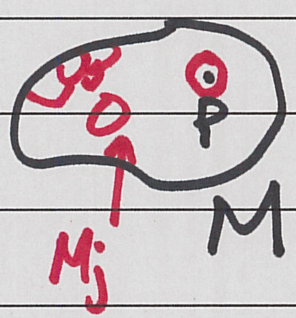
Lecture 16 MATH 6702. March 2, 2020

Physical Interpretation / Derivation of Poisson Eqn / Laplace's Eqn. / Heat Eqn.

① Players
② Modeling
③ Math Techniques
④ Physical "laws"

Integration $\boxed{1}$ You can integrate on just about anything.

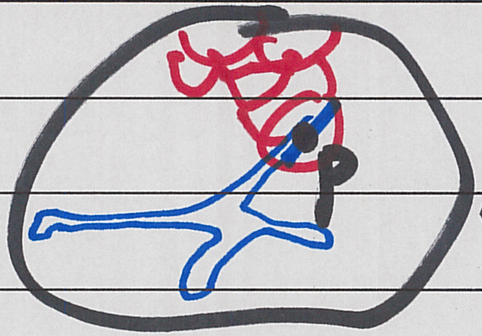


$M \xrightarrow{f} \mathbb{R}$

Need to be able to cut M up in little pieces

$$M = \bigcup M_j$$

Need to be able to cut up M.



$M \xrightarrow{f} \mathbb{R}$

and <u>measure the pieces</u>

pieces M_j

$M = \bigcup_j M_j$ (partition)

measure of pieces

$$\int_M f = \lim \sum_j \underline{\underline{f(p_j^*)}}\, \mu(M_j)$$

The limit is as the diam of the largest piece tends to 0.

$\boxed{2}$ Then you need formula in specific cases.

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$a \quad x_1, x_2 \dots \quad x_n \; b$

$$\lim \sum f(x_j^*)(x_{j+1} - x_j)$$

$$f(x) = x^n \xrightarrow[\text{FTC} \\ + \text{perm}]{} \int_a^b x^n\,dx = \frac{1}{n+1} x^{n+1} \Big|_a^b$$

$\boxed{3}$ parameterization + scaling factor is the main new thing. LATER

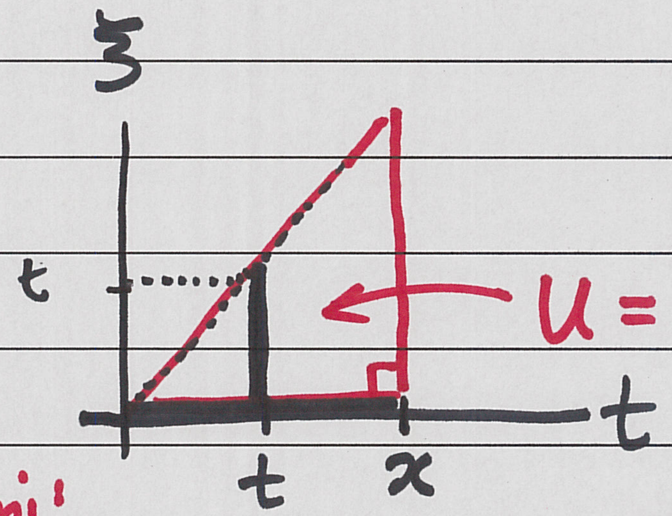
[redacted]

Exam Problem : $\begin{cases} u'' = f(x) \\ u(0) = 0 = u(L) \end{cases}$

$$u'(x) = u'(0) + \int_0^x f(\xi)\, d\xi$$

$$u(x) = u'(0)\, x + \underbrace{\int_0^x \int_0^t \underline{f(\xi)}\, d\underline{\xi}\, d\underline{t}}_{\text{Iterated integral(s)}}$$

formula from Fubini's Theorem.



