Lecture 19 MATH 6702 March 11, 2020

Exam 2

Problem 2: get an answer that has an integral in it. (don't bee surprised.)

HW 9. $\Phi(x,y) = \Phi_0(x^2+y^2)$

fundamental solution

Next problem: What PDE does $\Phi$ solve?

[redacted]

Heat Equation (from last time):

$$u_t = \Delta u + f$$

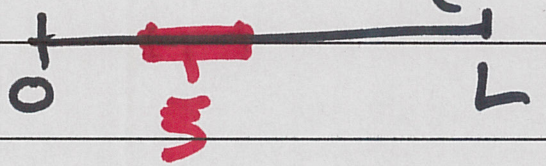
$\hookrightarrow$ steady state $-\Delta u = f$

1D

**ODE**

$$\begin{cases} u'' = 0 \\ u\big|_{x=0,L} = u_0, u_L \end{cases} \longleftrightarrow \begin{cases} -u'' = f \\ u\big|_{x=0,L} = 0 \end{cases}$$

$$\begin{cases} \text{"}-\Delta G = \delta\text{"} \\ G\big|_{x=0,L} = 0 \end{cases}$$



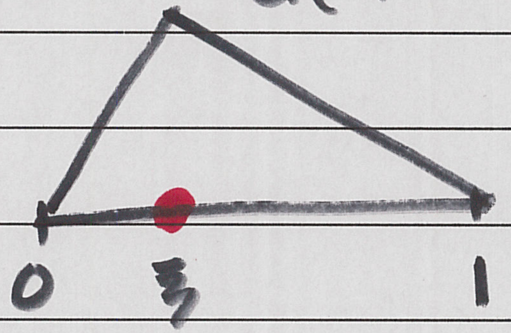

[redacted]

PDE $\quad u_t = \overset{k}{\Delta} u$ , $\quad$ 1-D Heat Eqn.

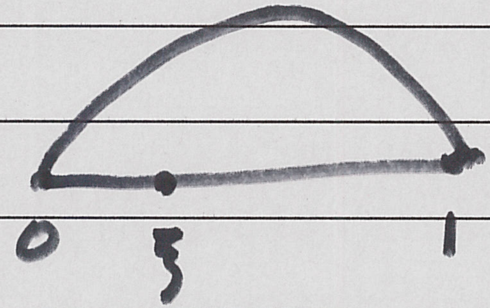
$$G(x,\xi,t) = \sum_{j=1}^{\infty} \underset{\substack{\uparrow \\ \text{large}}}{\frac{2}{L}} \sin\frac{j\pi\xi}{L} \sin\frac{j\pi x}{L}\, e^{-\overset{k}{}\left(\frac{j\pi}{L}\right)^2 t}$$

$\uparrow$ helps when $t>0$.

$G(x,\xi)$



$G(x,\xi,t)$ $\leftarrow$ Formally satisfies $u_t = \Delta u = u''$ $t>0$.



$\xi = \frac{1}{4}$

$t = 1$

[redacted]

$G = G(x, \xi, t)$ What is $\xi$ doing here?

Guess

$$\begin{cases} G_t = G_{xx} \\ G(0,\xi,t) = 0 \ , \ G(L,\xi,t) = 0 \\ \text{"} G(x,\xi,0) = \delta_\xi \text{"} \end{cases}$$

"$\Delta G = \delta_\xi$" for steady state.

[redacted]

Existence and Uniqueness for weak solutions of

$$\begin{cases} -\Delta u = f \\ u|_{\partial U} = 0 \end{cases}$$

<u>Weak formulation</u> $u \in W^{1,2}(U)$

$W^{1,2}(U)$ = The set of all $L^2$ functions $u$ with first order weak derivatives

$D_j u = g_j$

$$\left( -\int u D_j \phi = \int g_j \phi \right)$$

$\forall \phi$.

[redacted]

$$\int (-\Delta u)\phi = \int f\phi$$

$$\boxed{\int Du \cdot D\phi = \int f\phi \qquad \forall \phi}$$

weak 1st partials

Also consider $\phi Du$

$$\operatorname{div}[u D\phi] = Du \cdot D\phi + u\Delta\phi$$

$$\operatorname{div} \mathbb{V} = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}$$

"Integration by parts"

$$\int_U \operatorname{div}[u D\phi] = \int Du \cdot D\phi + \int u \Delta\phi$$

divergence thm $\;\shortparallel\;$ $\int_{\partial U} u D\phi \cdot N$ $\;\shortparallel\;$ $0$

[redacted]

We're looking for $u \in W^{1,2}(U)$ with

$$\int_U Du \cdot D\phi = \int_U f\phi \quad \forall \phi \in C_c^\infty(U)$$

weak gradient. "$-\Delta u = f$"

**Boundary Condition:**

$W^{1,2}(U) = H^1(U)$ is an inner product space.
Hilbert space.

$\langle \cdot , \cdot \rangle : H^1(U) \times H^1(U) \to \mathbb{R}$

symmetric, positive definite

$\langle v, v \rangle \geq 0$, "$=$" if and only if $v = 0$.

[redacted]

You can do geometry in an inner product space.

$$\langle v, w \rangle = \|v\| \, \|w\| \cos\theta$$

angle between functions!

$$\int_{[0,L]} \sin\frac{j\pi x}{L} \sin\frac{k\pi x}{L}$$

$$\parallel$$

$$\left\langle \sin\frac{j\pi x}{L}, \sin\frac{k\pi x}{L} \right\rangle = \begin{cases} 0, & j \neq k \leftarrow \text{orthogonal} \\ \frac{L}{2}, & j = k \end{cases}$$

$$\langle u, v \rangle_{H^1} = \int uv + \int Du \cdot Dv$$

$$H^1 = W^{1,2}$$

[redacted]

$$C_c^\infty(U) \subseteq H^1(U)$$

$$\begin{cases} -\Delta u = f \\ u|_{\partial U} \equiv 0 \end{cases}$$

$$\overline{C_c^\infty(U)} \subseteq H^1(U)$$

closed subspace

$$u \in H_0^1(U) = W_0^{1,2}(U)$$

Convergence of sequences

Hilbert Space $\langle \cdot , \cdot \rangle$

$\| \cdot \| = \sqrt{\langle \cdot , \cdot \rangle}$

$d(v,w) = \| v - w \|$

[redacted]

# Hilbert Space?

Start with an inner product space

→ norm

→ distance (metric space)

0 is not in the space.

0 1

$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

look at sequences: $\{v_j\}_{j=1}^{\infty}$

**Cauchy Condition**: A sequence ↑ is Cauchy if for any $\varepsilon > 0$, there is some $N$ such that $j, k > N \Rightarrow \|v_j - v_k\| < \varepsilon$.

[redacted]

A Metric Space is (metrically) <u>complete</u> if every Cauchy sequence converges.

$$\left.\begin{array}{l} H_0^1(U) \\ H^1(U) = W^{1,2}(U) \\ L^2 \end{array}\right)$$ are complete metric spaces.

Complete normed space = Banach Space

Complete inner product space

=

Hilbert space.

[redacted]

$$C^0[a,b] \subseteq L^1[a,b]$$

$\|u\|_{L^1}$

$C^0[a,b]$ is not complete in $\|\cdot\|_{L^1}$

[redacted]