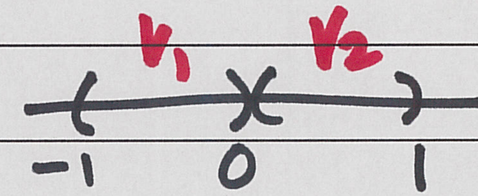
Lecture 3 MATH 6702. Jan 13, 2020

Objective: Construct (and discuss) "spaces" $C^{k,\alpha}$

$C^k \supseteq C^{k,\alpha} \supseteq C^{k+1}$ ← inclusion problem

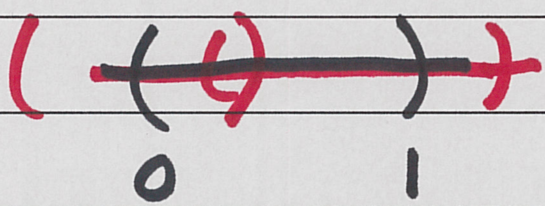
← also metric problem.

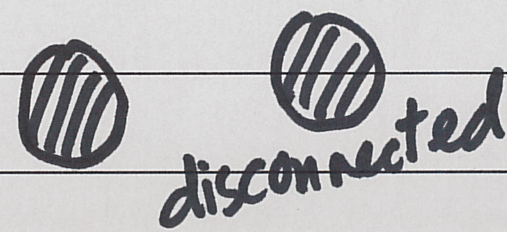
<u>Connected</u> in $\mathbb{R}^n$



$U = (-1,0) \cup (0,1)$ not connected

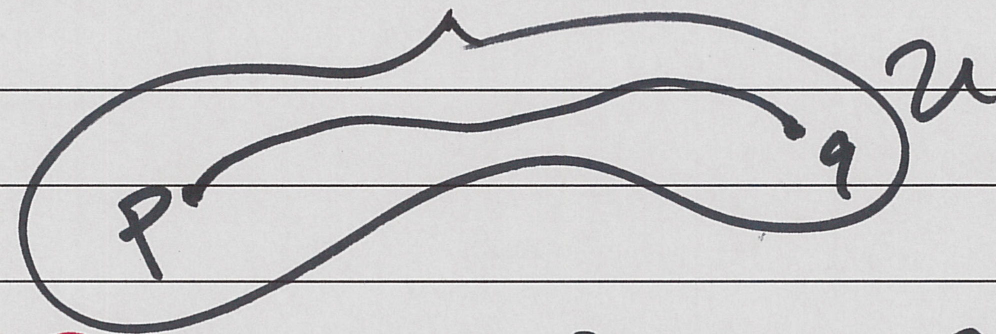
$(-1,0)$ is connected

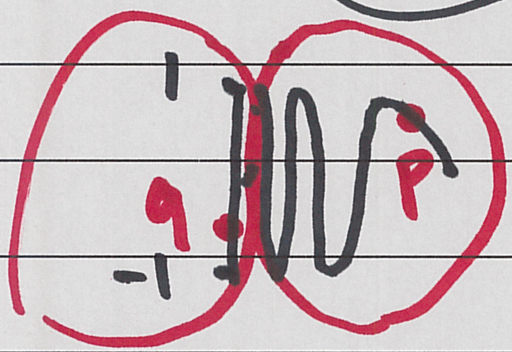




[redacted]

How about a path from p to q in U.





$\sin\frac{1}{x}$

$\{(0,y) : |y| \leq 1\} \cup \{(x, \sin\frac{1}{x}) : x > 0\}$

is connected

[redacted]

The closure of a set:

If $E \subseteq \mathbb{R}^n$, then there is a special set called the closure of E.

$$\overline{E} = \bigcap_{\substack{C \text{ closed} \\ C \supseteq E}} C$$

← unique smallest set containing E.

↖ $C = \mathbb{R}^n$ works here.

Bounded Set: $E \subseteq \mathbb{R}^n$ is bounded if there is some M so that

$$|x| \leq M \text{ for all } x \in E.$$

[redacted]

$$|\mathbf{x}| = \left(\sum x_i^2\right)^{1/2} \quad \text{where } \mathbf{x} = (x_1, \dots, x_n)$$

is the Euclidean norm on $\mathbb{R}^n$.

