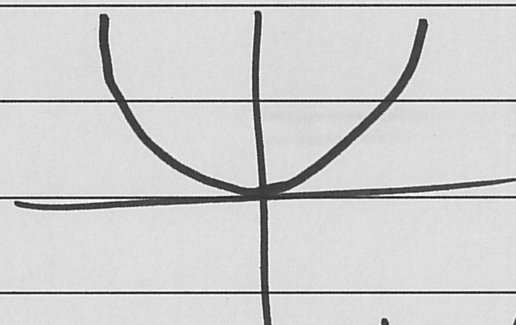
MATH 6702 Lecture 5 Jan. 22, 2020

$\mathbb{X} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

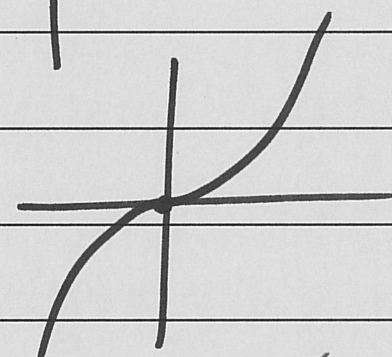
$\mathbb{X}_0 = (x_1^0, x_2^0, \dots, x_n^0) \longrightarrow \mathbb{P} = (p_1, p_2, \dots, p_n)$

$f(x) = |x|^3$

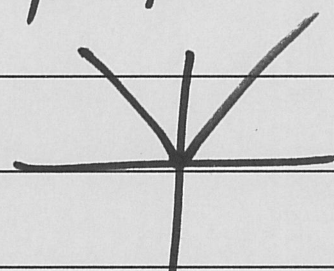


$f \in C^2(\mathbb{R}) \setminus C^3(\mathbb{R})$.

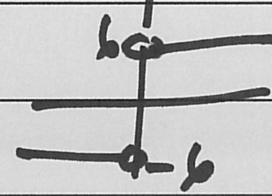
$$f'(x) = \begin{cases} 3x^2, & x > 0 \\ -3x^2, & x < 0 \end{cases}$$



$$f''(x) = \begin{cases} 6x, & x > 0 \\ -6x, & x < 0 \end{cases}$$



$$f''(x) = \begin{cases} 6 & x > 0 \\ -6 & x < 0 \end{cases}$$



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<u>Lipschitz</u> $\underline{|f(x)-f(\xi)|} \le C|x-\xi|$

ζ ξ
ZETA xi

locally Lipschitz -vs- Globally Lipschitz.

$|f(x)-f(x_0)| \le C|x-x_0|$ when $|x-x_0|<\delta$.

$C_b^0(a,b)$ ← bounded in sup norm.

Simple case(s)

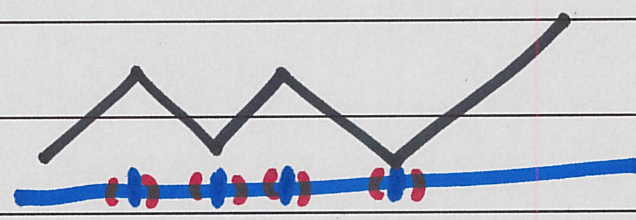
semi norm
$$[f] = \sup_{x,x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|} < \infty$$

U open with $\bar{U}$ compact.

$f(x) = x^{1/2} \leftarrow$ not C^1

but $C^{0,1/2}$ means "$1/2$ a derivative."

$\mathrm{Lip} = C^{0,1}$



Rademacher's Theorem: Every Lipschitz function has a derivative at almost all points.

If N is the set of points of nondifferentiability, for any $\epsilon > 0$ then there are open balls $B_1, B_2, \ldots$ with $N \subseteq \bigcup B_j$ and $\sum m(B_j) < \epsilon$.

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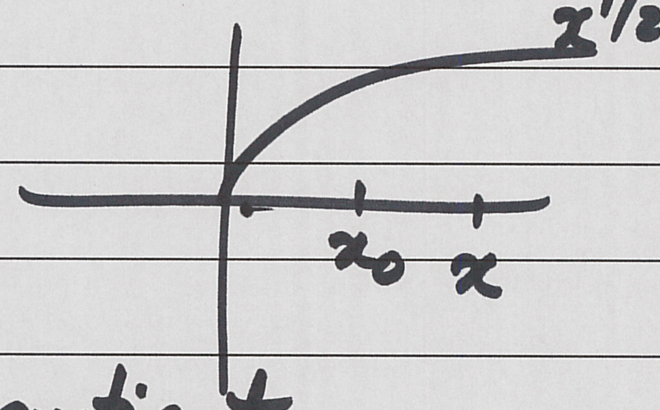
<u>Question</u>: Is it true that $f(x) = |x|^{1/2}$

is $1/3$ Hölder continuous?

