Monday Jan. 6, 2020.

Introduction:

$\left\{ \begin{array}{l} \text{2nd order linear partial differential eqns. (PDE)} \\ \text{constant coefficients:} \end{array} \right.$

- elliptic Laplace's PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$
- parabolic Heat PDE $u_t = \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$
- hyperbolic Wave eqn. $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

Objective: classify types and...
know something about solns.
(solve)?

[redacted]

MATH 6702 MATH METHODS II
(SPRING 2020)

Book:
- Ch. 4 Partial Derivatives
- Ch. 5 Multiple Integrals
- Ch. 6 Vector Analysis

$\left.\begin{array}{l}\text{Ch. 4}\\ \text{Ch. 5}\\ \text{Ch. 6}\end{array}\right\}$ Calc III

- Ch. 7 Fourier Series

(ch. 8 ODE)

- Ch. 9 Calculus of Variations ✓

(Tensors)

(special functions)

(series solns of ODE)

- * Ch. 13 Partial Differential Eqns.

[redacted]

Regularity:

Continuity classes C^k $\quad k = 0, 1, 2, \ldots$

$\uparrow$ simple differentiability classes.

Say f is a real valued function on an interval I.

$I = (a,b)$ or $[a,b]$

f is **continuous** at $x_0 \in I$ if

Given $\varepsilon > 0$, there is some $\delta > 0$ for which

$$\left.\begin{array}{l} x \in I \\ |x - x_0| < \delta \end{array}\right\} \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

If f is continuous at every point $x_0 \in I$, then we write $f \in C^0(I)$.

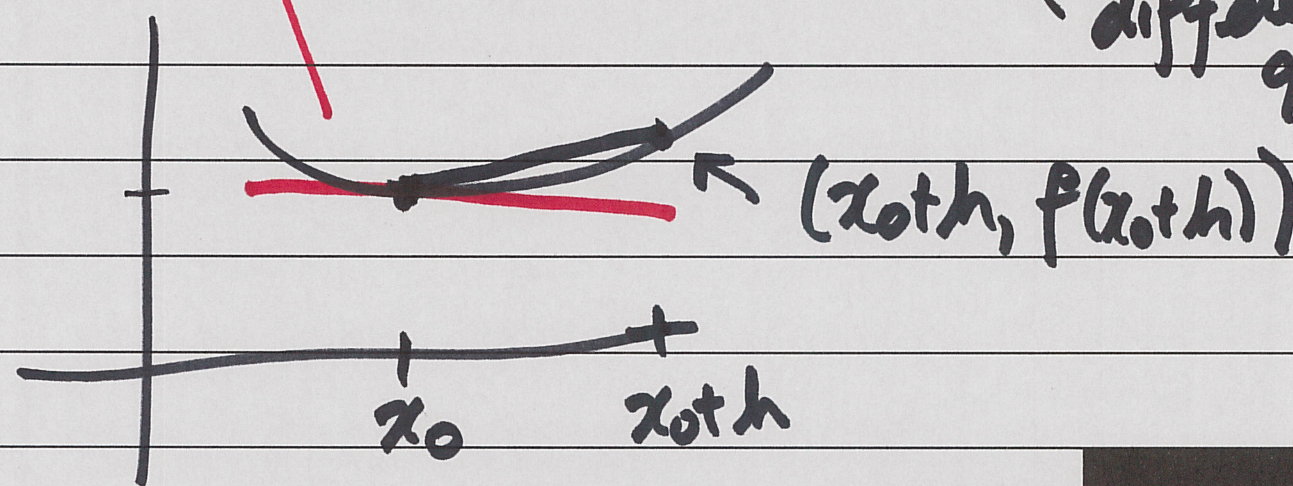
[redacted]

Also, $C^0(a,b)$, $C^0[a,b)$.

$f:(a,b)\to\mathbb{R}$ is <u>differentiable</u> at $x_0\in(a,b)$ if

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \quad \text{exists}$$

$\uparrow$ difference quotient



$f\in \mathrm{Diff}(a,b)$ if f is differentiable at every point $x_0\in(a,b)$.

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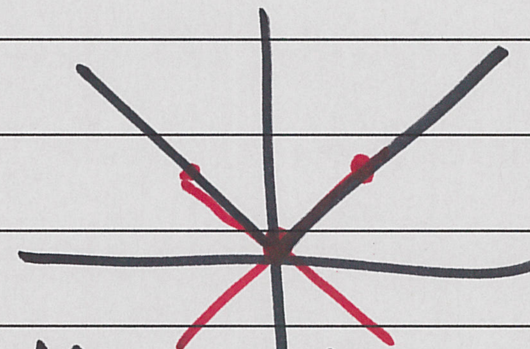
Continuously Differentiable ?

$f(x) = |x|$



not <u>differentiable</u>

If $f \in \mathrm{Diff}(a,b)$, then the <u>derivative</u>

$f' : (a,b) \longrightarrow \mathbb{R}$

continuously differentiable means

$f' \in C^0(a,b)$.

