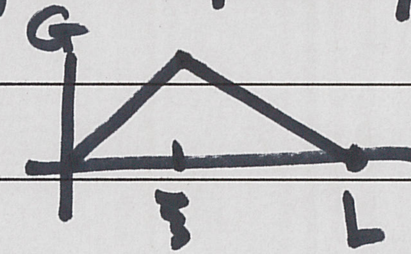
Lecture 15 MATH 6702 Feb. 26, 2020

Organization:

1. (today) $u_\delta \leftarrow H^1$ approximation of G
   $(G(x,\xi)$ for ξ fixed$)$

2. physical interpretation
   1-D Poisson, 1-D heat PDE
   (3-D too)



"$-\Delta G = \delta_{\xi}$"

* 3. Integration

4. Existence/Uniqueness Theorem of Weak Solutions

What about uniqueness for classical solutions of $\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} \equiv 0 \end{cases}$

$u \in C^2(\overline{\Omega})$

[redacted]

$$\begin{cases} \Delta u = f & \text{on } \mathcal{U} \leftarrow \text{bounded open} \\ u|_{\partial \mathcal{U}} \equiv 0 \end{cases}$$

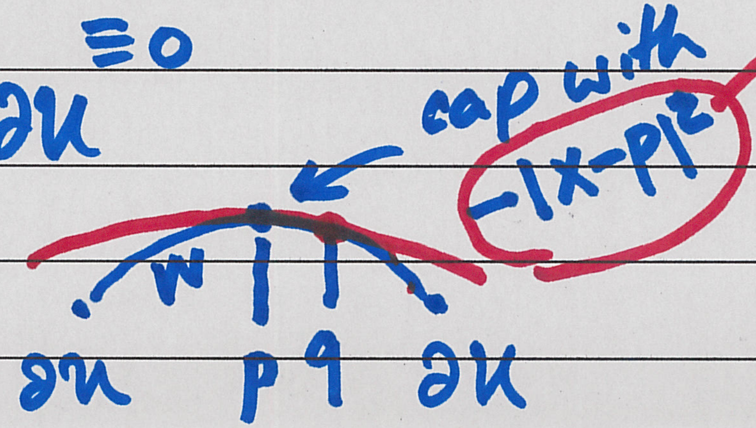
<u>Uniqueness</u>: This problem has (at most) <u>one</u> classical solution.

<u>Why?</u> Put $w = u - v$ where u and v are classical solns.

$$\begin{cases} \Delta w = 0 & \text{on } \mathcal{U} \\ w|_{\partial \mathcal{U}} \equiv 0 \end{cases}$$

$\Delta(-|x-p|^2) < 0$

<u>Idea:</u>



[redacted]

The Meaning of G as a solution of "$-\Delta u = \delta_\xi$":

<u>Consider</u>

$$\mu^{(x)} = \begin{cases} \frac{1}{2\delta}, & |x-\xi|<\delta \\ 0, & |x-\xi|>\delta \end{cases}$$

$\int \mu = 1$

$$= \frac{1}{2\delta}\chi_{[\xi-\delta,\,\xi+\delta]}.$$

Associated Functional $M[\phi] = \int \mu\phi = \frac{1}{2\delta}\int_{\xi-\delta}^{\xi+\delta} \phi(x)\,dx.$

$$\lim_{\delta\to 0} M[\phi] = \lim_{\delta\to 0} \underbrace{\frac{1}{2\delta}\int_{\xi-\delta}^{\xi+\delta}\phi(x)\,dx}_{\text{AVG value of } \phi} = \phi(\xi) = \delta_\xi[\phi].$$

[redacted]

Poisson Problem

$$(*)\quad \begin{cases} -\Delta u = \mu & \text{on } (0,L) \\ u(0) = u(L) = 0 \end{cases}$$

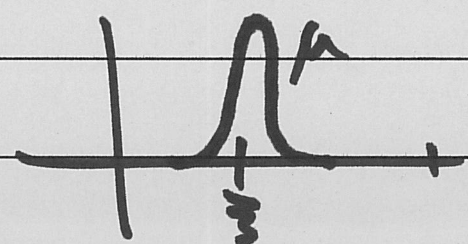
$-u''$

solution (weak solution) $u = u_\delta$

<u>Guess</u>: $u_\delta \longrightarrow G(x,\xi)$.

Solve $(*)$ for $u = u_\delta$.

Might want to try $\mu \in C_c^\infty(0,L)$.