Again take $0 < x_0 < x$

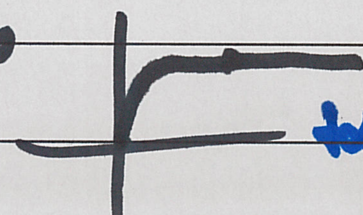


$$\frac{x^{1/2} - x_0^{1/2}}{(x - x_0)^{1/3}}$$ ← Hölder $1/3$ quotient.

$$\frac{d}{dx_0}\left[\frac{x^{1/2} - x_0^{1/2}}{(x - x_0)^{1/3}}\right] < 0. \quad \text{(exercise)}$$

… worst/largest when $x_0 = 0$,

but then $\dfrac{x^{1/2}}{x^{1/3}} = x^{1/6}$



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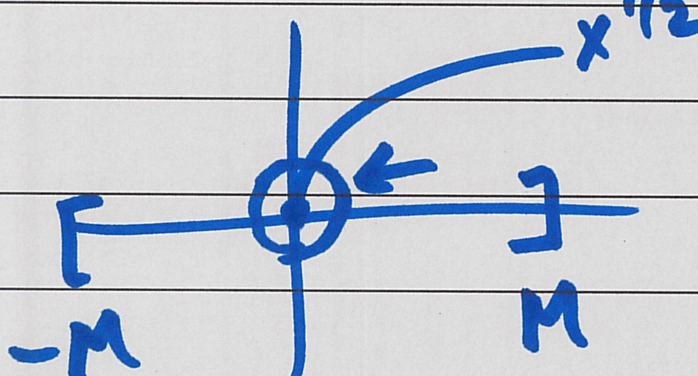
Conclusion: $f(x) = |x|^{1/2}$ has

$$f \in C^{0,1/3}[0,M] \text{ for } M \text{ finite.}$$

$$\text{But } f \notin C^{0,1/3}(\mathbb{R}).$$



Exercise $g(x) = |x|^{1/3}$

$$g \notin C^{0,1/2}[-M,M].$$

[redacted]

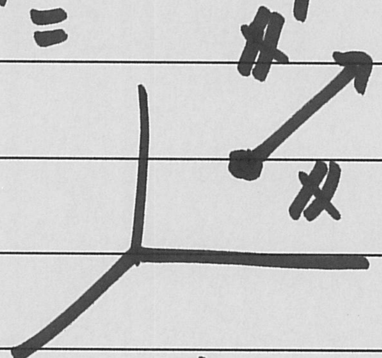
$C^{1,1/2}(U)$, $U \in \mathbb{R}^n$, $\bar{U}$ <u>compact</u>.

↑ means all partial derivatives

$u_{x_1}, u_{x_2}, \ldots, u_{x_n} \in C^{0,1/2}$

ODEs and PDEs.

$X' =$



C^1 path

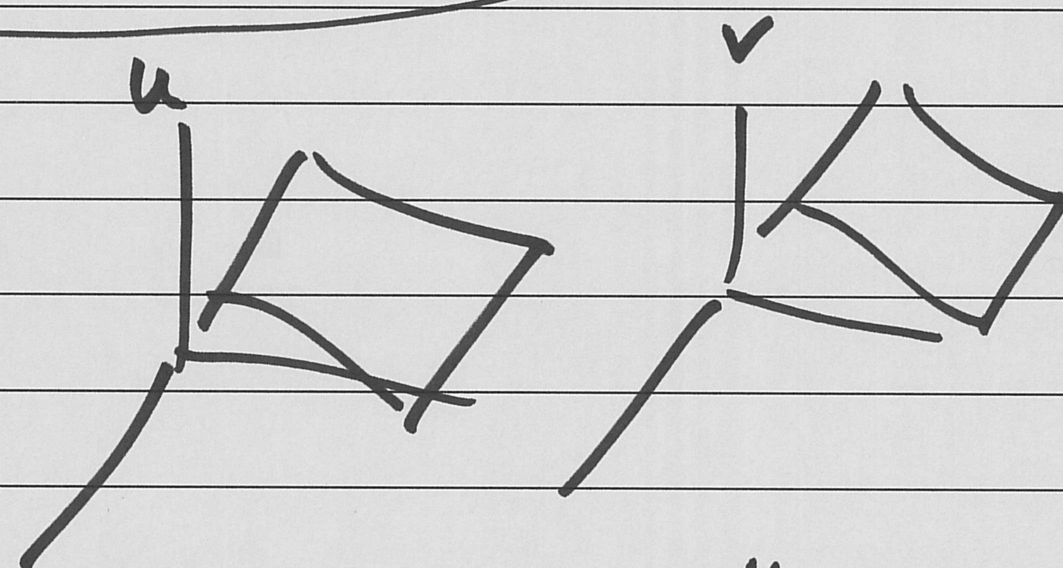
If you specify $X'(t)$
you can
find X.

[redacted]

11.

$$\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial v}{\partial y}.$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$



To find u means the "specified" partials fit together

(well).

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