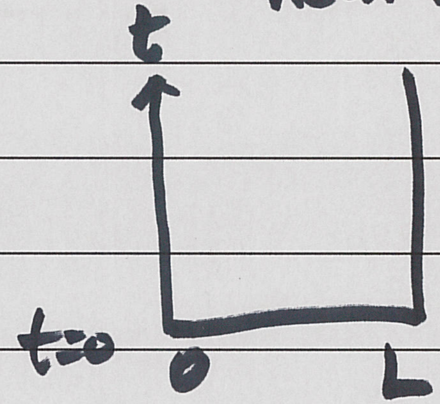
Lecture 7 Math 6702 Jan 29, 2020

~~Heat~~ 1-D Heat PDE

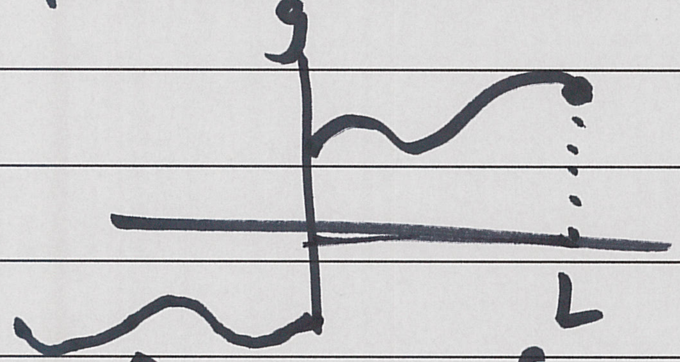
$$\begin{cases} u_t = u_{xx} \quad \text{on} \quad [0,L] \times (0,T] \\ u(x,0) = g(x) \\ u(0,t) = 0 = u(L,0) \end{cases}$$

$\uparrow$ Heat Domain



$\leftarrow$ Heat Boundary

Expand g with a Fourier Series



$$g = \sum_{j=1}^{\infty} g_j \sin \frac{j\pi x}{L}$$

$\leftarrow$ odd extension of g.

$$g_j = \frac{2}{L} \int_0^L g(\xi) \sin \frac{j\pi \xi}{L} \, d\xi$$

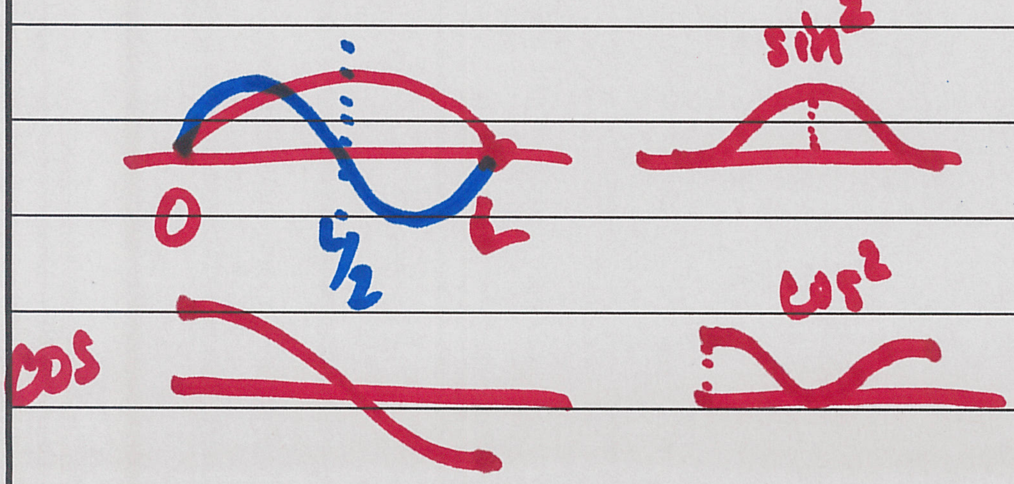
[redacted]

$$g = \sum_{j=1}^{\infty} g_j \sin \frac{j\pi x}{L}$$

$$\int_0^L g(x) \sin \frac{k\pi x}{L}\, dx = \int_0^L \sum_{j=1}^{\infty} g_j \sin \frac{j\pi x}{L} \sin \frac{k\pi x}{L}\, dx$$

$$= \sum_{j=1}^{\infty} g_j \underbrace{\int_0^L \sin \frac{j\pi x}{L} \sin \frac{k\pi x}{L}\, dx}$$

<u>If $j = k$</u> $\quad \int_0^L \sin^2 \frac{j\pi x}{L}\, dx = \int_0^L \cos^2 \frac{j\pi}{L} dx = \frac{L}{2}$



$j \neq k$
All zero

[redacted]

$$\int_0^L g(x) \sin\frac{k\pi x}{L}\,dx = \sum_{j=1}^{\infty} g_j \int_0^L \sin\frac{j\pi x}{L} \sin\frac{k\pi x}{L}\,dx$$

$$= g_k \cdot \frac{L}{2}$$

$$\Rightarrow \quad g_k = \frac{2}{L}\int_0^L g(x) \sin\frac{k\pi x}{L}\,dx$$

The Fourier expansion will converge (nicely) to any continuous function.

