Lecture 17 MATH 6702 March 4, 2020

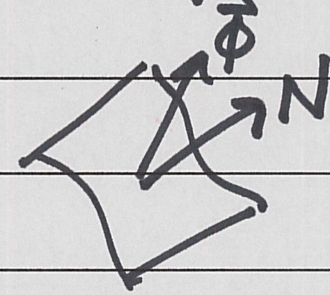
Θ thermal energy density

3 players : Θ , $\vec{\Phi}$ heat energy flux field.

Q forcing/source density

from last time: volumetric flux integral

$$\int_{\mathcal{U}} \vec{\Phi} \cdot N$$



rate of volume crossing $\mathcal{U}$

surface

$[\vec{\Phi}] = \frac{L^3}{L^2 T}$

2D $[\vec{\Phi}_2] = \frac{L}{T}$

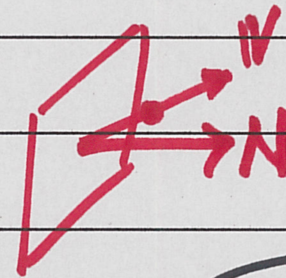
1D $[\vec{\Phi}_1] = \frac{L}{T}$

Mass Flux

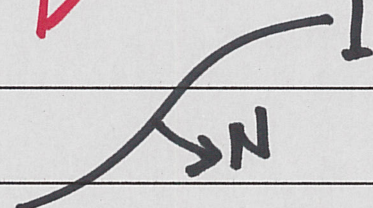
$[\vec{\Phi}] = \frac{M}{L^2 T}$ ← gives rate of mass crossing a surface $\mathcal{U}$.

3D

$\frac{\text{Mass}}{\text{Time}} \propto \frac{\text{volume}}{\text{time}}$ $\int_{\mathcal{U}} \vec{\Phi} \cdot N$

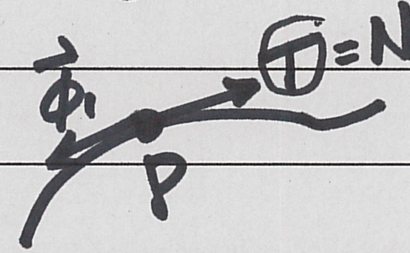


2D $[\vec{\Phi}_2] = \frac{M}{L \cdot T}$

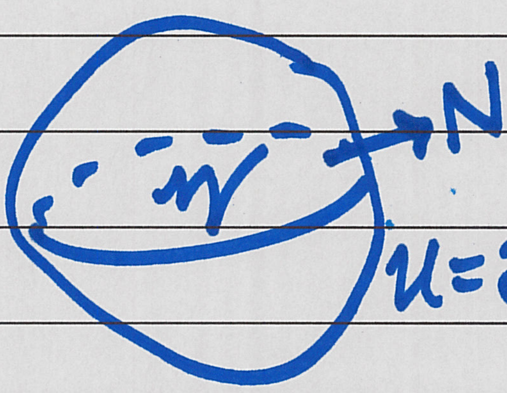


Γ rate of mass crossing Γ

1D $[\vec{\Phi}_1] = \frac{M}{T}$



$\vec{\Phi}_1 \cdot N$ rate of mass crossing P in direction N.

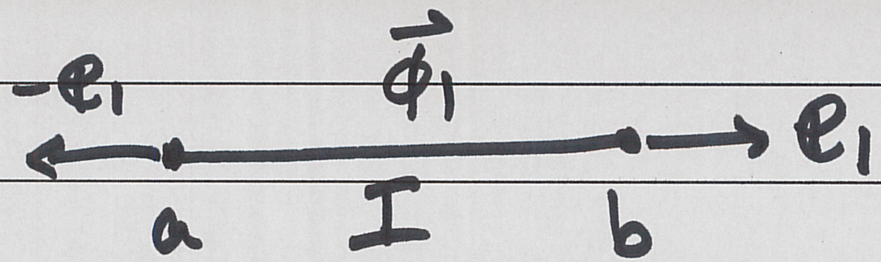


N ← outward unit normal

$\mathcal{U} = \partial \mathcal{V}$

$\int_{\partial \mathcal{V}} \vec{\Phi} \cdot N$

↑ rate at which mass exits $\mathcal{V}$



rate at which mass is exiting I through $\partial I = \{a, b\}$

$$\vec{\phi}_1(b) \cdot e_1 - \vec{\phi}_1(a) \cdot e_1 \leftarrow$$ 1D exit rate for mass

Heat Energy Flux $\vec{\Phi}$

$$[\vec{\Phi}] = \frac{[\text{energy}]}{L^2 \cdot T}$$

$$[\vec{\phi}_2] = \frac{[\text{energy}]}{LT}$$

$$[\vec{\phi}_1] = \frac{[\text{energy}]}{T}$$

$\uparrow$ 1D heat flux.

rate of heat energy crossing $\mathcal{U}$:

$$\int_{\mathcal{U}} \vec{\Phi} \cdot N$$

[redacted]

Relations between density and flux
— conservation and the flow of mass

Remember θ, $[\theta] = \frac{[\text{energy}]}{L^3}$

For modeling leading to the heat equation, we assume there is no flow of mass.
- One consequence is that θ can be thought of as a <u>material density</u> or a <u>spatial density</u>.

