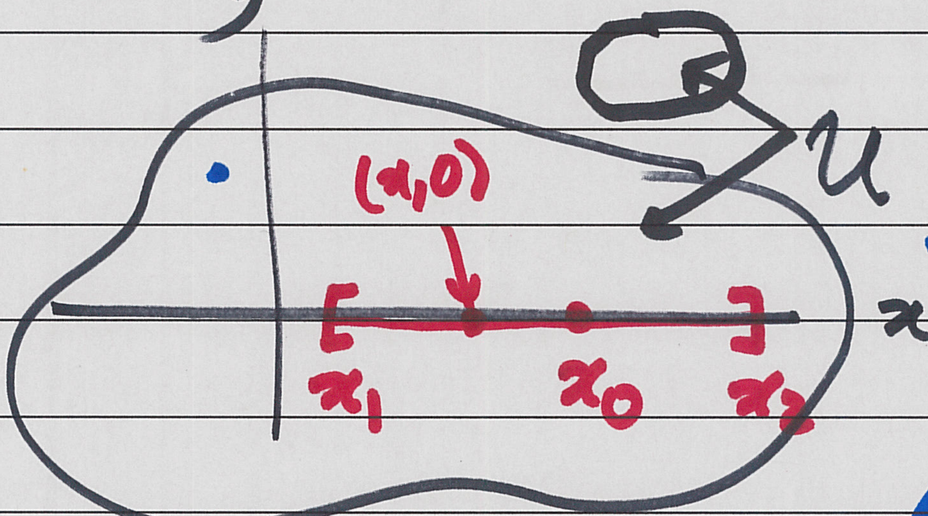
Lecture 14 MATH 6702 Feb. 24, 2020

Question: Assignment #6.

$L[u] = a\, u_x + b\, u_y + c\, u = f$

1st order linear



$\mathcal{U} \xrightarrow{u} \mathbb{R}$

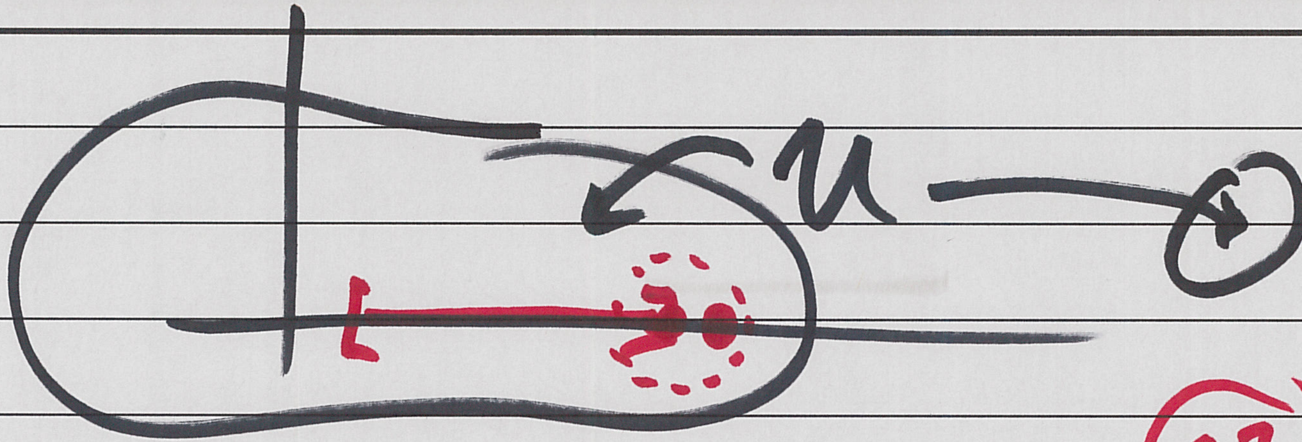
for any $f_0 \in C^0(\bar{\mathcal{U}})$

1. If you can solve $\begin{cases} L[v] = f_0 \\ \boxed{v|_I = 0} \end{cases}$,

then how about $\begin{cases} Lu = f \text{ on } \mathcal{U} \\ \boxed{u|_I = u_0} \end{cases}$? $u_0 = u_0(x)$

[redacted]

②



$$\begin{cases} Lu = f \\ u\big|_I = u_0 = u(x) \end{cases} \rightsquigarrow \begin{cases} \\ v\big|_I = 0 \end{cases}$$

??

