Lecture 9 MATH 6702 February 5, 2020

$\Delta u = 0$

Hans Lewy System

Poisson's Equation with homogeneous boundary conditions

We have (at least formally) two Green's functions:

$$G(x,\xi) = -\frac{2}{L}\sum_{j=1}^{\infty}\left(\frac{L}{j\pi}\right)^2 \sin\frac{j\pi\xi}{L}\sin\frac{j\pi x}{L} \qquad (1)$$

for $\begin{cases} x'' = f \\ x(0)=0,\ x(L)=0 \end{cases}$ i.e., $Lx = x''$ and $[0,L]$

$$G(x,\xi,t) = \frac{2}{L}\sum_{j=1}^{\infty}\sin\frac{j\pi\xi}{L}\sin\frac{j\pi x}{L}e^{-k\left(\frac{j\pi}{L}\right)^2 t} \qquad (2)$$

for $\begin{cases} u_t = u'' \\ u(x,0) = g(x) \\ u(0,t)=0,\ u(L,t)=0 \end{cases}$ i.e., $Lu = u_t - \Delta u$ and $[0,L]$

[redacted]

What does $Lx = x''$ do to G?

$$G(x,\xi) = -\frac{2}{L}\sum_{j=1}^{\infty}\left(\frac{L}{j\pi}\right)^2 \sin\frac{j\pi\xi}{L}\sin\frac{j\pi x}{L} \qquad (1)$$

properties: ① The series in (1) converges for all x and ξ.

② The series in (2) converges for $t>0$ (but not for $t=0$)

Your job: Find other properties of $G(x,\xi)$ in (1). and (2)

Note: ① The PDE for (1) is linear but $\underbrace{\text{not homogeneous}}_{\text{forced}}$.

$f \longmapsto \int Gf$ on the non-homogeneity

② $Lu = u_t - \Delta u$ is linear and the PDE (2) is homogeneous.

initial condition $g \longmapsto \int Gg$

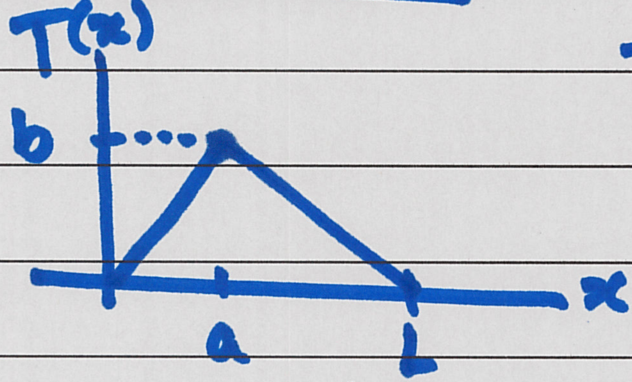
[redacted]

Formal computation for (2): $\begin{cases} G_t = k G_{xx} \\ G(0,\xi,t) = 0, \; G(L,\xi,t) = 0 \end{cases}$

Does this really hold?
If so, what is the initial condition?

$$G(x,\xi) = -\frac{2}{L} \sum_{j=1}^{\infty} \left(\frac{L}{j\pi}\right)^2 \sin\frac{j\pi\xi}{L} \sin\frac{j\pi x}{L}$$

<u>Tent Function:</u>



$$T(x) = \begin{cases} \frac{b}{a}x, & 0 \le x \le a \\ -\frac{b}{L-a}(x-L), & a \le x \le L \end{cases}$$

$$= \sum_{j=1}^{\infty} T_j \sin\frac{j\pi x}{L}$$

$$T_j = \frac{2}{L} \int_0^L T(x) \sin\frac{j\pi x}{L} \, dx.$$

[redacted]

