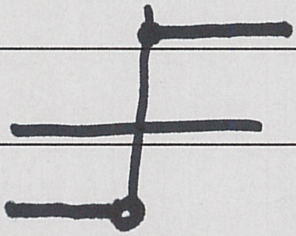
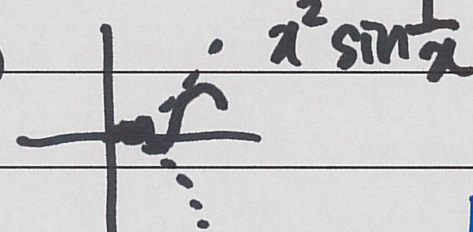
Math Methods II Lecture 2 Jan 8, 2020

<u>Last Time</u>: $C^{0,\alpha}$ $C^{1,\alpha}$ <u>problems</u>: INCLUSION METRIC

$$C^0 \supseteq C^1 \supseteq C^2 \supseteq \cdots \supseteq C^k \supseteq \cdots$$

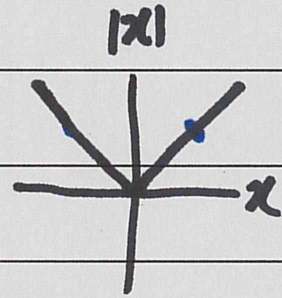
Diff





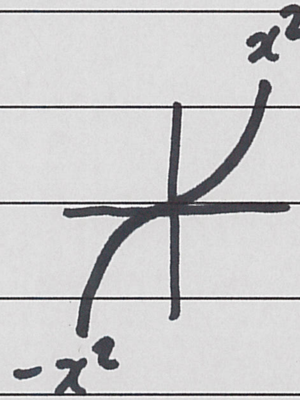
monotone

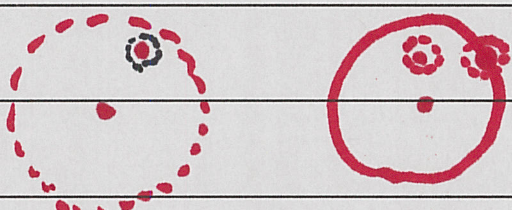
BV



<u>Lipschitz</u> <u>continuous</u>
= "essentially" differentiable

$$|f(x_2)-f(x_1)| \le \Lambda |x_2-x_1|$$





$\{x \in \mathbb{R}^2 : |x-0| \le 1\}$

Existence / Uniqueness for ODE's

$x' = f(x)$ or $x' = f(x,t)$ ← ODE's for $x = x(t)$

usual: If $f \in C^1$, then the ODE has unique solutions

i.e., the BVP $\begin{cases} x' = f(x) \\ x(t_0) = x_0 \end{cases}$

better: $f \in \text{Lip} \Rightarrow$ Existence and uniqueness.

$x' = |x|$

$f \in C^0$ (Peano)

$f \in \text{Lip}$ (Lipschitz)

Regularity for ODE

$x' = f(x)$

$x \in \text{Diff} \Rightarrow x \in C^0,\ f \in C^0 \Rightarrow x \in C^1.$

Why not 2^{nd} order ODE $\quad y'' = y$?

$$\left(y = c_1 e^{-t} + c_2 e^{t} \right)$$

Every ODE is equivalent to a first order <u>system</u>.

$$\begin{cases} y' = z \\ z' = y \end{cases} \qquad \begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$\mathbb{X}' = \mathbb{F}(\mathbb{X}, t)$$

Existence + uniqueness

[redacted]

Regularity for Laplace's PDE :

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \quad , \; u = u(x,y) .$$

Is it possible that $\dfrac{\partial^2 u}{\partial x \partial y}$ does not exist ?

Is it possible that $\dfrac{\partial^2 u}{\partial x^2}$ or $\dfrac{\partial^2 u}{\partial y^2}$ is not C^0 ?

Is it possible that $u \notin C^4$?

$\dfrac{\partial^4 u}{\partial x^4}$ No!

$$f \in \text{Diff} \implies f \in C^0.$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists. (and is finite)}$$

$$\implies \lim_{h \to 0} |f(x+h) - f(x)| = 0$$

continuity.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

harmonic function

$\uparrow$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Cauchy – Riemann PDEs

u, v conjugate harmonic functions.

