MATH 6702 Lecture 4 Jan 15, 2020

<u>Highlights from last time</u>:

- C^0 / L^∞ norm $\|f\|_{C^0} = \sup_x |f(x)|$
- Euclidean norm and distances
- Normed vector space
- metric space
- Path connectedness

[redacted]

Supremum of a set of numbers.

$A \subseteq \mathbb{R}$ and A is <u>bounded above</u>

$\sup A$ = <u>least</u> upper bound of A

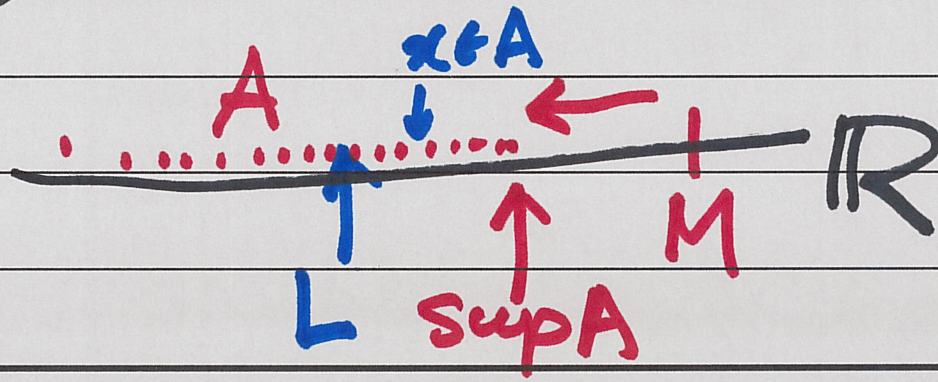
(supremum)

if M is an upper bound

$x \leq \sup A \leq M$

<u>or</u> $\sup A = +\infty$ if A is not bounded above.

Bounded above: $x \leq M$ whenever $x \in A$.



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Continuous functions are not always a metric space under the sup norm :

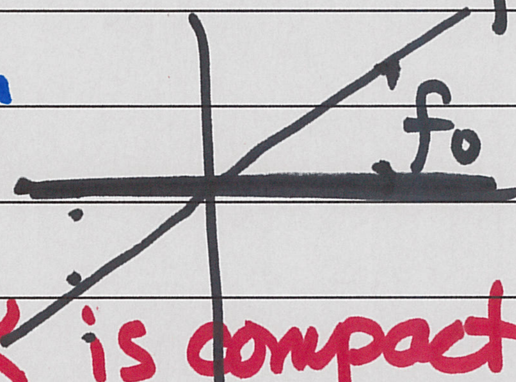
$\underline{C^0(\mathbb{R})} \leftarrow \quad d: C^0 \times C^0 \longrightarrow [0, \infty)$

$d(f, g)$

$f_0(x) \equiv 0$,

$f_1(x) = x$

$d(f_0, g_0) = \sup_{x \in \mathbb{R}} |f_1(x) - f_0(x)| = +\infty$



"continuous"

$C^0(K)$ where K is compact.

cts. $\Rightarrow$ bounded, metric ✓

$C_b^0(\mathbb{R})$ ✓

[redacted]

Simply Connected

← deformation homotopy

$H : [a,b] \times [0,1] \longrightarrow U$ continuous.