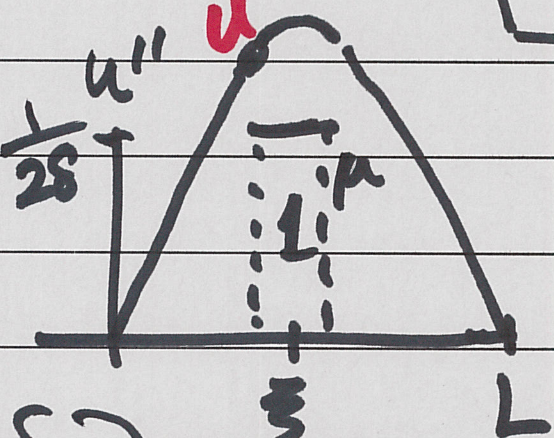


[redacted]

<u>Weak Soln</u> $-\int u''\phi = \int \mu\phi$

$\boxed{\int u'\phi'} = \int \mu\phi \leftarrow$ weak formulation

Make a guess for u':



$$u'(x) = \begin{cases} c_1 \cancel{x}, & 0 \le x \le \xi-\delta \\ -\frac{1}{2\delta}x + b, & \xi-\delta \le x \le \xi+\delta \\ c_2, & \xi+\delta \le x \le L \end{cases}$$

$u'' = -\frac{1}{2\delta}$

$u' = -\frac{1}{2\delta}x + b \leftarrow$

$u = -\frac{1}{4\delta}x^2 + bx + c$

[redacted]

Guess for u (more detailed)

$$u(x) = u_\delta = \begin{cases} c_1 x, & 0 \le x \le \xi - \delta \\ -(x-\xi)^2/(4\delta) + a(x-\xi) + b, & \xi - \delta \le x \le \xi + \delta \\ c_2(x-L), & \xi + \delta \le x \le L. \end{cases}$$

For continuity:

$$\begin{cases} c_1(\xi - \delta) = -\dfrac{\delta}{4} + a(-\delta) + b \\ c_2(\xi + \delta - L) = -\dfrac{\delta}{4} + a\delta + b \end{cases}$$

↑ With this $u = u_\delta \in W^1$

To check: $-\int u_\delta \phi' = \int u_\delta' \phi$

[redacted]

$$\int u' \phi'$$

$$u' = \begin{cases} c_1 & \cdots \\ -(x-\xi)/(2\delta) + a & \cdots \\ c_2 & \cdots \end{cases}$$

$$c_1 \int_0^{\xi-\delta} \phi'(x)\,dx + \int_{\xi-\delta}^{\xi+\delta} \left[-\frac{x-\xi}{2\delta} + a \right] \phi'(x)\,dx$$

$$+ c_2 \int_{\xi+\delta}^{L} \phi'(x)\,dx$$

$$\overset{?}{=} \frac{1}{2\delta} \int_{\xi-\delta}^{\xi+\delta} \phi(x)\,dx$$

$$c_1 \phi(\xi-\delta) - \frac{1}{2\delta} \int_{\xi-\delta}^{\xi+\delta} \overset{\eta}{(x-\xi)}\, \overbrace{\phi'(x)\,dx}^{d\mu}$$

$$+ a \left(\phi(\xi+\delta) - \phi(\xi-\delta) \right)$$

$$- c_2 \phi(\xi+\delta).$$

$$(c_1 \mp a)\,\phi(\xi-\delta) - \frac{(x-\xi)}{2\delta}\phi \Big|_{\xi-\delta}^{\xi+\delta} + \frac{1}{2\delta} \underline{\int_{\xi-\delta}^{\xi+\delta} \phi(x)\,dx} + \cdots$$

[redacted]

Need

$$(c_1 - a)\,\phi(\xi - \delta) - \boxed{\frac{x-\xi}{2\delta}\,\phi(x)\Big|_{\xi-\delta}^{\xi+\delta}} + (a - c_2)\,\phi(\xi+\delta) = 0$$

$$\frac{1}{2}\phi(\xi+\delta) + \frac{1}{2}\phi(\xi-\delta)$$

$$\boxed{\left(c_1 - a \overset{-}{+} \frac{1}{2}\right)\phi(\xi-\delta) + \left(a - c_2 \overset{-}{+} \frac{1}{2}\right)\phi(\xi+\delta) = 0}$$

← We want/need this for every $\phi \in C_c^\infty(0,L)$.

Fundamental Lemma

$$\int f\phi = 0 \ \forall \phi \Rightarrow f = 0.$$

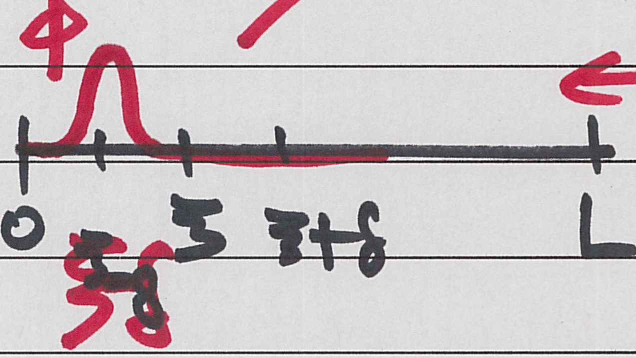
[redacted]

(10)

Key Point:

$$\left(c_1 - a \cancel{+} - \frac{1}{2}\right)\phi(\xi-\delta) = \left(c_2 - a \cancel{-} + \frac{1}{2}\right)\phi(\xi+\delta)$$

$$\forall \phi \in C_c^\infty$$



$\longleftarrow$ pick ϕ with $\xi-\delta \in \operatorname{supp}\phi$ and $\xi+\delta \notin \operatorname{supp}\phi$.