$C^1(a,b) = \{ f \in \mathrm{Diff}(a,b) : f' \in C^0(a,b) \}$

[redacted]

$f(x) = |x|$

$$g(x) = \int_0^x |t|\, dt$$

g is differentiable, $g \in \mathrm{Diff}(\mathbb{R})$

$g'(x) = |x|$

FTC

Find a function which is differentiable but not continuously differentiable.

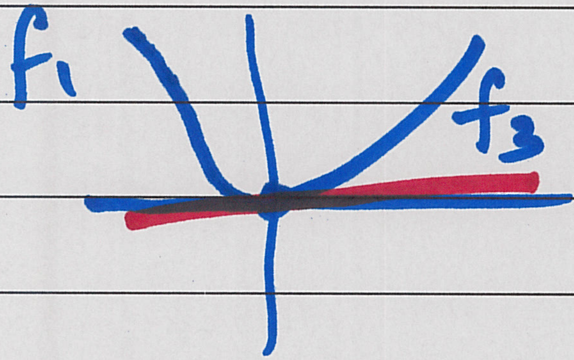
$\mathrm{Diff}(a,b) \setminus C^1(a,b)$.

f (above) is in C^1.

[redacted]

Try 2 : Define f by cases.

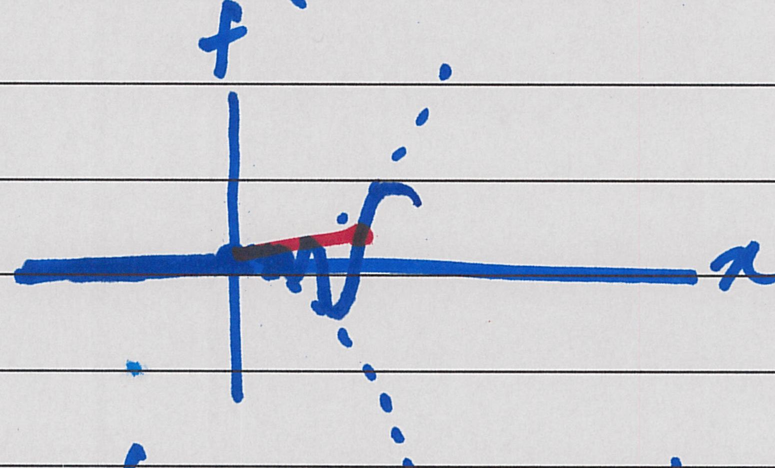
$$f(x) = \begin{cases} f_1(x), & x < 0 \\ f_2(x), & x = 0 \\ f_3(x), & x > 0 \end{cases}$$



Exercise: Show if f is differentiable at $x_0 \in (a, b)$, then f is continuous at x_0.

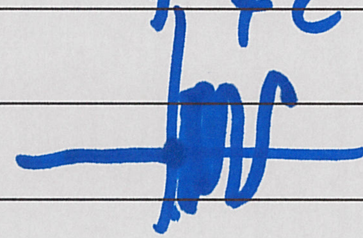
[redacted]

$$f(x) = \begin{cases} 0, & x \le 0 \\ x^2 \sin\frac{1}{x}, & x > 0 \end{cases}$$



hope: $f'(0) = 0$ ✓

$f' \notin C^0$



$\underline{x>0}$ $f'(x) = 2x \sin\frac{1}{x} + x^2 \cdot \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$

$= \underbrace{2x \sin\frac{1}{x}}_{\text{gets small as } x \searrow 0.} - \cos\frac{1}{x}$

$\lim\limits_{x=\frac{1}{2k\pi}\to 0} f'(x) = -1$, $\lim\limits_{x=\frac{1}{(2k+1)\pi}\to 0} f'(x) = +1$

[redacted]

(corrected signs JM)

$\cancel{C^k(I)}$

$C^0(I)$, $C'(a,b)$

$f \in C'(I)$ __means__ there is an open interval
$\uparrow$ any interval $\quad J = (\alpha, \beta)$ with $I \subseteq J$

__and__ there is a function $\phi \in C'(J)$ with

$\phi\big|_I = f$

$\uparrow$
restriction; ϕ is an extension.

[redacted]

$\cancel{f \in C^k(I)}$

$k \geq 1$ $C^k(I) = \{ f \in C^{k-1}(I) : f^{(k-1)} \in C^1(I) \}$

(inductively okay)

