Lecture 13 MATH 6702 Feb. 19, 2020.

$$\begin{cases} \Delta u = f \text{ on } U \\ u|_{\partial U} = 0 \end{cases} \longrightarrow$$

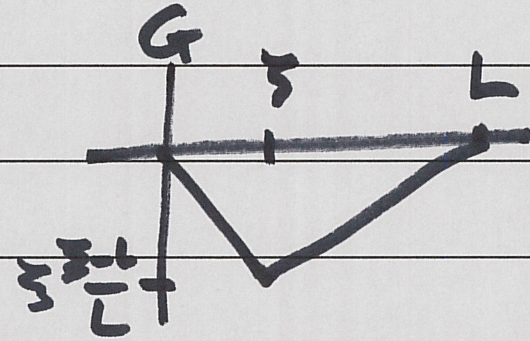
Existence + Uniqueness for weak solutions

$f \in L^2$

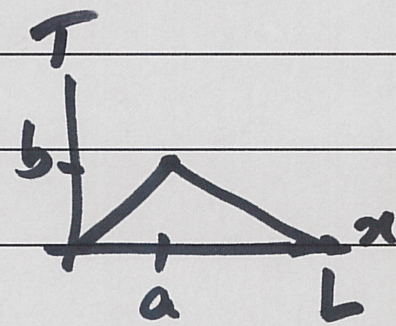
$u \in W^{1,2}$

<u>1D</u> $\begin{cases} u'' = f \text{ on } [0,L] \\ u(0) = u(L) = 0 \end{cases}$

$$G(x,\xi) = \begin{cases} \frac{\xi - L}{L} x, & 0 \le x \le \xi \\ \frac{\xi}{L}(x-L), & \xi \le x \le L \end{cases}$$



$$T(x) = \begin{cases} \frac{b}{a} x, & 0 \le x \le a \\ \frac{b}{a-L}(x-L), & a \le x \le L \end{cases}$$

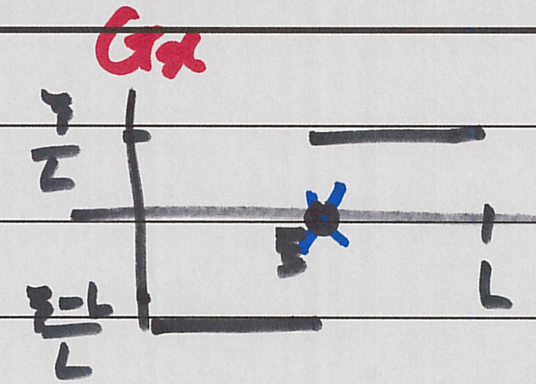


Weak Derivatives

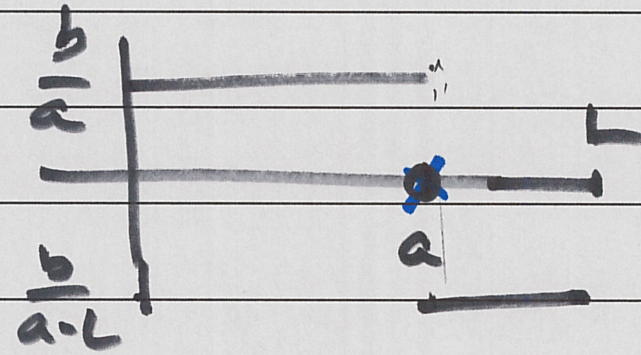
[redacted]

②

$$G_x = \begin{cases} \dfrac{\xi - L}{L}, & 0 \le x < \xi \\ \dfrac{\xi}{L}, & \xi < x \le L \end{cases}$$



$$T' = \begin{cases} \dfrac{b}{a}, & 0 \le x < a \\ \dfrac{b}{a-L}, & a < x \le L \end{cases}$$



$$-\int_0^L G(x,\xi)\,\phi'(x)\,dx = \int_0^L G_x(x,\xi)\,\phi(x)\,dx \quad \forall \phi \in C_c^\infty(0,L)$$

$$-\int_0^L T(x)\,\phi'(x)\,dx = \int_0^L T'(x)\,\phi(x)\,dx \quad \forall \phi \in C_c^\infty(0,L)$$

Weak Derivatives.

[redacted]

Functional Representation of Functions

"operator version" "distribution version"

$$\mathcal{D}' = \mathcal{L}^0\left(\underline{C_c^\infty(\mathcal{U})}\right) = \left[\, C_c^\infty(\mathcal{U}) \,\right]^*$$

↑ continuous and linear functionals

↑ dual space.

Functions G T G_x — T'

operator version $\phi \mapsto \int_{(0,L)} G\phi$

$\phi \mapsto \int_{(0,L)} G_x \phi = -\int_{(0,L)} G\phi'$

$\phi \mapsto \int_{(0,L)} T\phi$

f , $\phi \mapsto \int_{(0,L)} f\phi$ functional representation of f

[redacted]

G_x and T' do not make sense classically.

G_x and T' <u>do</u> make sense weakly (as weak derivatives)

G_{xx} and T'' do not make sense even weakly.