Analog of open intervals in $\mathbb{R}^n$

= open connected sets

Analog of finite open intervals in $\mathbb{R}^n$

= open, bounded connected sets.

[redacted]

Also, closed and bounded = <u>compact</u>

Continuity: $u : U \to \mathbb{R}^k$ is continuous at $\mathbb{p} \in U \subseteq \mathbb{R}^n$ if

Given any $\varepsilon > 0$, there is some $\delta > 0$ such that

$$\left.\begin{array}{l} \mathbb{x} \in U \\ |\mathbb{x} - \mathbb{p}| < \delta \end{array}\right\} \Longrightarrow |u(\mathbb{x}) - u(\mathbb{p})| < \varepsilon.$$

The Euclidean norm $|\mathbb{x}| = |\mathbb{x} - \mathbb{0}|$ satisfies

(i) $|c\mathbb{x}| = |c|\,|\mathbb{x}|$ non-negative homogeneity

(ii) $|\mathbb{x}| = 0 \iff \mathbb{x} = \mathbb{0}$ positive definite

(iii) $|\mathbb{x} + \mathbb{p}| \leq |\mathbb{x}| + |\mathbb{p}|$ triangle inequality

(norms).

These make sense in any <u>vector space</u>

[redacted]

Whenever you have a norm $\|\cdot\|: V \to [0,\infty)$ on a vector space, then you can define a distance

$$d(v,w) = \|v-w\| \leftarrow \text{norm induced distance}$$

$$d : V \times V \to [0,\infty)$$

(i) $d(v,w) = d(w,v)$ (symmetric)

(ii) $d(v,w) = 0 \iff v = w$ (positive definite)

(iii) $d(v,w) \le d(v,z) + d(z,w)$

$v, w, z \in V$

(triangle inequality for distance)

[redacted]

A set X is a <u>metric space</u> if there is a distance function

$$d : X \times X \to [0,\infty) \quad \text{satisfying}$$

(i) $d(x,y) = d(y,x)$ for $x, y \in X$

(ii) $d(x,y) = 0 \iff x = y$

(iii) $d(x,y) \leq d(x,z) + d(z,y)$

Generalise Continuity: If $f : X_1 \to X_2$ where X_1 and X_2 are metric spaces with distances d_1 and d_2, then f is <u>continuous</u> at $x \in X_1$ if for any $\varepsilon > 0$, there is a δ such that

$$\left.\begin{array}{l} p \in X_1 \\ d_1(p,x) < \delta \end{array}\right\} \Rightarrow d_2(f(p), f(x)) < \varepsilon .$$

[redacted]

Distance on $C^0(I)$:

(I ← interval)

$$d(f,g) = \sup_{x \in I} |f(x) - g(x)|$$

↑ distance on C^0.

want "max" but

maybe – unbounded $\sup = +\infty$ ✓

maybe – (a,b) ← bounded by b

but $b \notin (a,b)$.

sup = least upper bound.

$d : C^0 \times C^0 \to [0,\infty)$

[redacted]

$$C_b^0(I) = \{ u \in C^0(I) : \sup_{x \in I} |u(x)| < \infty \}$$

↑ bounded continuous functions.

$$\|u\|_{C^0} = \sup_{x \in I} |u(x)|$$

C_b^0 is a normed (and metric) space.

<u>norm</u>: norm of uniform convergence.

L^∞ norm.

sup norm

C^0 norm

[redacted]

Exercise The uniform limit of continuous functions is continuous.

$$\|u_j - u\|_{C^0} \longrightarrow 0 \quad \text{as } j \nearrow \infty.$$

$$\{u_j\} \subseteq C^0$$

Another metric space (special case)

$C^0(K)$ where $K \subseteq \mathbb{R}^n$ is compact.