④

$$T_j = \frac{2}{L}\int_0^L T(x)\sin\frac{j\pi x}{L}\,dx$$

$$\frac{L}{2}T_j = \int_0^a \frac{b}{a}x\sin\frac{j\pi x}{L}\,dx + \int_a^L \frac{b}{L-a}(L-x)\sin\frac{j\pi x}{L}\,dx$$

$$\int_0^a x\sin\frac{j\pi x}{L}\,dx = -\frac{L}{j\pi}x\cos\frac{j\pi x}{L}\Big|_0^a + \int_0^a \frac{L}{j\pi}\cos\frac{j\pi x}{L}\,dx$$

$$= -\frac{aL}{j\pi}\cos\frac{j\pi a}{L} + \left(\frac{L}{j\pi}\right)^2\sin\frac{j\pi x}{L}\Big|_0^a$$

$$= -\frac{aL}{j\pi}\cos\frac{j\pi a}{L} + \left(\frac{L}{j\pi}\right)^2\sin\frac{j\pi a}{L}$$

$$\int_a^L (L-x)\sin\frac{j\pi x}{L}\,dx = -\frac{L}{j\pi}(L-x)\cos\frac{j\pi x}{L}\Big|_a^L - \int_a^L \frac{L}{j\pi}\cos\frac{j\pi x}{L}\,dx$$

$$= +\frac{L}{j\pi}(L-a)\cos\frac{j\pi a}{L} - \left(\frac{L}{j\pi}\right)^2\sin\frac{j\pi x}{L}\Big|_a^L$$

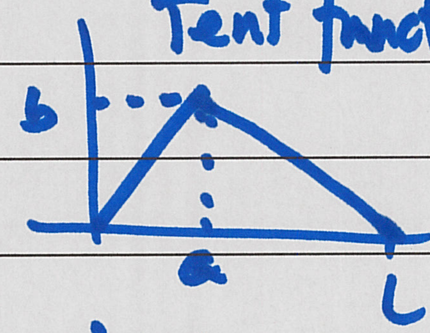
$$= \frac{L}{j\pi}(L-a)\cos\frac{j\pi a}{L} + \left(\frac{L}{j\pi}\right)^2\sin\frac{j\pi a}{L}$$

$$\frac{L}{2}T_j = \left(\frac{b}{a}+\frac{b}{L-a}\right)\left(\frac{L}{j\pi}\right)^2\sin\frac{j\pi a}{L}$$

[redacted]

$$T_j = \frac{2b}{L} \cdot \frac{L}{a(L-a)} \left(\frac{L}{j\pi}\right)^2 \sin\frac{j\pi a}{L}$$

Tent function

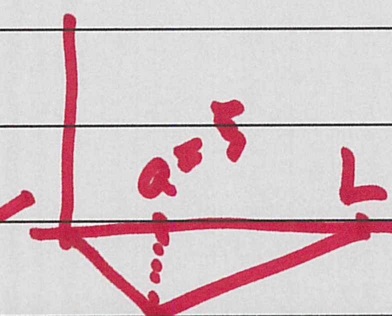


$$= \frac{2b}{a(L-a)} \left(\frac{L}{j\pi}\right)^2 \sin\frac{j\pi a}{L}$$

$$T(x) = \sum_{j=1}^{\infty} \frac{2b}{a(L-a)} \left(\frac{L}{j\pi}\right)^2 \underbrace{\sin\frac{j\pi a}{L}} \sin\frac{j\pi x}{L}$$

COMPARE TO

$$G(x,\xi) = -\frac{2}{L} \sum_{j=1}^{\infty} \left(\frac{L}{j\pi}\right)^2 \underbrace{\sin\frac{j\pi \xi}{L}} \sin\frac{j\pi x}{L}$$



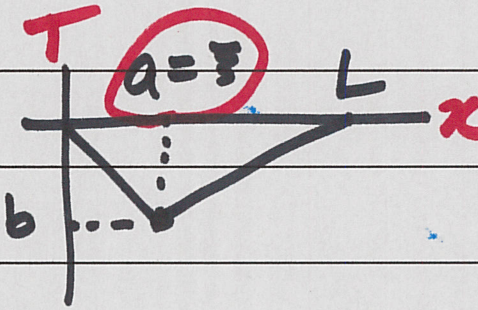
$\underline{a = \xi}$: $\dfrac{2b}{\xi(L-\xi)} = -\dfrac{2}{L}$ (?)