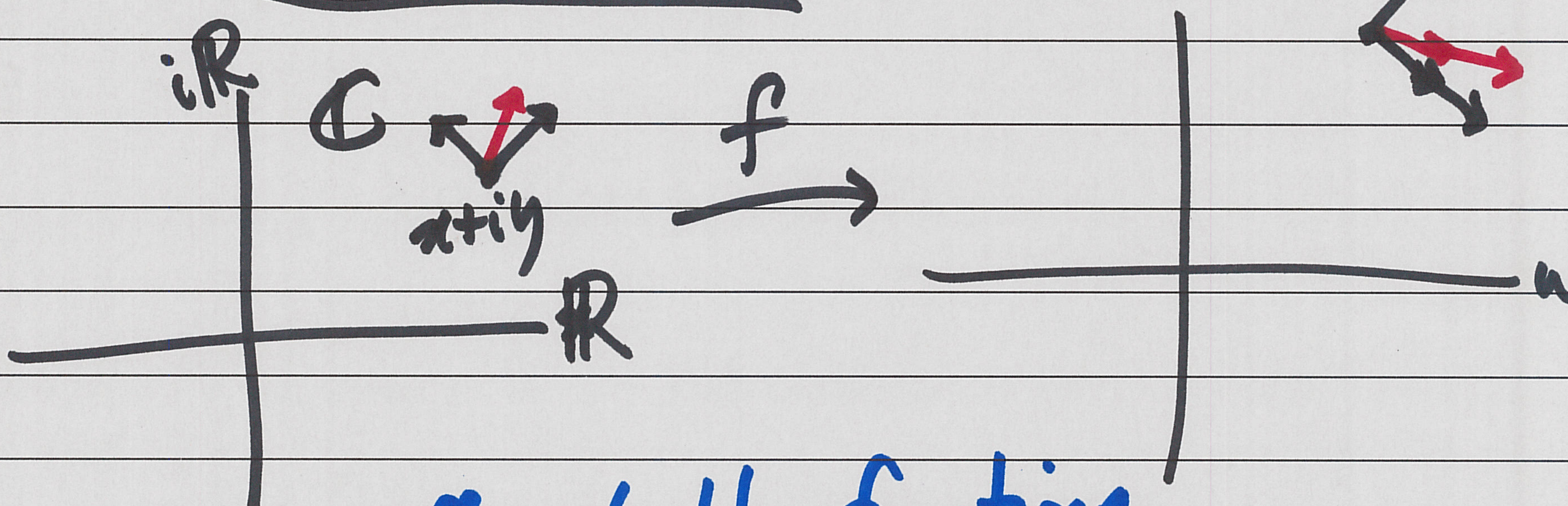
$u = u(x, y)$

$$\boxed{f(x+iy) = u + iv}$$

complex differentiable functions are <u>analytic</u>.

[redacted]

Complex Functions



complex differentiable functions

||

conformal maps

[redacted]

$$e^x$$

$$f(x+iy) = e^{x+iy} = e^x(\cos y + i\sin y) = e^x\cos y + i e^x \sin y$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{u} \qquad\qquad \overset{\shortparallel}{v}$$

Euler's formula

$$u(x,y) = e^x \cos y \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u_x = e^x \cos y, \quad u_{xx} = e^x \cos y$$

$$u_y = -e^x \sin y, \quad \underline{u_{yy} = -e^x \cos y}$$

[redacted]

8.

$$\sin(x+iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{1}{2i}\left[(\cos x + i\sin x)e^{-y} - (\cos x - i\sin x)e^{y}\right]$$

$$= \frac{1}{2i}\left[(e^{-y} - e^{y})\cos x + i(e^{-y} + e^{y})\sin x\right]$$

$$= \underbrace{\frac{e^{y} - e^{-y}}{2}\cos x}_{v}\, i + \underbrace{\frac{e^{y} + e^{-y}}{2}\sin x}_{u}$$

$$\boxed{\begin{aligned} u &= \sin x \cosh y \\ v &= \cos x \sinh y \end{aligned}}$$

[redacted]

A set E is <u>closed</u> if its complement

$\mathbb{R}^n \setminus E$ is open.

The union of any collection of open sets is open.

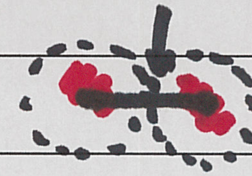
$$\bigcup_\alpha U_\alpha$$

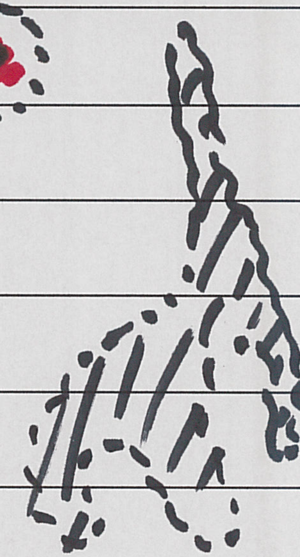
Any intersection of closed sets is closed.

DeMorgan $\left(\bigcap E_\alpha\right)^c = \bigcup E_\alpha^c$ open.

closure $\overline{A}$ = intersection of closed sets E with $A \subseteq E$

[redacted]





$\underline{\underline{\partial A}} = \overline{A} \cap \overline{A^c}$

$B_r(p) = \{x : |x-p| < r\}$

$\overline{B_r(p)} = \{x : |x-p| \leq r\}$

$\partial B_r(p) = \{x : |x-p| = r\}$

<u>**Connected**</u> in $\mathbb{R}^n$: U is <u>connected</u> if whenever V_1 and V_2 are disjoint ope [redacted]
sets with $U \subseteq V_1 \cup V_2$, then
$V_1 = \phi$ or $V_2 = \phi$.

Correction: $V_1 \cap U$ $V_2 \cap U$

A set $E \subseteq \mathbb{R}^n$ is <u>bounded</u> if there is some M such that

$$|\mathbf{x}| < M \text{ for every } \mathbf{x} \in E.$$

<u>Exercise</u> If E is bounded, then $\bar{E}$ is bounded.

If E is connected, then $\bar{E}$ is connected.

[redacted]