$\leftarrow$ done

homogeneous Cauchy data

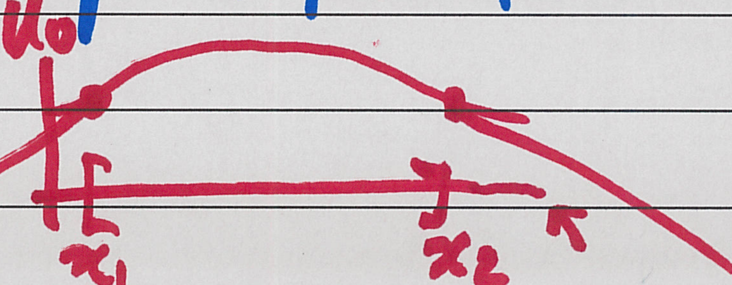
<u>IDEA 0</u>: "$v = u - u_0$"

$v(x,y) = u(x,y) - \underline{\underline{u_0(x)}}$

$\to$ only defined for $x_1 \le x \le x_2$.

<u>IDEA 1</u>: Extend u_0

$\bar{u}_0 \in C^1(\mathbb{R})$



[redacted]

3

$$V = u - \bar{u}_0$$

$$\bar{u}_0(x) = \begin{cases} u_0(x_1) + u_0'(x_1)(x - x_1), & x \le x_1 \\ u_0(x), & x_1 \le x \le x_2 \\ u_0(x_2) + u_0'(x_2)(x - x_2), & x \ge x_2 \end{cases}$$

$$\begin{cases} V_x = u_x - \bar{u}_0' \\ V_y = u_y \end{cases} \longrightarrow \quad u_x = V_x + \bar{u}_0'$$

know (I want) $au_x + bu_y + cu = f$

$$a(V_x + \bar{u}_0') + b V_y + c(V + \bar{u}_0) = f$$

$$aV_x + bV_y + CV = \underbrace{f - a\bar{u}_0' - c\bar{u}_0}_{f_0}$$

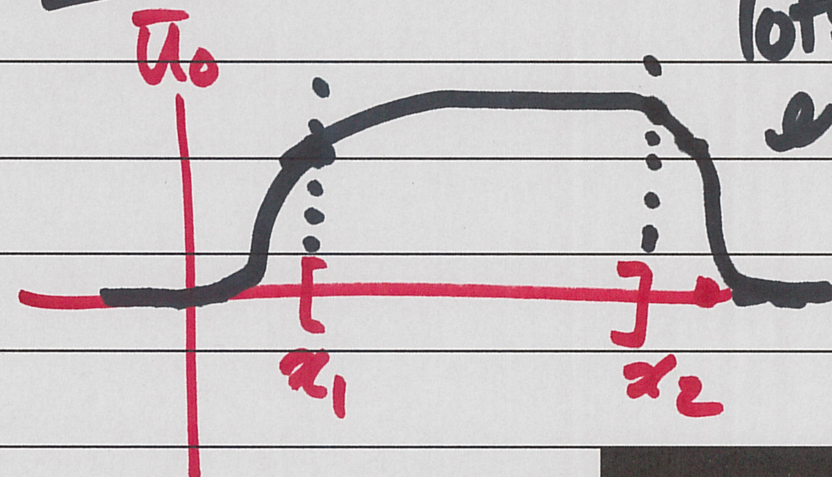
[redacted]

Extend $\bar{u}_0$ by:

$$\text{and}\begin{cases} \bar{u}_0'(x) \equiv u_0'(x_1) & \text{for } x \le x_1 \\ \bar{u}_0'(x) \equiv u_0'(x_2) & \text{for } x \ge x_2 \end{cases}$$

