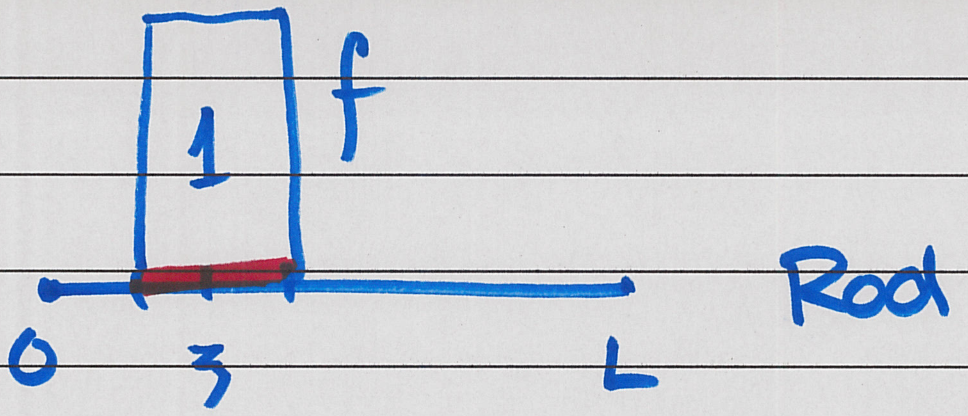
$$\boxed{-u_{xx} = f}$$ Poisson's ODE



Remember the Green's function!

Forcing

$$f(x) = \frac{1}{2\delta}\chi_{[\xi-\delta,\,\xi+\delta]}, \quad \int f = 1$$



f = thermal input density , $[f] = \frac{[\text{energy}]}{L\,T}$

units of f?

$\int_{\cancel{[0,L]}} f = 1$?

$[\xi-\delta, \xi+\delta]$

derivation:

$\frac{d}{dt}\int_{I=[a,b]} \theta_1 = \text{exit} + \int f$

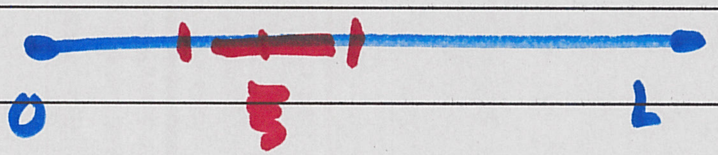
$=$

Important interpretation: f supplies 1 unit of energy/time uniformly across $[\xi-\delta, \xi+\delta]$

[redacted]

rate of heat energy in = 1

heat must exit.



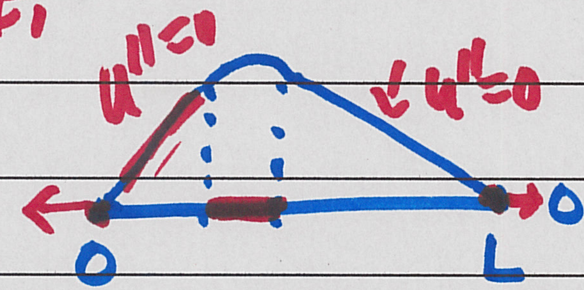
Fourier's Law at endpoints : $\vec{\Phi} = -u' e_1$

heat goes out at $x = 0$

$\vec{\Phi} = -K u' e_1$

rate: $\frac{L-\xi}{L}$

Weak solution

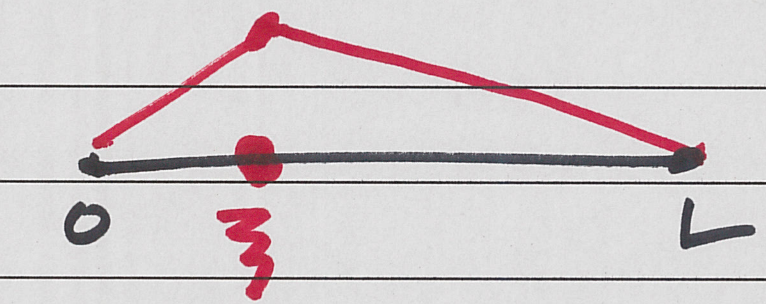


heat goes out at L , $u_x < 0$

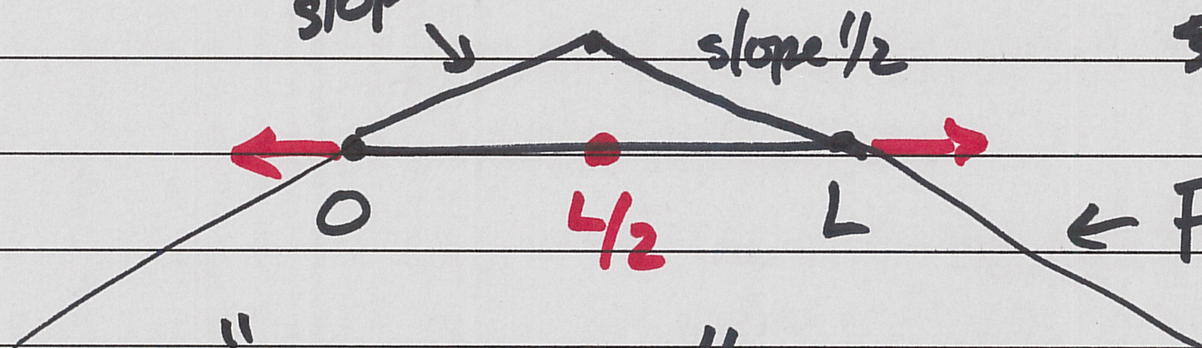
rate $\frac{\xi}{L}$

$\frac{L-\xi}{L} + \frac{\xi}{L} = 1$

[redacted]



$$u = \int_{\xi \in [0,L]} G(x,\xi)\, \underline{\underline{f(\xi)}}\, d\xi$$



← Fundamental Solution. Φ

"$-\Delta G = \delta_{L/2}$"

$$G\Big|_{x=0,L} = 0$$

[redacted]