With <u>mass flow</u> these become different:

usual: Introduce a spatial velocity field $\mathbb{V}$, $[\mathbb{V}] = \frac{L}{T}$

Then there is a natural mass flux field associated with the flow of mass via $\mathbb{V}$:

$$\vec{\Phi} = \rho \mathbb{V} \leftarrow \text{mass flux field.}$$

$\rho = \rho(x, t)$ mass density

$\rho = \rho(x, t)$ spatial mass density

Note: $[\vec{\Phi}] = \frac{M}{L^2 T}$ $\begin{cases} [\rho] = \frac{M}{L^3}. \\ [\mathbb{V}] = \frac{L}{T} \end{cases}$

Conservation of mass is expressed by

$$\boxed{\frac{d}{dt}\int_{\mathcal{V}} \rho = -\int_{\partial\mathcal{V}} \rho \mathbb{V} \cdot N}$$

Integral form.

$$\frac{d}{dt}\int_{\mathcal{W}} \rho = -\int_{\partial\mathcal{W}} \rho \mathbb{V}\cdot N$$

Divergence Theorem (Gauss' Theorem)

For any vector field $\mathbb{W} \in C^1(\overline{\mathcal{W}})$

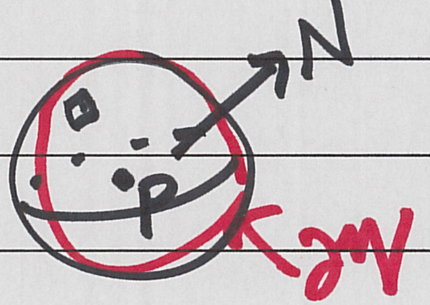
$$\int_{\partial\mathcal{W}} \mathbb{W}\cdot N = \int_{\mathcal{W}} \operatorname{div} \mathbb{W}$$

$$\left.\begin{array}{l}\rho \in C^1(\overline{\mathcal{W}}) \\ \mathbb{V} \in C^1(\overline{\mathcal{W}})\end{array}\right\} \quad \begin{array}{l}\displaystyle\int_{\mathcal{W}} \rho_t = -\int_{\mathcal{W}} \operatorname{div}(\rho\mathbb{V}) \\ \displaystyle\int \big(\underbrace{\rho_t + \operatorname{div}(\rho\mathbb{V})}\big)\, \underline{\underline{\chi_{\mathcal{W}}}} = 0\end{array}$$

[redacted]

Divergence Theorem

What is the divergence of a vector field?



$$\int_{\partial\gamma} \mathbb{W} \cdot N \Leftarrow \text{flux integral.}$$

Imagine a limit in which $\gamma \to \{p\}$

$$\lim_{\gamma \to \{p\}} \int_{\partial\gamma} \mathbb{W} \cdot N = 0$$

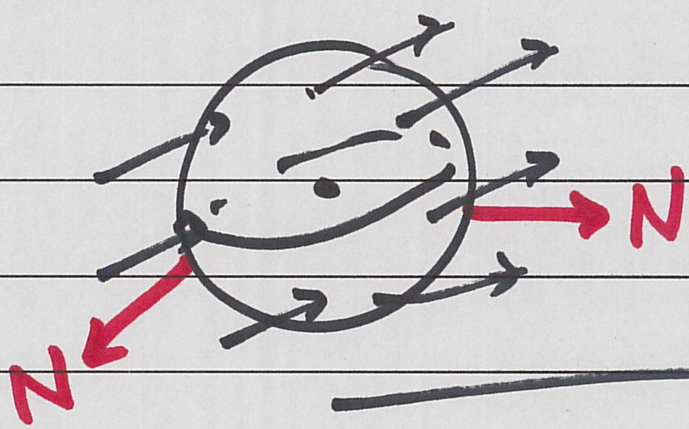
$\sim |\mathbb{W}(p)|$

$$\left| \int_{\partial\gamma} \mathbb{W} \cdot N \right| \leq \int_{\partial\gamma} |\mathbb{W}|$$

$$\leq (|\mathbb{W}| + 1)\, \text{Area}(\partial\gamma)$$

[redacted]

<u>Try</u> $\displaystyle \lim_{\mathcal{V} \to \{p\}} \frac{1}{\text{Area}(\partial \mathcal{V})} \int_{\partial \mathcal{V}} \mathbb{W} \cdot N = 0$