<u>Okay</u>

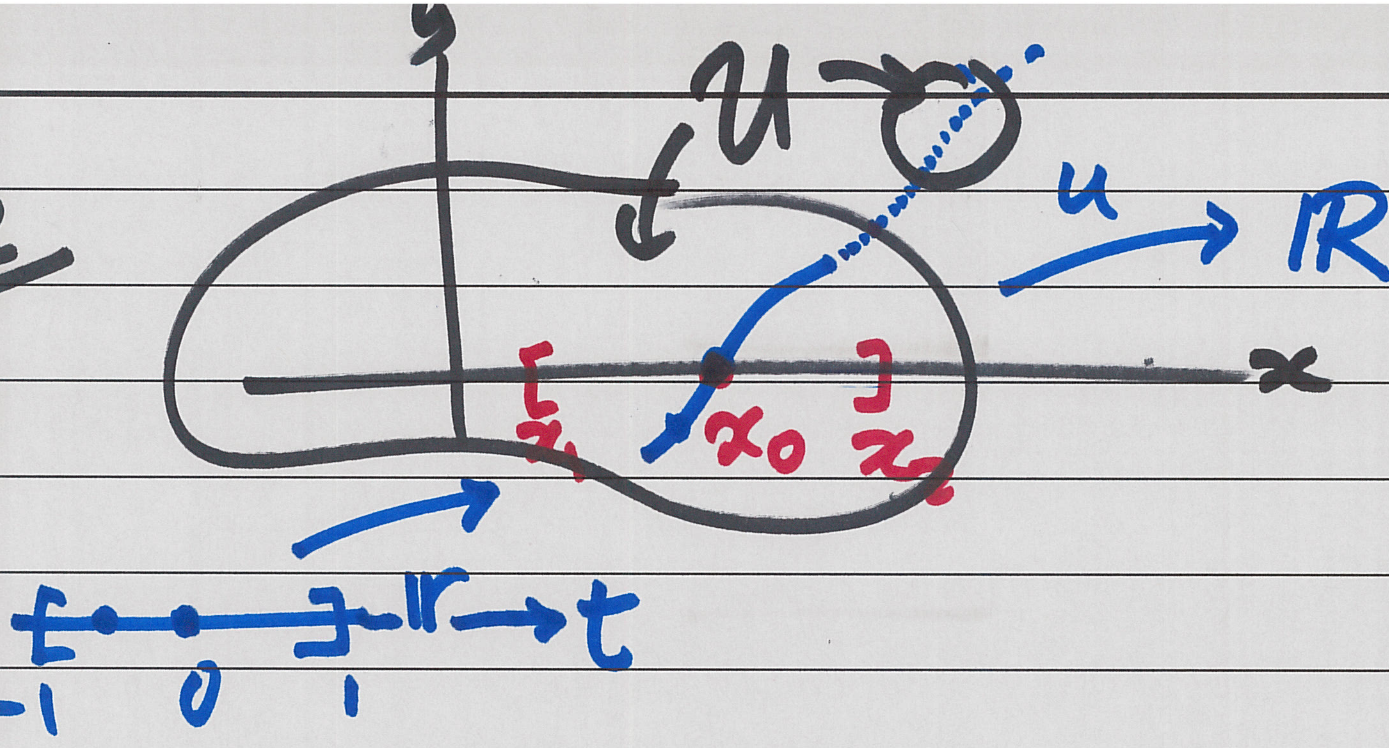
<u>U</u>nique<u>ness</u>?



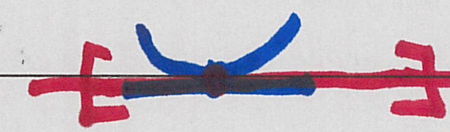
lots of extensions.

[redacted]

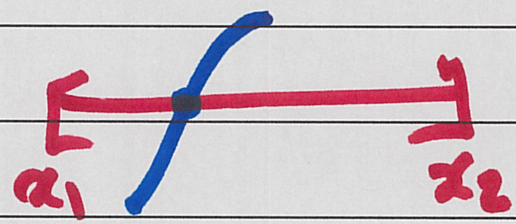
#2



Problem situation:

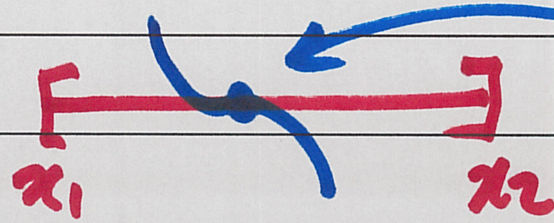


$u \equiv u_0 \Rightarrow u_x \big|_I = u_0'$



Cauchy Data

non characteristic (okay)



characteristic/
singular
(not okay)