Real space where Fourier series converge:

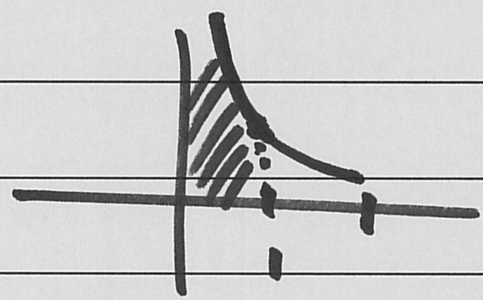
$L^2[0,L]$. $\leftarrow$ square integrable

$\int g^2 < \infty$

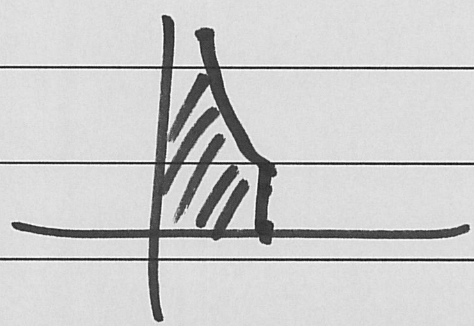
[redacted]

non-integrable

$$\int_0^1 \frac{1}{x}\,dx = \ln x \Big|_0^1 = \infty$$



integrable $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_0^1 = 2 < \infty$$

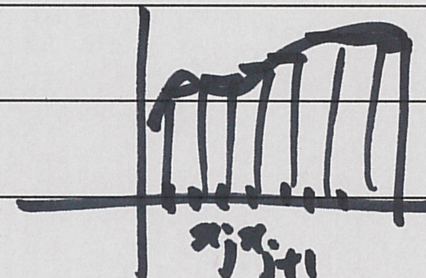


[redacted]

Non-measurable

Riemann Integral

$$\int_{[a,b]} f = \lim \sum f(x_j^*)(x_j - x_{j-1})$$



If $f \in C^0[a,b]$, then this integral/limit exists.

Integration of more general functions requires the ability to measure more general sets than intervals.

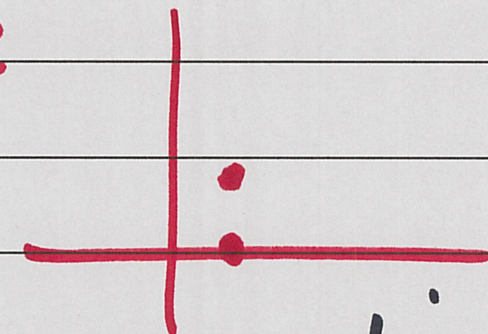
[redacted]

Non-measurable set

$[0,1] \cap \mathbb{Q} \leftarrow$ What's the measure?

Some answers: $\mathbb{Q}$ are countable

$\uparrow$ $\frac{m}{n}$



$\varepsilon > 0$

$\left(\frac{m}{n} - \frac{\varepsilon}{4}, \frac{m}{n} + \frac{\varepsilon}{4}\right)$

$\frac{\varepsilon}{4}$

The measure of the rational numbers is zero.

$[0,1] \cap \mathbb{Q} \subseteq \bigcup_{j=1}^{\infty} \left(r_j - \frac{\varepsilon}{2^{j+1}}, r_j + \frac{\varepsilon}{2^{j+1}}\right)$

$\mu([0,1] \cap \mathbb{Q}) \leq \varepsilon$

[redacted]

$\mu([0,1] \setminus \mathbb{Q}) = 1$

measure

m

interval

$$\left[\begin{array}{ll} \text{(i)} & \mu(I) = \text{length}(I) \\ \text{(ii)} & \mu(I+t) = \mu(I), \quad I+t = \{x+t : x \in I\} \\ & \qquad\qquad\qquad\qquad\qquad \text{translate} \\ \text{(iii)} & \mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j) \end{array}\right.$$

disjoint

Theorem (Lebesgue) There does not exist a measure on all sets.