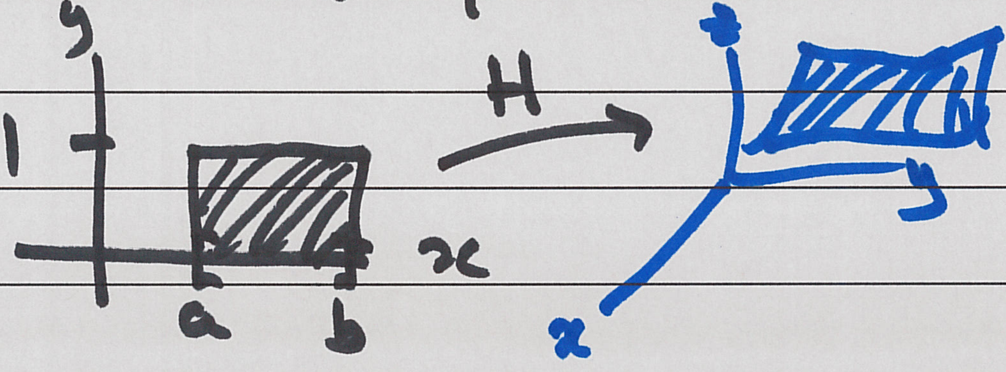
$[a,b]$ ↑ domain of a path

$[0,1]$ ← Deformation interval

$\gamma : [a,b] \to U \subseteq \mathbb{R}^n$

$[a,b] = \{ x \in \mathbb{R} : a \le x \le b \}.$

$[a,b] \times [0,1] = \{ (x,y) : x \in [a,b],\ y \in [0,1] \}.$



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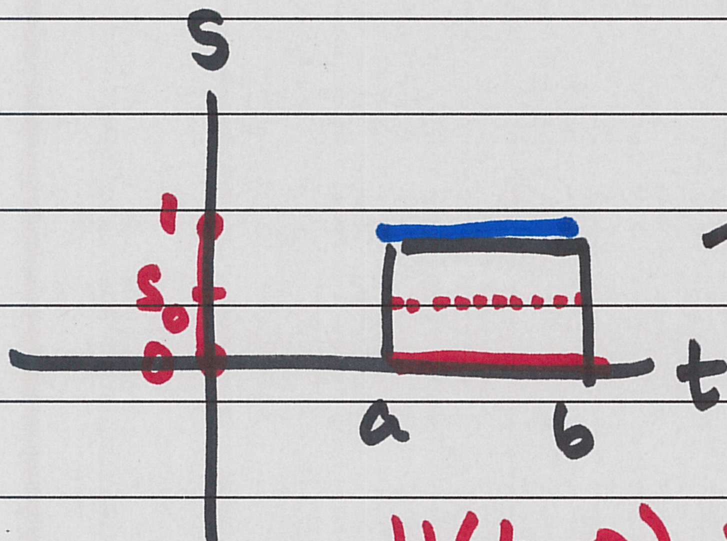
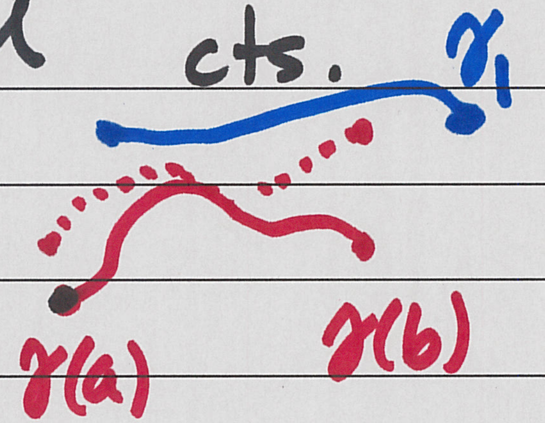
$H : [a,b] \times \underbrace{[0,1]}_{s} \longrightarrow \mathcal{U}$ cts.



$\xrightarrow{H}$



$\gamma_{s_0}(t) = H(t, s_0)$

$$\underline{\underline{H(t,0) = \gamma(t)}}$$

A deformation of the path γ.