[redacted]

b $\frac{d}{dt} u \circ \mathbb{r} = \frac{d}{dt}\left[u(r_1(t), r_2(t)) \right]$

$\uparrow$ x slot $\quad \uparrow$ y slot

<u>Chain Rule</u> : $\frac{d}{dt} u \circ \mathbb{r} = \boxed{u_x \cdot r_1' + u_y r_2'}$

5 Compare to $\boxed{a u_x + b u_y} + cu = f$

$\mathbb{r}' = (a, b) \leftarrow$ ODE for $\mathbb{r}$

$$\begin{cases} r_1' = a(r_1, r_2) \\ r_2' = b(r_1, r_2) \end{cases} \quad (4)$$

[redacted]

ODE

$$\begin{cases} \mathbb{X}' = \mathbb{F}(\mathbb{X}, t) \\ \mathbb{X}(t_0) = \mathbb{X}_0 \end{cases}$$

nonautonomy

(a) $\begin{cases} \mathbb{r}' = (a, b) = \mathbb{F}(\mathbb{r}) \\ \mathbb{r}(0) = (x_0, 0) \end{cases}$

existence/uniqueness?

Need Lipschitz $\underline{\underline{\text{or}}}$ $\underline{\underline{C^1}}$ (for $\mathbb{F}$)

$a, b \in C^1$ will do it!

[redacted]

Exam:

$u = u(x, y, z)$

$$L\begin{bmatrix} u \\ v \end{bmatrix} = \begin{pmatrix} u_x - v_y + 2y u_x + 2z v_{\cancel{x} z} \\ v_x + u_y + 2y v_x - 2z u_{\cancel{x} z} \end{pmatrix}$$

Domain $\mathbb{R}^2$ or $\mathbb{R}^3$ or $\mathbb{R}^n$ ✓

maybe domain of u or v

Domain of L — Not what goes in to u or v but what $\begin{pmatrix} u \\ v \end{pmatrix}$ is . . .

where $\begin{pmatrix} u \\ v \end{pmatrix}$ is

← the thing that goes in to L.

[redacted]

$\binom{u}{v}$ should be $\underline{\underline{\text{in}}}$ some set of

<u>pairs of functions.</u>

Cartesian Product $\qquad \mathbb{R},\ \mathbb{R}\times\mathbb{R}=\mathbb{R}^2$

Domain $C^1(\mathbb{R}^3)\times C^1(\mathbb{R}^3)$

$C^1(\mathcal{U})\times C^1(\mathcal{U}),\ \mathcal{U}\subseteq\mathbb{R}^3$

Codomain $C^0 \times C^0$

#2 Exam 1

$$W(\xi,\eta,T) = u(\underset{x}{\underset{\parallel}{\alpha\xi}}, \underset{y}{\underset{\parallel}{\alpha\eta}}, \underset{t}{\underset{\parallel}{\beta T}})$$

$$(x,y,t) \in \underset{\substack{\uparrow \\ x^2+y^2<1}}{B_1(0)} \times \underset{\substack{\uparrow \\ t}}{[0,\infty)}$$

heat domain for u

$\xi^2+\eta^2 < \dfrac{1}{\alpha^2}$

$B_{1/\alpha}(0)$.

$\Rightarrow$ heat domain for W:

$$B_{1/\alpha}(0) \times [0,\infty)$$

[redacted]

$$w_T = \beta u_t \quad \text{not } \beta u_T$$

$$w_T(\xi, \eta, T) = \beta u_t(\underset{x}{\underset{\parallel}{\alpha\xi}}, \underset{y}{\underset{\parallel}{\alpha\eta}}, \underset{t}{\underset{\parallel}{\beta T}})$$

"slots"

$$\Delta w = \alpha^2 \Delta u$$

ξ, η spatial Laplacian on $C^2(B_{1/\alpha}(0))$

(b) Can Solve $\quad w_T = \Delta w + f_0$

<u>Solve</u> $\quad u_t = k\Delta u + f$

<u>scale in time</u>

$$\boxed{w(x,y,T) = u\left(x,y,\tfrac{1}{k}T\right)}$$

$$\Rightarrow w_T = \frac{1}{k} u_t$$

$$= \frac{1}{k}\left(\underbrace{k\Delta u}_{=\,\Delta w} + f\right)$$