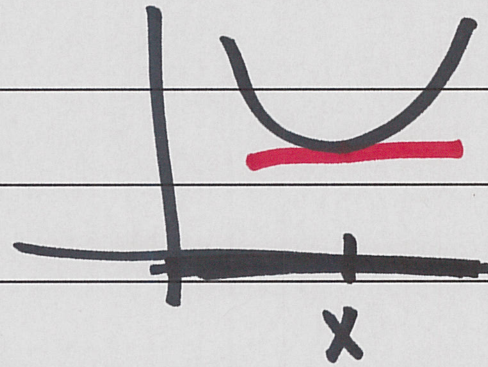
↑ metric space under

$$\|u\|_{C^0} = \sup_{x \in K} |u(x)| < \infty$$

[redacted]

A Calculus of Variations Problem:

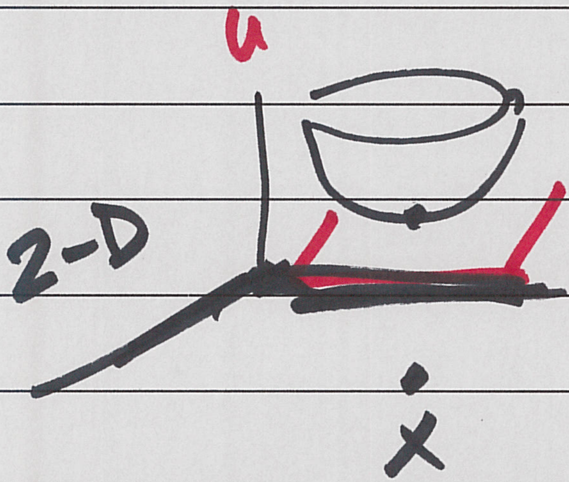
Remember calculus:



$y = (x-3)^2 + 2$

↑ minimize.

(1-D)



$\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = (0,0)$

⋮

C^0 or C^1 ← vector spaces.

infinite dimensional

[redacted]

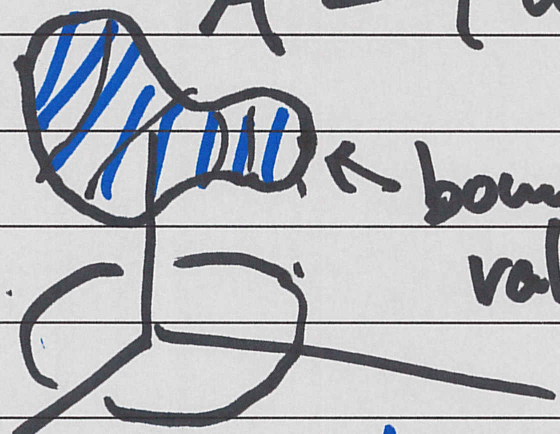
Let $B_1(0) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$.

Let $X = C^1(\overline{B_1(0)})$

↖ continuous 1^{st} partials on $\overline{B_1(0)}$

Let $g_0 \in X$.

$A = \{ u \in X : u|_{\partial B_1(0)} = g_0 \}$ admissible class.



← boundary values from g.

minimize $\text{Area}(u) = \int_{B_1(0)} \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$ on A.

[redacted]

paths and path connected sets

A path in X (metric space) is just a continuous function $\gamma : [a,b] \to X$.

Fun fact: These can be complicated.

Theorem (Peano) There is a continuous path $\gamma : [a,b] \to \mathbb{R}^3$ such that for every

(x,y,z) with $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

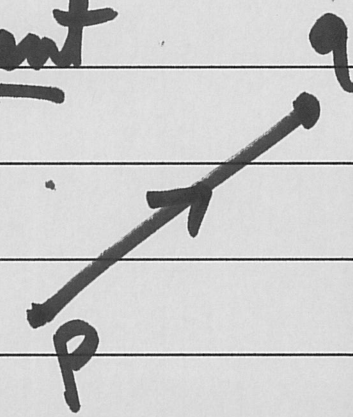
there is some $t \in [a,b]$ with $\gamma(t) = (x,y,z)$.

Theorem An *open* subset of $\mathbb{R}^n$ is connected if and only if it is path connected.

path connected: Given p and q in the set, there is a path $\gamma : [a,b] \to X$ with

$\gamma(a) = p$ and $\gamma(b) = q$.

Special Path(s): Segment



$\gamma(t) = (1-t)p + tq$

↑ convex combination

$\gamma : [0,1] \to \mathbb{R}^n$.

[redacted]