[redacted]

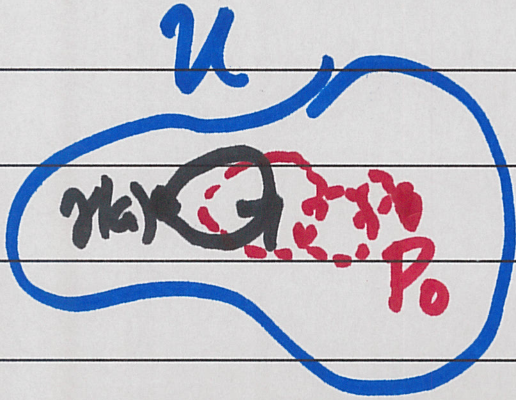
$\mathcal{U} \subseteq \mathbb{R}^2$ is <u>simply</u> <u>connected</u> if

any loop $\gamma : [a,b] \to \mathcal{U}$ $\quad (\gamma(a) = \gamma(b))$

has a deformation $H : [a,b] \times [0,1] \to \underline{\underline{\mathcal{U}}}$

<u>such that</u> $\quad H(a,s) = H(b,s)$ for all s.
(loops)

$H(t,1) \equiv p_0 \in \mathcal{U}$. (constant)



Every loop can be contracted to a point.

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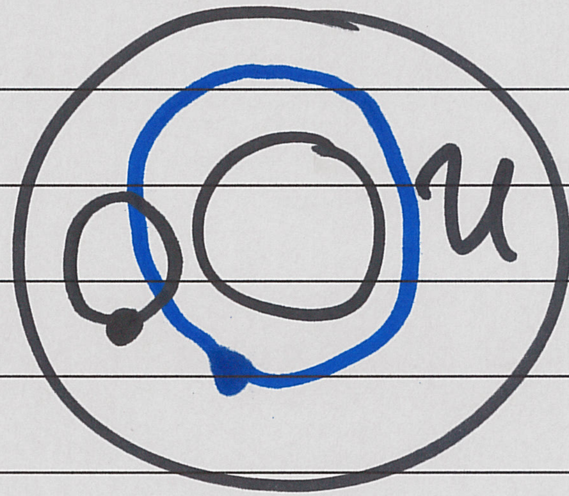
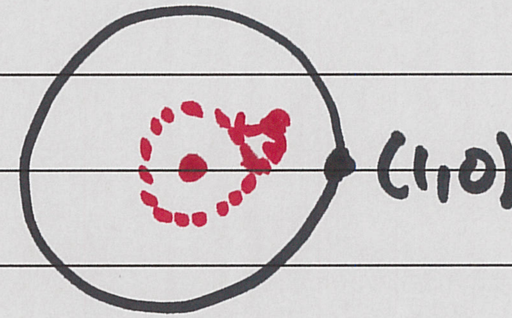
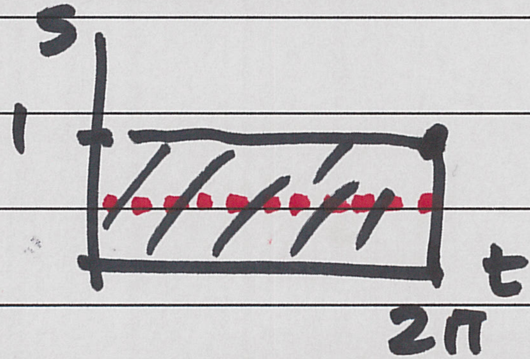