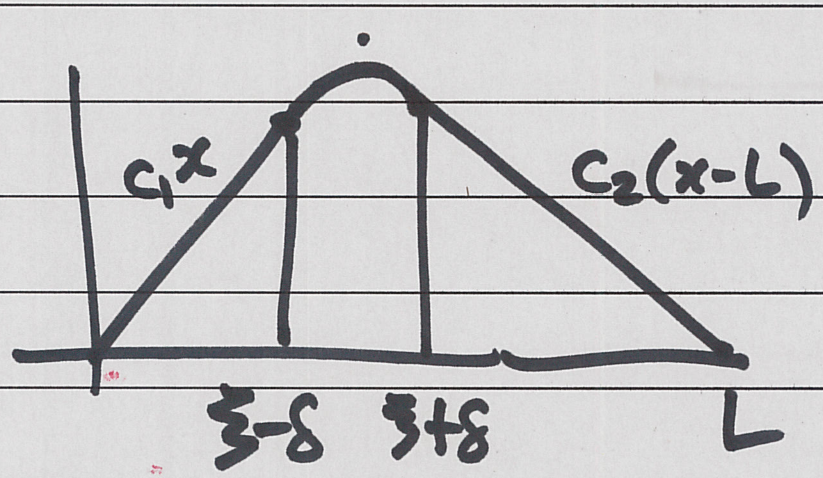
$$\Rightarrow c_1 - a - \frac{1}{2} = 0$$

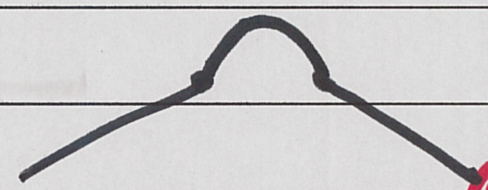
$$c_1 = a + \frac{1}{2}$$

Also, $c_2 = a - \frac{1}{2}$.

[redacted]

Eqns. For constants:





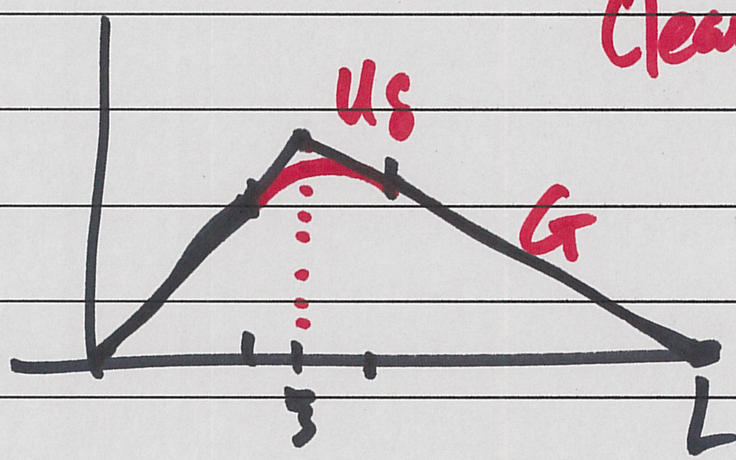
This algebraic system for a, b, c_1, c_2 has a unique solution.

$$\begin{cases} c_1 = a + \frac{1}{2} \\ c_2 = a - \frac{1}{2} \end{cases}$$ ← weak solution

$$\begin{cases} c_1(\xi-\delta) = -\frac{f}{4} - a\delta + b \\ c_2(\xi+\delta-L) = -\frac{f}{4} + a\delta + b \end{cases}$$ ← continuity, weak deriv.

[redacted]

$$u_\delta(x) = \begin{cases} \frac{L-\xi}{L}x, & 0 \le x \le \xi-\delta \\ -\frac{(x-\xi)^2}{4\delta} + \frac{L-2\xi}{2L}(x-\xi) - \frac{4\xi^2-4L\xi+\delta L}{4L}, & \xi-\delta \le x \le \xi+\delta \\ -\frac{\xi}{L}(x-L), & \xi+\delta \le x \le L. \end{cases}$$



Clearly $u_\delta \to G$ (as $\delta \searrow 0$)
(pointwise uniformly)

Regularity of u_δ?

$u = u_\delta \in C^1[0,L]$.

[redacted]