$U = \{(t,\xi) : 0 < \xi < t,\ 0 < t < x$

Fubini:

$$\int_{(t,\xi)\in U} \bar{f}(t,\xi) = \lim \sum_j \bar{f}(p_j^*)\, \mu(U_j)$$

$$\bar{f}(t,\xi) = f(\xi)$$

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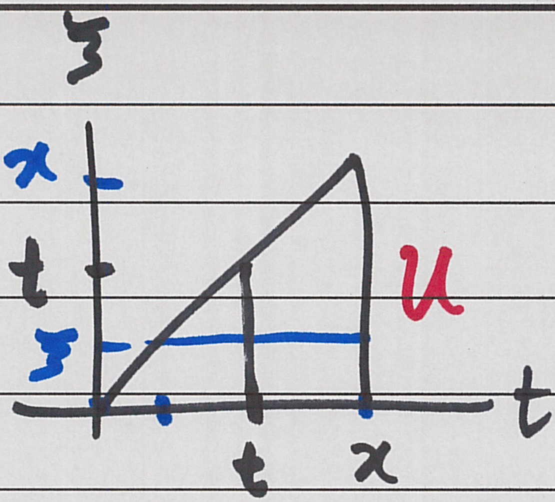

Fubini:

$$\int_U f = \int_0^x \int_0^t f(\xi)\, d\xi\, dt$$

$$= \int_0^x \int_\xi^x f(\xi)\, dt\, d\xi$$

$$= \int_0^x f(\xi)(x-\xi)\, d\xi.$$

Apply to ODE

$u'' = f(x)$

$$u(x) = u'(0)x + \int_0^x \int_0^t f(\xi)\, d\xi\, dt$$

$$= u'(0)x + \int_0^x f(\xi)(x-\xi)\, d\xi$$

use $u(L) = 0$

$$\Longrightarrow \quad u'(0) = -\frac{1}{L}\int_0^L f(\xi)(L-\xi)\, d\xi$$

[redacted]

$u'' = f$

$$u(x) = u'(0)\,x + \int_0^x f(\xi)(x-\xi)\,d\xi$$

$$= -\frac{x}{L}\int_0^L f(\xi)(L-\xi)\,d\xi + \int_0^x f(\xi)(x-\xi)\,d\xi$$

$$= \int_0^L \Big[\underbrace{\frac{\xi - L}{L}x + \chi_{[0,x]}(\xi)\,(x-\xi)}_{} \Big] f(\xi)\,d\xi$$

$$= \begin{cases} \frac{x-L}{L}\xi, & 0 \le \xi \le x \\ \frac{\xi - L}{L}x & x \le \xi \le L \end{cases}$$

Want:

$$u(x) = \int_0^L G(x,\xi)\, f(\xi)\, d\xi$$

$$G(x,\xi) = \begin{cases} \frac{\xi - L}{L}x & , 0 \le x \le \xi \\ \frac{\xi}{L}(x-L) & , \xi \le x \le L \end{cases}$$

[redacted]

