## Assignment 4 Problem 5

$\mathbb{r} : [a,b] \longrightarrow \mathbb{R}^n$, $C^1$

reparameterize by arclength, i.e., give me a unit speed parameterization.

Should include $\mathbb{r}'(t) \neq 0$.

$$u = \frac{\mathbb{r}'(t)}{\|\mathbb{r}'(t)\|} \leftarrow \text{unit vector in same direction as } \mathbb{r}'$$

$\gamma : [0, \ell] \to \mathbb{R}^n$, $|\gamma'| = 1$

$\ell$ ↑ length of the path.

$\gamma = \gamma(s)$ ← arclength.

[redacted]

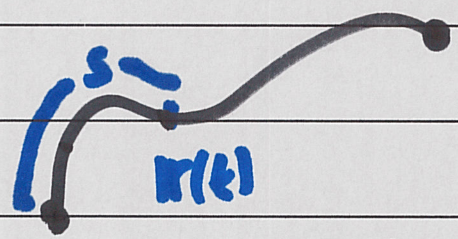
**Hint:** look at the arclength.

**Bogus Solution:** $\gamma(s) = \mathbb{r}(a) + \int_0^s \frac{\mathbb{r}'(T)}{|\mathbb{r}'(T)|}\, dT$

$\mathbb{r} : [a,b] \longrightarrow \mathbb{R}^n$

$t$

Correct Approach:




$s = \int_0^t |\mathbb{r}'(T)|\, dT$ start: This defines $s = s(t)$

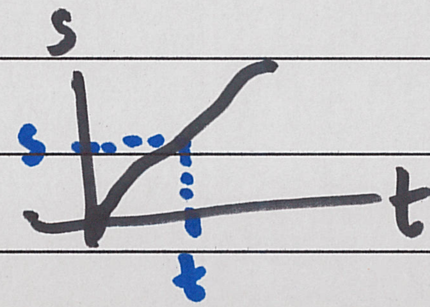
inverse $\phi = \phi(s)$

$\mathbb{r} : [a,b] \longrightarrow \mathbb{R}^n$

$\frac{ds}{dt} = |\mathbb{r}'(t)| > 0$

$\gamma(s) = \mathbb{r}(t)$

$t = t(s)$




[redacted]

$$s = \int_0^{\phi(s)} |\mathbf{r}'(\tau)|\, d\tau$$

$\phi(s)$ = the "$t$" for which $\underbrace{s}_{\uparrow} = \int_0^t |\mathbf{r}'(\tau)|\, d\tau.$

(arclength along the path is $s$.)

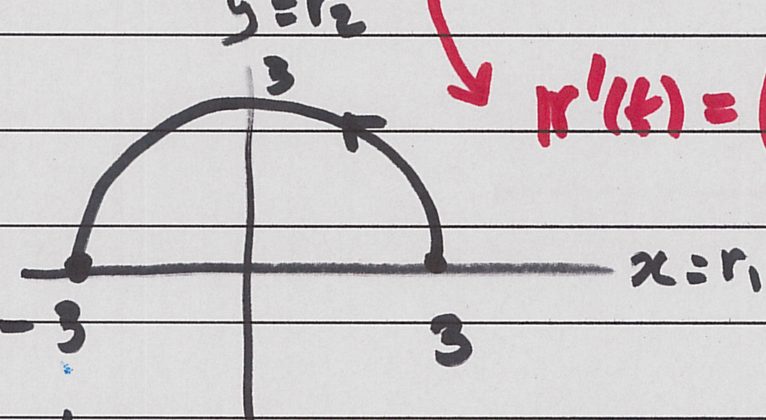
$$1 = |\mathbf{r}'(\underbrace{\phi(s)}_{= t})| \cdot \underbrace{\phi'(s)}_{= \frac{d\phi}{ds}} \implies \phi'(s) = \frac{1}{|\mathbf{r}'(t)|}$$

Set $\gamma : [0, \ell] \to \mathbb{R}^n$ where $\ell = \int_a^b |\mathbf{r}'(\tau)|\, d\tau$ [redacted]

by $\gamma(s) = \mathbf{r} \circ \phi(s)$.

<u>Example</u> $\mathbb{r}(t) = (3\cos 2t, 3\sin 2t)$

$[a,b] = [0, \pi/2]$.




$\mathbb{r}'(t) = (-6\sin 2t, 6\cos 2t)$

$|\mathbb{r}'(t)| = 6$

$s = \int_0^t 6\, d\tau = 6t$ , $\ell = \int_0^{\pi/2} 6\, d\tau = 3\pi$

$\phi(s) = s/6$

$\gamma(s) = 3\left(\cos\frac{s}{3}, \sin\frac{s}{3}\right)$.