Take $b = -\dfrac{1}{L}\xi(L-\xi)$.

[redacted]

This tells us something about $G = G(x, \xi)$:

$$T(x) = \begin{cases} \frac{b}{a} x , & 0 \le x \le a \\ -\frac{b}{L-a}(x-L) , & a \le x \le L \end{cases}$$

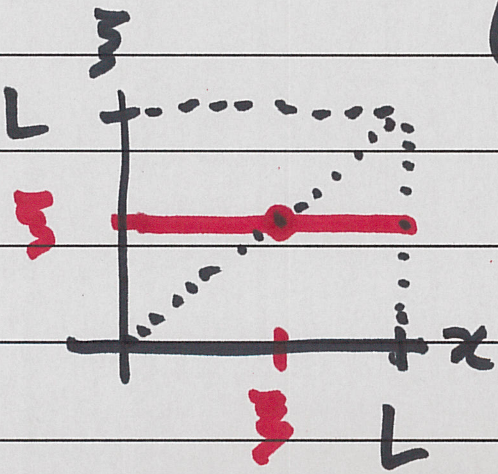


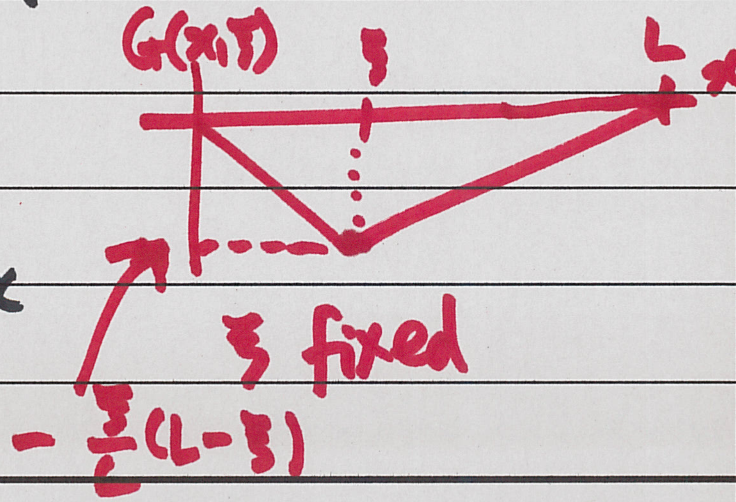
If we put $a = \xi$ and $b = -\frac{\xi(L-\xi)}{L}$ for ξ fixed,

Then

$$G(x, \xi) = \begin{cases} -\frac{L-\xi}{L} x , & 0 \le x \le \xi \\ -\frac{\xi}{L}(L-x) , & \xi \le x \le L \end{cases}$$

Explicit formula for $G(x, \xi)$ from (1).