3rd Solution

$$\int_0^x \underbrace{\int_0^t f(\xi)\,d\xi}_{\phi(t)}\,dt \qquad d\psi = dt$$

integrate by parts

$$= t\int_0^t f(\xi)\,d\xi\,\Big|_0^x - \int_0^x t\,f(t)\,dt$$

$$= x\int_0^x f(\xi)\,d\xi - \int_0^x \xi\,f(\xi)\,d\xi$$

$$= \int_0^x (x-\xi)\,f(\xi)\,d\xi$$

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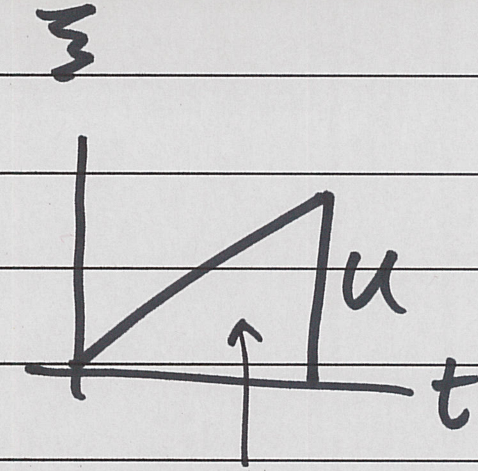
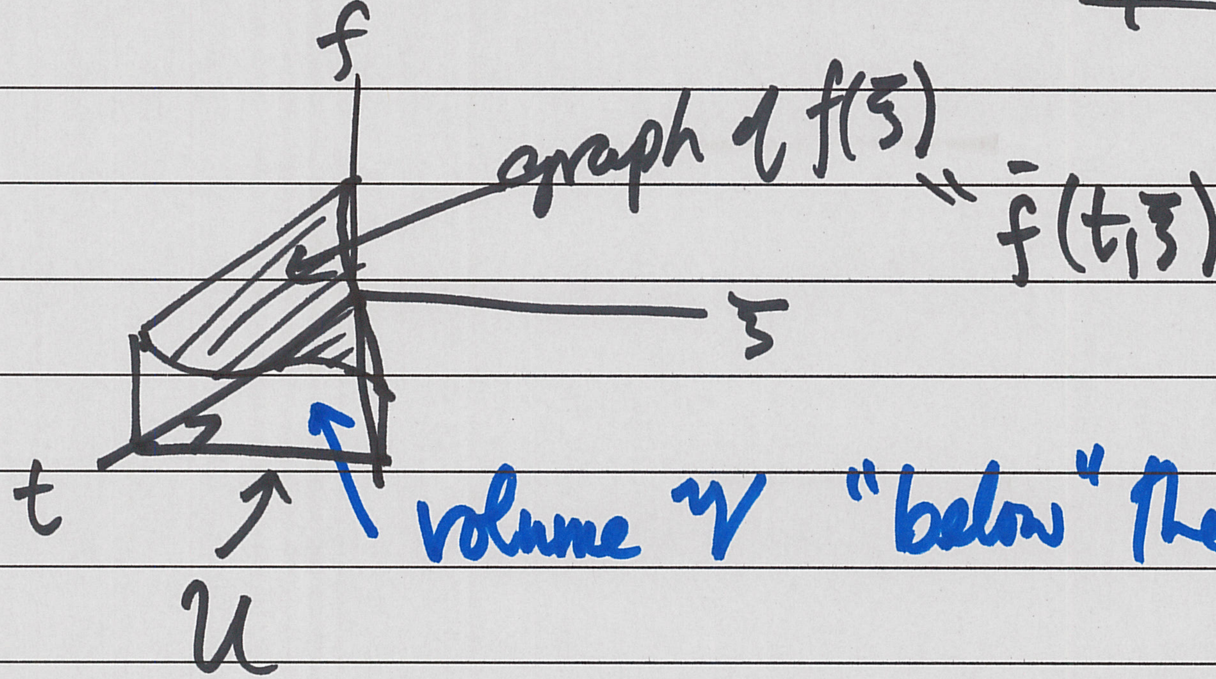
Integral Over a Volume?

$$\int_0^x \int_0^t f(\xi)\, d\xi\, dt$$





$$\int_0^x \int_0^t f(\xi)\, d\xi\, dt = \int_U f = \int_V 1 .$$

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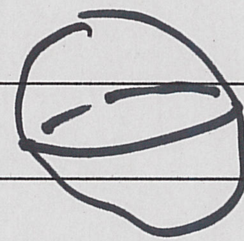
Heat Energy Density

θ = heat energy density

$$[\theta] = \frac{[\text{energy}]}{L^3} = \frac{[\text{work}]}{L^3} = \frac{[\text{force}]}{L^2} = \frac{M}{LT^2}$$

↗ units of θ

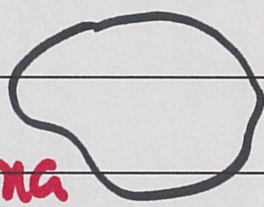


$\mathcal{V}$

total heat energy in $\mathcal{V}$

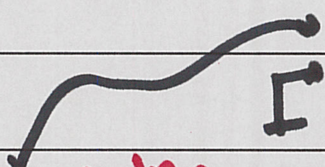
$$\int_{\mathcal{V}} \theta$$

<u>Lower dimensions</u> :



lamina

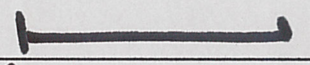
$\mathcal{U} \subseteq \mathbb{R}^2 \quad [\theta_2] = \frac{[\text{energy}]}{L^2}$, total NRG $= \int_{\mathcal{U}} \theta_2$



Γ

wire

$[\theta_1]$, $\int_{\Gamma} \theta_1$



0 L

rod.