[redacted]

$$\begin{cases} u_t = u_{xx} \quad \text{on } [0,L] \times (0,T] \\ u(x,0) = g \\ u(0,t) = u(L,t) = 0 \quad \text{homogeneous boundary conds.} \end{cases}$$

~~Try~~ $u(x,t) = A(x)\,B(t)$ ← separated variables solution.

$$AB' = A''B$$

$$\frac{B'}{B} = \frac{A''}{A} = \lambda \quad (\text{constant})$$

separation constant

$$A(0)\,B(t) = 0 = A(L)\,B(t) \quad \forall t.$$

Two point boundary value prob.

$$\begin{cases} A'' = \lambda A \\ A(0) = 0 = A(L) \end{cases}$$

[redacted]

$$\begin{cases} A'' = \lambda A \text{ on } [0, L] \\ A(0) = 0,\ A(L) = 0 \end{cases}$$

Sturm-Liouville Problem

λ is unknown

If $\underline{\lambda < 0}$, $\lambda = -\mu$

$A(x) = a \cos\sqrt{\mu}\, x + b \sin\sqrt{\mu}\, x$

$A(0) = 0 \Rightarrow a = 0.$

$A(L) = 0 \Rightarrow \underline{\underline{b}}\, \underbrace{\sin\sqrt{\mu} L}\ = 0$

$\sqrt{\mu_j} = \dfrac{j\pi}{L}$; $\mu_j = \dfrac{j^2\pi^2}{L^2}$

$\sin \dfrac{j\pi x}{L}$

[redacted]

$\underline{\lambda = 0}$ $\quad A'' = 0$

$A = ax + b$

$A(0) = 0 \Rightarrow b = 0$

$A(L) = 0 \Rightarrow aL = 0 \Rightarrow a = 0.$

$\times$ no solution.

$\underline{\text{check } \lambda > 0}$ $\quad A = a e^{\sqrt{\lambda} x} + b e^{-\sqrt{\lambda} x}$

$\overset{\text{or}}{=} \alpha \cosh \sqrt{\lambda} x + \beta \sinh \sqrt{\lambda} x.$

$\times$ no solution

[redacted]

$$(1)\rightarrow \begin{cases} u_t = u_{xx} \quad \text{on} \quad [0,L]\times(0,T] \\ u(x,0) = g(x) = \sum g_j \sin\frac{j\pi x}{L} \end{cases}$$
$$(2)\rightarrow \quad u(0,t) = 0 = u(L,t)$$

$u_j(x) =$

$$A_j(x) = \sin\frac{j\pi x}{L}$$

$$A_j'' = \lambda_j A_j = -\frac{j^2\pi^2}{L^2} A_j$$

$$\frac{B_j'}{B_j} = \lambda = -\frac{j^2\pi^2}{L^2}$$

$$B_j' = -\frac{j^2\pi^2}{L^2} B_j$$

$$B_j = c_j e^{-\frac{j^2\pi^2}{L^2}t}$$

[redacted]

$$A_j(x)B_j(t) = c_j e^{-\frac{j^2\pi^2}{L^2}t}\sin\frac{j\pi x}{L}$$

satisfy (1) and (3).

Linear PDE $u_t = u_{xx}$

homogeneous boundary conditions $u(0,t) = 0 = u(L,t)$

$$\Rightarrow \quad u = \sum_{j=1}^{\infty} c_j e^{-\frac{j^2\pi^2}{L^2}t} \sin\frac{j\pi x}{L}$$

satisfy (1) and (3).

Only need $u(x,0) = g(x) = \sum_{j=1}^{\infty} g_j \sin\frac{j\pi x}{L}$

i.e., $\displaystyle\sum_{j=0}^{\infty} c_j \sin\frac{j\pi x}{L} = \sum_{j=0}^{\infty} g_j \sin\frac{j\pi x}{L}$

[redacted]

$$u(x,t) = \sum_{j=1}^{\infty} g_j \, e^{-\frac{j^2\pi^2}{L^2}t} \sin\frac{j\pi x}{L}$$

$$g_j = \frac{2}{L}\int_0^L g(\xi)\sin\frac{j\pi\xi}{L}\,d\xi.$$

$$u(x,t) = \sum_{j=1}^{\infty} \frac{2}{L}\int_0^L g(\xi)\sin\frac{j\pi\xi}{L}\,d\xi \; e^{-\frac{j^2\pi^2}{L^2}t} \sin\frac{j\pi x}{L}$$

$$= \int_0^L \underbrace{\left(\frac{2}{L}\sum_{j=1}^{\infty} e^{-\frac{j^2\pi^2}{L^2}t}\sin\frac{j\pi\xi}{L}\sin\frac{j\pi x}{L}\right)}_{G(x,\xi,t)} g(\xi)\,d\xi$$

independent of g.

Green's Function.

[redacted]