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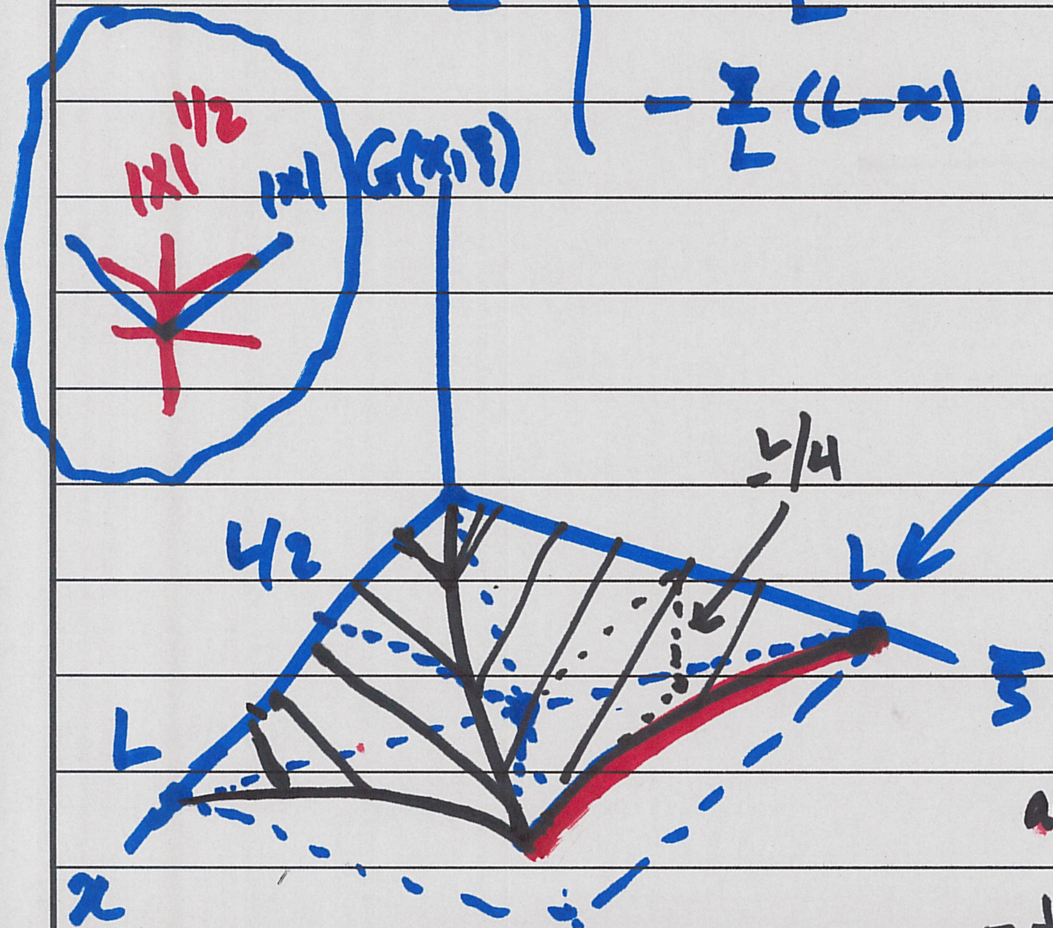
$$G(x,\xi) = \frac{2}{L}\sum_{j=1}^{\infty}\left(\frac{L}{j\pi}\right)^2 \sin\frac{j\pi\xi}{L}\sin\frac{j\pi x}{L}$$

$$= \begin{cases} -\frac{L-\xi}{L}x, & 0 \le x \le \xi \\ -\frac{\xi}{L}(L-x), & \xi \le x \le L \end{cases} \leftarrow -\frac{(L-x)^2}{L}$$



$x+\xi = L$

$\xi = L-x$

$-\frac{x^2}{L} = G(x, L-x)$

along $\xi = x$

$-\frac{1}{L}x(L-x)$

Not C^1! , C^0? (yes)

[redacted]

Let's see if we can make sense of G_x globally. (We need a new kind of derivative.)

(G is not C^1, but G is $C^{0,1}([0,L]\times[0,L])$
Lipschitz.

We can do this with **weak derivatives**:

$L^1(a,b)$ = integrable function on (a,b).
i.e., measurable and $\int |f| < \infty$.

$L^1_{loc}(a,b)$ = locally integrable
i.e., measurable and $\int_K |f| < \infty$
if $K \subseteq (a,b)$ is compact.

[redacted]

Def'n (A) A set E is <u>compactly contained</u> in (an open set) $\mathcal{U}$ if

(i) $\bar{E}$ is compact, and
(ii) $\bar{E} \subseteq \mathcal{U}$.

write $E \subset\subset \mathcal{U}$.

$\subset\subset$ = compactly contained

(B) Given $u : \mathcal{U} \to \mathbb{R}$

$$\operatorname{supp} u = \overline{\{x \in \mathcal{U} : u(x) \neq 0\}}$$

← closure

↖ support of u.

(C) Given $u : \mathcal{U} \to \mathbb{R}$, we say u is <u>compactly supported</u> if

(i) $\operatorname{supp} u$ is compact, and
(ii) $\operatorname{supp} u \subseteq \mathcal{U}$

$\Big\}$ $\underline{\underline{\operatorname{supp} u \subset\subset \mathcal{U}}}$

[redacted]