$$\int_0^L \theta_1(x,t)\, dx$$

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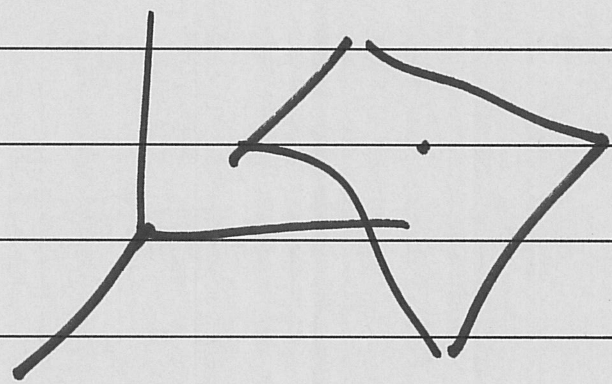
Heat Energy Flux

$\vec{\Phi}$ = heat energy flux field, $[\vec{\Phi}] = \dfrac{[\text{energy}]}{L^2 \cdot T}$

"flux" means "per time"



roughly $\vec{\Phi}$ tells how much heat energy crosses a surface "at a point"

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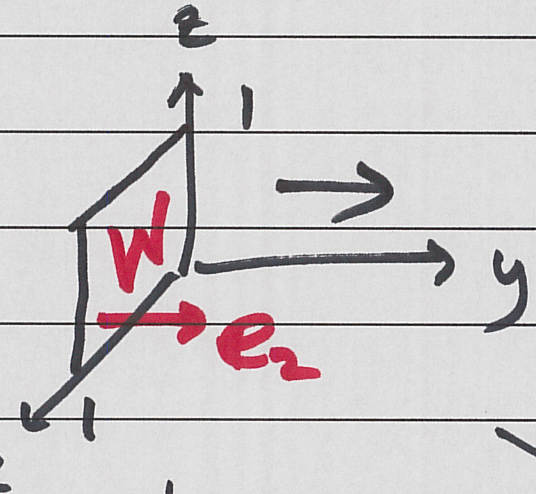
Volume Flux

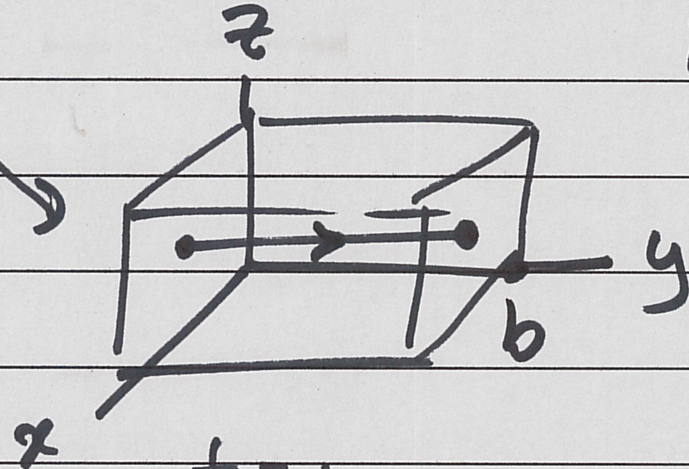
volume flux field is like a velocity field.



$t=0$

$$\vec{\phi} = b\,e_2 = b(0,1,0)$$

$$[\vec{\phi}] = \frac{L^3}{L^2 \cdot T}$$



$t=1$

rate of volume going to the right across W

$$\int_W \vec{\Phi}\cdot e_2 = b \leftarrow b \cdot \text{area}(W)$$

$$\left[\int_W \vec{\Phi}\cdot e_2\right] = \frac{[\text{vol}]}{T} = \frac{L^3}{T} \;\text{“}\; \text{!}$$