$\gamma(t) = (\cos t, \sin t)$

$[a,b] = [0, 2\pi]$

$$\boxed{H(t,s) = (1-s)(\cos t, \sin t)}$$



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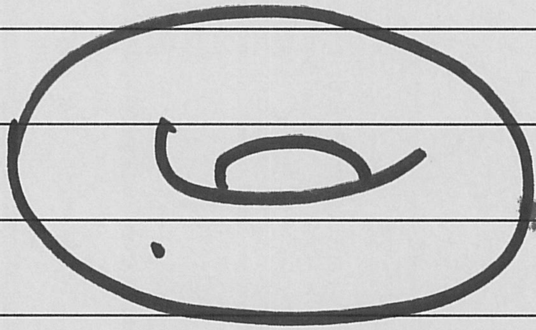
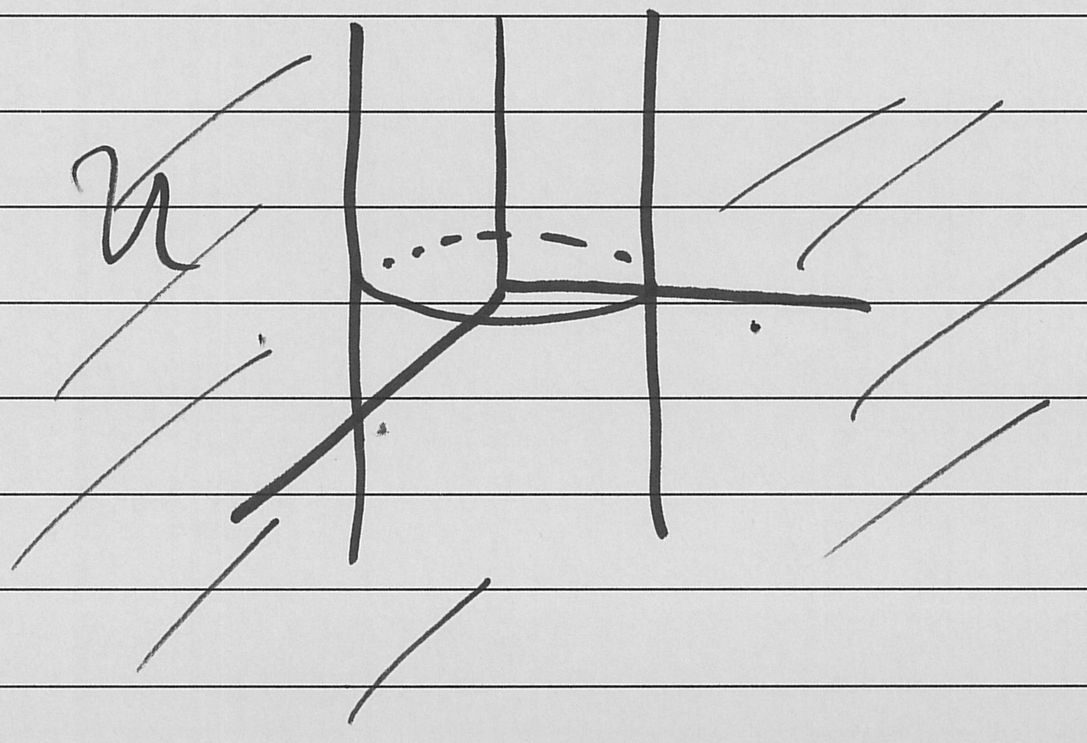
8.

Not simply connected in $\mathbb{R}^3$?



$U = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 > 1\}$



Homework 1.

1. $y'' = 0 \leftarrow$ solve the ODE.

$$y(x) = ax + b \quad , \; a, b \text{ arbitrary}.$$

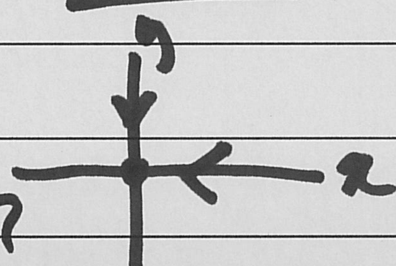
2. $y'' = \sin(x^2)$

$y = f_1$ is a solution. (Solve the ODE).

$$y(x) = f_1 + \underline{\underline{ax+b}}$$

4. continuity

$$\frac{x^2}{x^2+y^2}, \; \underline{\underline{\frac{0^2}{0^2+0^2}?}}$$



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Homework 2

1. $y'' = 0$ equivalent system.

$$\begin{cases} y' = z \\ z' = 0 \end{cases}$$

2. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0$ Wave eqn.

$$\left(w_1 = \frac{\partial u}{\partial x}, \quad w_2 = \frac{\partial u}{\partial y}. \right)$$