<u>For PDE</u> $u = u(x_1, x_2, \ldots, x_n)$

maybe even $u : \mathcal{U} \to \mathbb{R}^k$

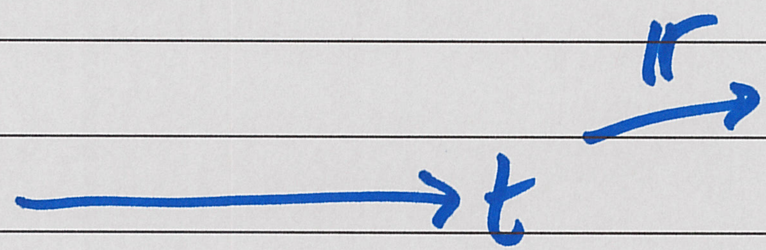
$\mathcal{U} \subseteq \mathbb{R}^n$

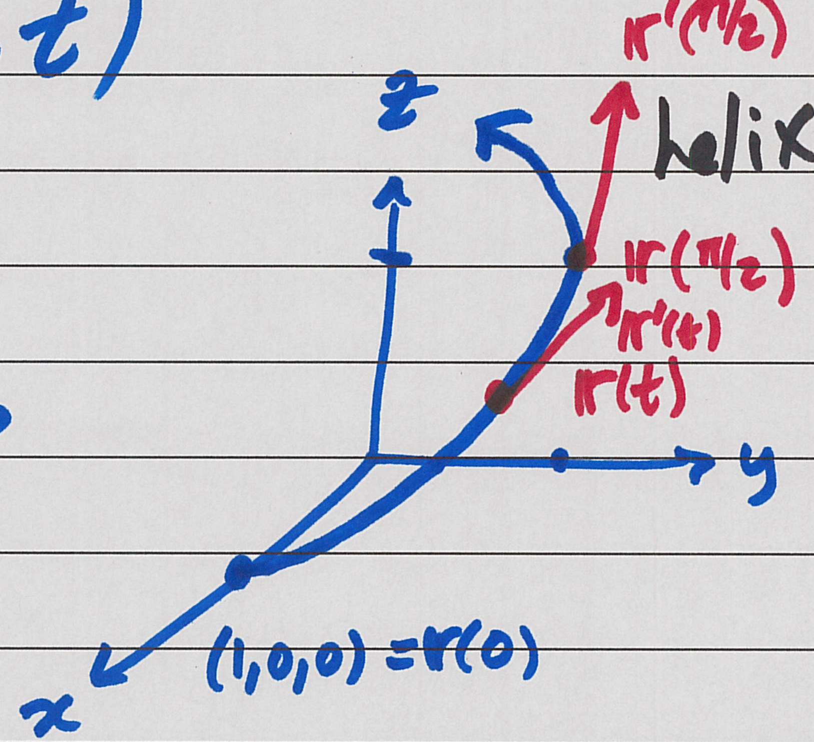
but usually $u : \mathcal{U} \to \mathbb{R}$

[redacted]

$\mathbb{r}(t) = (\cos t, \sin t, t)$

$\mathbb{r} : \mathbb{R}^1 \to \mathbb{R}^3$

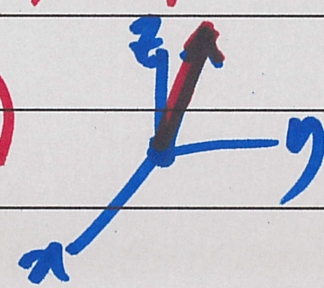




$\mathbb{r}(\pi/2)$

$\mathbb{r}'(t) = (-\sin t, \cos t, 1)$

$\mathbb{r}'(\pi/2) = (-1, 0, 1)$



[redacted]

What about $\mathbb{r} : (a,b) \to \mathbb{R}^k$

vector valued function of one variable?

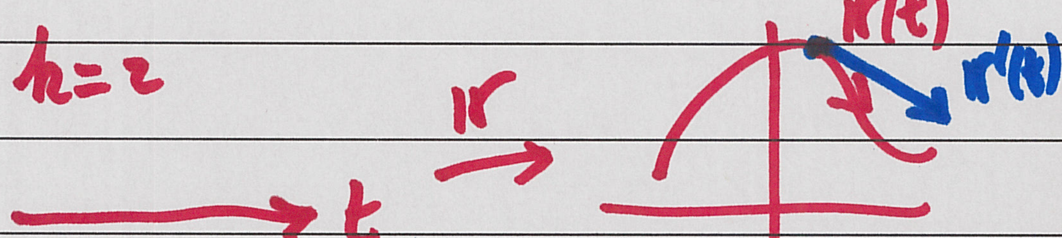
<u>ODE</u>

Autonomous

$$\underline{\mathbb{r}'} = \mathbb{F}(\underbrace{\mathbb{r}, t}_{\mathbb{R}^{k+1}}) = \underline{\underline{\mathbb{F}(\mathbb{r})}}$$

<u>Image</u> is a curve.

$k = 2$



derivative is a tangent vector to the curve

[redacted]

$$\boxed{u : \mathcal{U} \to \mathbb{R} \qquad \mathcal{U} \subseteq \mathbb{R}^n.}$$

$\mathcal{U}$ is a subset of $\mathbb{R}^n$.

Which sets (like intervals) do we want to use?

$$\mathbb{R}^n = \{ (x_1, \dots, x_n) : x_1, x_2, \dots, x_n \in \mathbb{R} \}$$

$\parallel$

$\mathbf{x}$

n-dimensional Euclidean space.

[redacted]