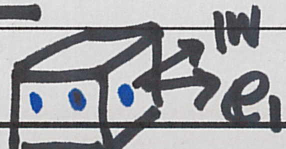


Definition of Divergence

$$\boxed{\lim_{\mathcal{V} \to \{p\}} \frac{1}{\text{vol}(\mathcal{V})} \int_{\partial \mathcal{V}} \mathbb{W} \cdot N = \operatorname{div} \mathbb{W}}$$

<u>In coordinates:</u> $\operatorname{div} \mathbb{W} = \dfrac{\partial W_1}{\partial x} + \dfrac{\partial W_2}{\partial y} + \dfrac{\partial W_3}{\partial z}$

<u>(rectangular)</u>



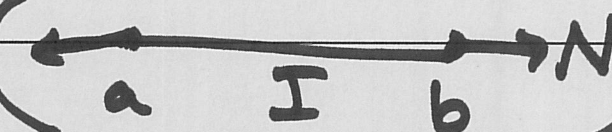
$\mathbb{W} = (W_1, W_2, W_3)$

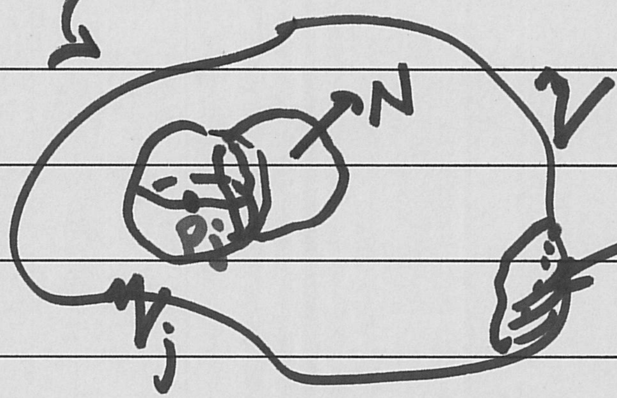
[redacted]

Divergence Theorem

$$\int_{\partial \mathcal{V}} \mathbb{W}\cdot N = \int_{\mathcal{V}} \operatorname{div} \mathbb{W}$$

$$\int_I f' = f\Big|_{\partial I}$$





$$\int_{\mathcal{V}} \operatorname{div} \vec{\mathbb{W}} \approx \sum \underbrace{\operatorname{div} \mathbb{W}(p_j)} \operatorname{vol}(\mathcal{V}_j)$$

$$\approx \sum \int_{\partial \mathcal{V}_j} \mathbb{W}\cdot N$$

$$\approx \int_{\partial \mathcal{V}} \mathbb{W}\cdot N .$$

[redacted]

back to mass flow

$$\frac{d}{dt}\int_{\mathcal{V}} \rho = -\int_{\partial\mathcal{V}} (\rho \mathbf{v})\cdot N$$

$\rho, \mathbf{v}$

$\rho \in C^1(\bar{\mathcal{V}})$

$\mathbf{v} \in C^1(\bar{\mathcal{V}} \to \mathbb{R}^3)$

$$\int_{\mathcal{V}} \rho_t = -\int_{\mathcal{V}} \operatorname{div}(\rho \mathbf{v})$$

$$\Rightarrow \boxed{\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0}$$

differential form of conservation of mass.

Conservation of Heat Energy

$$\frac{d}{dt}\int_{\mathcal{V}} \theta = -\int_{\partial \mathcal{V}} \vec{\phi} \cdot N$$

$$\Rightarrow \boxed{\frac{\partial \theta}{\partial t} + \operatorname{div}(\vec{\phi}) = 0} \leftarrow \underline{\text{Heat}}\ \underline{\text{Equation}}$$

Q $\begin{cases} \text{internal energy, external energy} \\ \text{generated, extracted.} \end{cases}$

$$[Q] = \frac{[\text{energy}]}{L^3 T}$$

$\begin{cases} 2D\ Q_2 \\ 1D\ Q_1 \end{cases}$ $\int_{\mathcal{V}} Q$

[redacted]

$$\frac{d}{dt}\int_{\mathcal{V}} \theta = -\int_{\partial\mathcal{V}} \vec{\Phi}\cdot N + \int_{\mathcal{V}} Q$$

total heat energy in $\mathcal{V}$

exiting rate of energy through $\partial\mathcal{V}$

<u>Heat Eqn</u> $$\boxed{\frac{\partial\theta}{\partial t} = -\operatorname{div}\vec{\Phi} + Q}$$

<u>Physical</u> "Laws" Temperature u

specific heat law $\theta = c\rho u$

Fourier's law $\vec{\Phi} = -K\,Du$

[redacted]