<u>But</u> They do make sense <u>distributionally</u>

$$\phi \longmapsto \int_{(0,L)} G\phi'' \leftarrow \text{something in } \mathcal{D}'$$

$$\cancel{\phi \longmapsto \int_{(0,L)} G_{xx}\phi}$$

$$\phi \longmapsto -\int_{(0,L)} G_x\phi'$$

[redacted]

All distributions are differentiable

$L : C_c^\infty(\mathcal{U}) \longrightarrow \mathbb{R}$ is any distribution in $\mathcal{D}'$

There is another distribution defined by

$$L'[\phi] = -L[\phi']$$

Is this linear? $L' : C_c^\infty(\mathcal{U}) \longrightarrow \mathbb{R}$

L'

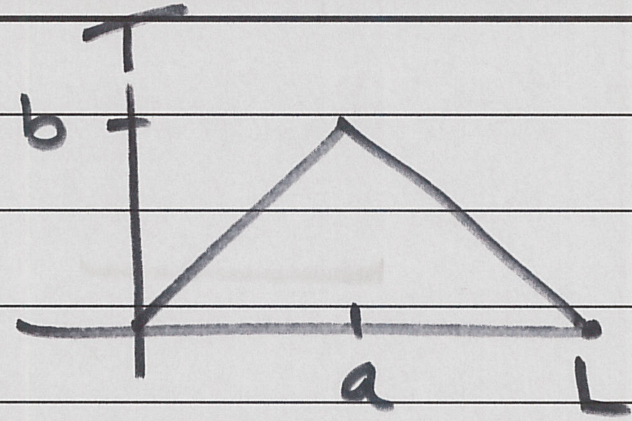
$$\begin{aligned} L'[a\phi + b\psi] &= -L[a\phi' + b\psi'] \\ &= -aL[\phi'] - bL[\psi'] \\ &= aL'\phi + bL'\psi. \end{aligned}$$

linear! ✓

$\boxed{\text{cts ?}}$

[redacted]

$\underline{\underline{T''}}$



$L[\phi] = \int T\phi''$

$= -\int T'\phi'$ (T' is a weak derivative)

$= -\int_0^a \frac{b}{a}\phi'\,dx - \int_a^L \frac{b}{a-L}\phi'\,dx$

$= -\frac{b}{a}\phi(a) + \frac{b}{a-L}\phi(a)$

[redacted]

$= -\left(\frac{b}{a} - \frac{b}{a-L}\right)\phi(a)$

L is a multiple of the evaluation functional on $\mathcal{H}$

(6)

$\phi \mapsto \phi(a)$ ← continuous linear functional

names:
- evaluation functional
- Dirac delta functional
- Dirac measure

$\delta_a[\phi] = \phi(a)$

linear: $\delta_a[\alpha\phi + \beta\psi] = \alpha\phi(a) + \beta\psi(a)$ ✓
$= \alpha\,\delta_a[\phi] + \beta\,\delta_a[\psi]$.

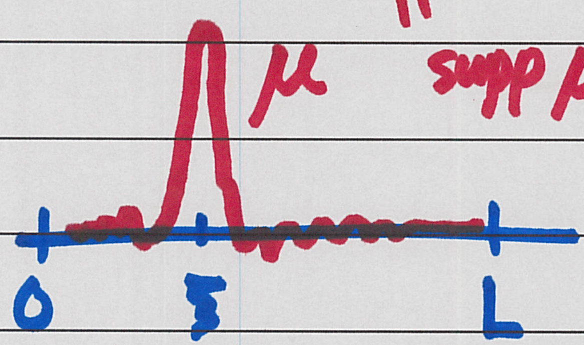
Gxx

$L[\phi] =$

[redacted]

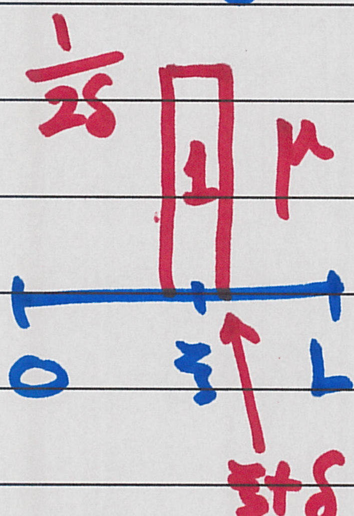
What does this mean?

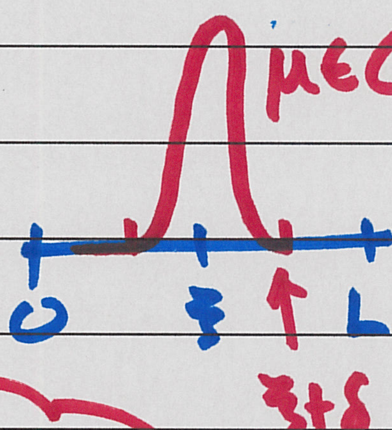
$\delta_\xi \leftarrow$ distribution (no corresponding function)



" supp μ concentrated at ξ. "

$\int \mu = 1.$





$\mu \in C_c^\infty(0,L)$

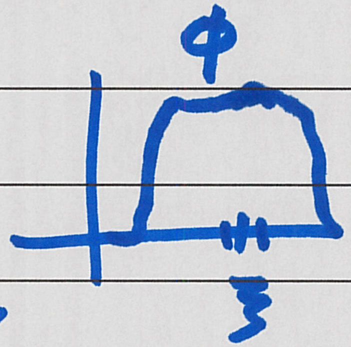
supp $\mu \subseteq (\xi-\delta, \xi+\delta)$, $\mu \geq 0$.

$\int \mu = 1.$

distrib. $\phi \longmapsto \int \mu \phi$

[redacted]

$$\int_0^L \mu_\delta \phi \, dx = \int_{\xi-\delta}^{\xi+\delta} \frac{1}{2\delta} \phi(x) dx$$



What happens as $\delta \searrow 0$

Good (wrong) answer : ∞

$$\frac{1}{2\delta} \int_{\xi-\delta}^{\xi+\delta} \phi(x)\, dx \xrightarrow{\delta \searrow 0} \phi(\xi)$$

What is this ?

Ans. : The average value of ϕ on $[\xi-\delta, \xi+\delta]$

[redacted]