$f \in \bigcap_{k=0}^{\infty} C^k(a,b)$

3- $$f(x) \sim \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j$$

Taylor expansion

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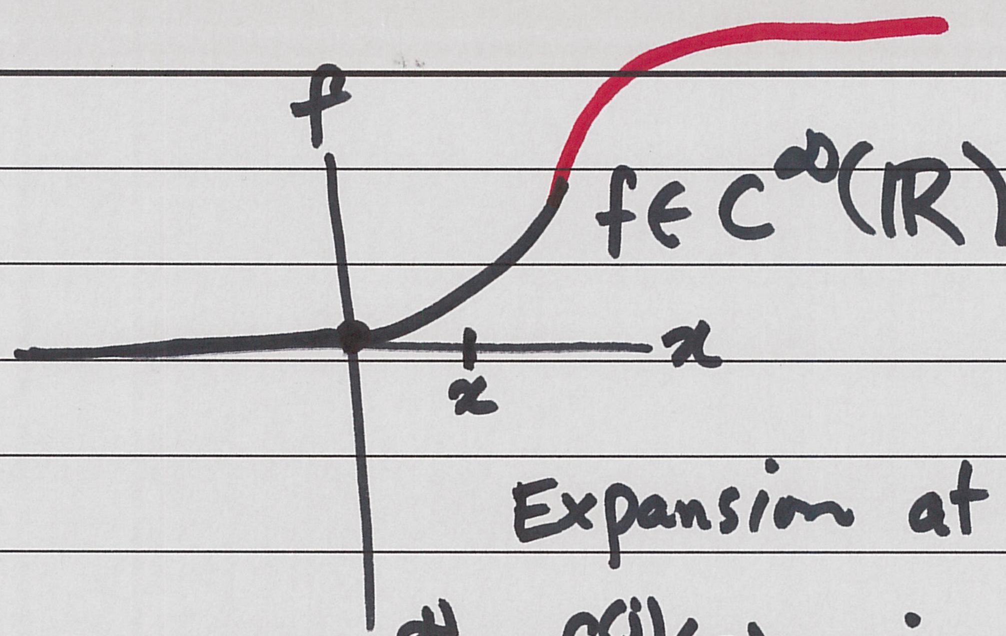
5.



Expansion at $x_0 = 0$

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j \equiv 0.$$

$$f(x) = \begin{cases} 0, & x \le 0. \\ e^{-\frac{1}{x^2}}, & x > 0. \end{cases}$$

$$f \notin C^{\omega} \subseteq C^{\infty}$$

[redacted]

What is required for the Taylor expansion to work?

$1^{\underline{st}}$ answer It always works.

remainder

$$f(x) = \sum_{j=0}^{k} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j + \frac{f^{(k+1)}(x^*)}{(k+1)!}(x-x_0)^{k+1}$$

$k^{\underline{th}}$ order Taylor approx

C^{k+1}

x^* somewhere between x and x_0.

$\boxed{C^{\infty} \supsetneq C^{\omega}}$

$2^{\underline{nd}}$ answer $f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$

[redacted]

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j$$

$\mathbb{x}, \mathbb{x}_0 \in \mathbb{R}^n$

$(x_1, x_2, \dots, x_n)$

$$\sum_{\beta} \frac{D^{\beta} u(\mathbb{x}_0)}{\beta!} (\mathbb{x}-\mathbb{x}_0)^{\beta}$$

$\beta = (\beta_1, \beta_2, \dots, \beta_n)$

natural nos. 0, 1, 2, 3, ..

$u : \mathbb{R}^2 \to \mathbb{R}$, $u = u(x,y)$

$D^{(1,0)} u = \frac{\partial u}{\partial x}$ $\quad \frac{\partial u}{\partial y}$ $\quad \beta = (\beta_1, \beta_2)$

$\frac{\partial^2 u}{\partial x^2}$ $\quad \frac{\partial^2 u}{\partial x \partial y}$ $\quad \frac{\partial^2 u}{\partial y \partial x}$ $\quad \frac{\partial^2 u}{\partial y^2}$

$D^{(2,0)} u$ $\quad D^{(1,1)} u$ $\quad D^{(1,1)} u$

[redacted]