[redacted]

## Some Organization:

heat equation

$$\begin{cases} u_t = \Delta u \text{ on } U \times (0,T) \\ u(x,0) = g(x) \\ u\big|_{\partial U} = h(x,t) \end{cases}$$

$U = [0,L]$ (1-D)

$\longrightarrow$ homogeneous boundary conds.

$u(0,t) = 0,\ u(L,t) = 0$

$$S : g \longmapsto \int_{\xi \in [0,L]} G(x,\xi,t)\, g(\xi)$$

$G$ = Green's function for

$Lu = u_t - \Delta u$ on $[0,L]$

$\downarrow$ $u\big|_{\partial U} = h(x)$ or $0$

steady state

Laplace's Equation

$$\begin{cases} \Delta u = 0 \\ u\big|_{\partial U} = h \end{cases}$$

$\leftarrow$ (exam)

operator + domain $\longrightarrow$ Green's fnc.

$\downarrow$

solution operator

$\hookrightarrow$ Poisson's eqn. + homo. bdy. conds.

$$\Delta u = -\Delta h = f\ ,\quad u\big|_{\partial U} = 0\ .$$

[redacted]

6.

Last time:

$$G(x,\xi,t) = \cancel{-\frac{2}{L}\sum_{j=1}^{\infty}\left(\frac{L}{j\pi}\right)^2} \sin\frac{j\pi\xi}{L}\sin\frac{j\pi x}{L}\, e^{-k\left(\frac{j\pi}{L}\right)^2 t}$$

$$\sum_{j=1}^{\infty}\frac{2}{L}$$

coefficients look big.

$$G = \sum_1 \frac{2}{L}\sin\frac{j\pi\xi}{L}\sin\frac{j\pi x}{L}\, e^{-\left(\frac{j\pi}{L}\right)^2 t}$$

exponentials may help when $t>0$.

Fix $\xi$: Plug $G$ into the PDE

$$G_{xx} = -\sum_1 \frac{2}{L}\left(\frac{j\pi}{L}\right)^2 \sin\frac{j\pi\xi}{L}\sin\frac{j\pi x}{L}\, e^{-k\left(\frac{j\pi}{L}\right)^2 t}$$

$$G_t = -k\sum_1 \frac{2}{L}\left(\frac{j\pi}{L}\right)^2 \sin\frac{j\pi\xi}{L}\sin\frac{j\pi x}{L}\, e^{-k\left(\frac{j\pi}{L}\right)^2 t}$$

$G(x,\xi,t)$ satisfies $G_t = kG_{xx}$.

$G: [0,L]\times[0,L]\times(0,T)$

[redacted]

Poisson's Eqn

$$\begin{cases} \Delta u = f & \text{on } U \\ u|_{\partial U} = 0 \end{cases}$$

$U = [0, L]$ (1-D)

$$\begin{cases} u'' = f & \text{on } [0, L] \\ u(0) = 0, \ u(L) = 0 \end{cases}$$

$$u'' = \sum_{j=1}^{\infty} f_j \sin\frac{j\pi x}{L}, \qquad f_j = \frac{2}{L}\int_0^L f(\xi)\sin\frac{j\pi\xi}{L}\, d\xi.$$

$$\begin{cases} u_j'' = f_j \sin\frac{j\pi x}{L} \\ u_j(0) = 0, \ u_j(L) = 0 \end{cases} \Longrightarrow u_j = -\left(\frac{L}{j\pi}\right)^2 f_j \sin\frac{j\pi x}{L}$$

$$u = \sum_{j=1}^{\infty} u_j = -\sum_{j=1}^{\infty}\left(\frac{L}{j\pi}\right)^2 f_j \sin\frac{j\pi x}{L}$$

[redacted]

8.

$$u(x) = -\sum_{j=1}^{\infty} \left(\frac{L}{j\pi}\right)^2 f_j \sin\frac{j\pi x}{L}$$

$$= -\sum_{j=1}^{\infty} \frac{2}{L}\int_0^L f(\xi)\sin\frac{j\pi\xi}{L}\,d\xi \;\sin\frac{j\pi x}{L}$$

$$= \int_0^L \left(-\frac{2}{L}\sum_{j=1}^{\infty} \sin\frac{j\pi\xi}{L}\sin\frac{j\pi x}{L}\right) f(\xi)\,d\xi$$

$$\left(\frac{L}{j\pi}\right)^2$$

$$G(x,\xi) = -\frac{2}{L}\sum_{j=1}^{\infty}\left(\frac{L}{j\pi}\right)^2 \sin\frac{j\pi\xi}{L}\sin\frac{j\pi x}{L}$$

Green's function for $\Delta$
on $[0,L]$

[redacted]