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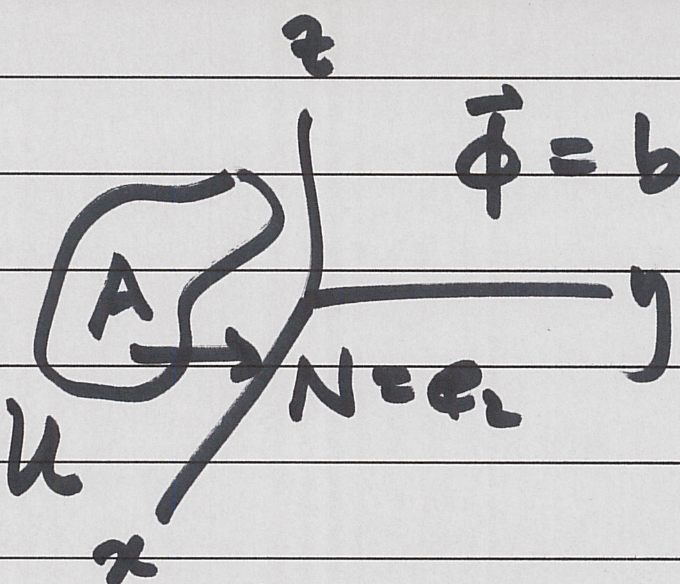
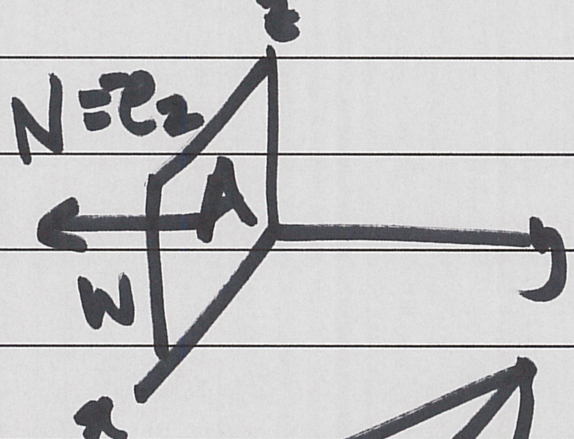


$\vec{\Phi} = b\,e_2$

flux integral

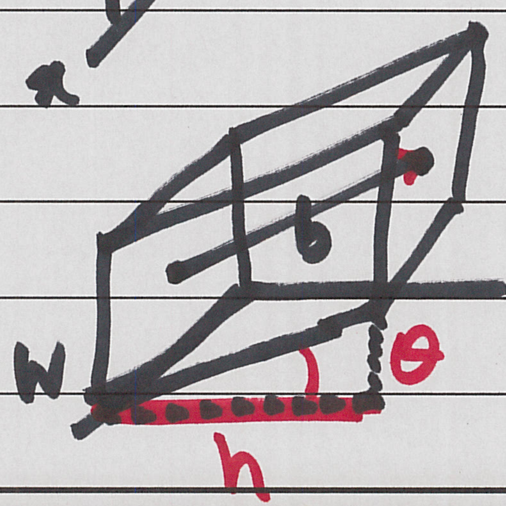
$\int_{\mathcal{U}} \vec{\Phi} \cdot e_2 = Ab.$

↑ units $\frac{[\text{vol}]}{T}$



rate of volume crossing $\mathcal{U}$ to the left.

$\int_{W} \vec{\Phi} \cdot e_2 = -Ab.$



$\vec{\Phi} = (0, b\cos\theta, b\sin\theta)$

$\vec{\Phi} \cdot e_2 = h = b\cos\theta$

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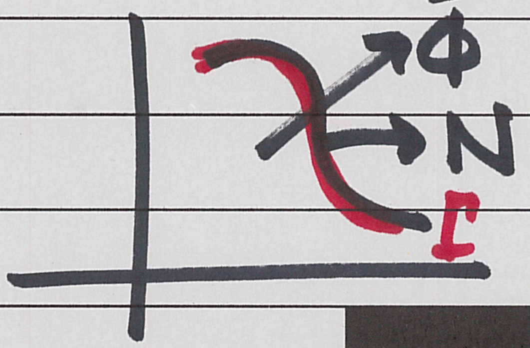
(Volumetric) Flux Integral

$$\int_{\mathcal{U}} \vec{\Phi} \cdot N = \text{rate of volume crossing } \mathcal{U} \text{ in the direction of } N$$

($\mathcal{U}$: surface with unit normal N)

area flux (2D)

$$[\vec{\Phi}_2] = \frac{[\text{area}]}{L \cdot T}$$



$$\int_I \vec{\Phi} \cdot N$$

1D $\vec{\Phi}_1 = \frac{[\text{length}]}{T}$

↑ at a point.

[redacted]