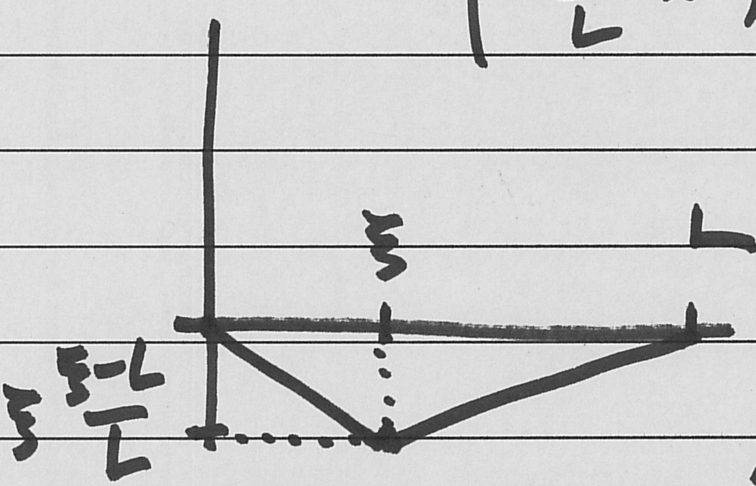
Lecture 12 Math 6702 Feb. 17, 2020

- 1-D Heat PDE $u_t = \Delta u = u''$
- Poisson's PDE $\Delta u = f \longrightarrow$ 1-D, $u'' = f$

ODE

$\rightsquigarrow$ Green's Function

$$G(x,\xi) = \begin{cases} \frac{\xi - L}{L} x, & 0 \le x \le \xi \\ \frac{\xi}{L} x, & \xi \le x \le L \end{cases}$$



Is G a solution of the original DE?

[redacted]

G has a weak derivative w.r.t. x for ξ fixed.

<u>Idea</u>: Say you have a classical operator

$$\frac{d}{dx} : C^1[0,L] \longrightarrow C^0[0,L].$$

There is a adjoint operator $\cancel{\phi \longmapsto -\int g\phi'}$

$$\cancel{L : W^1[0,L] \longrightarrow \mathbb{R},\ L[g] = -\int g\phi'}$$

$$\cancel{-\int \phi\phi' = \int}$$

$$\phi \longmapsto -\int f\phi'.$$ This operator "fakes" a weak derivative of f.

[redacted]

g is a weak derivative of f if

$$-\int f\phi' = \int g\phi. \quad \forall \phi \in C_c^\infty.$$

You can use this weak differentiation operator

$$\phi \longmapsto -\int f\phi'$$

To formulate weak versions of ODE's.

$$u' = 0 \longleftarrow \text{Classical Solution}$$

$u = \text{const.}$

A weak solution is one for which

$$-\int u\phi' = 0. \quad \forall \phi \in C_c^\infty$$

first derivative is 0 weakly

[redacted]

$$\int u\phi' = 0 \quad \forall \phi \in C_c^\infty$$ ← What can you say about such a function u?

<u>One guess</u>: $u \equiv 0$.

<u>Pros</u>: ① Definitely this works. ✓

② Motivated by Fundamental Lemma of calc. of variations: ✓

$$\int u\phi = 0 \quad \forall \phi \in C_c^\infty \Rightarrow u \equiv 0.$$

<u>Cons</u>: ① constants also work:

$$\int c\phi' = c\int \phi' = c\phi\Big|_\partial = 0.$$

[redacted]

$\int u\phi' = 0 \quad \forall\, \phi \in C_c^\infty[a,b]$. $\leftarrow$ u is a weak solution of $x' = 0$.

<u>Observation 1</u>: This is weaker because

$A = \{\phi' : \phi \in C_c^\infty\} \subsetneqq C_c^\infty$

Can you characterize these test functions?

<u>Ans.</u> Yes: $A = \{\psi \in C_c^\infty : \int \psi = 0\}$.

This is interesting because there is geometry.

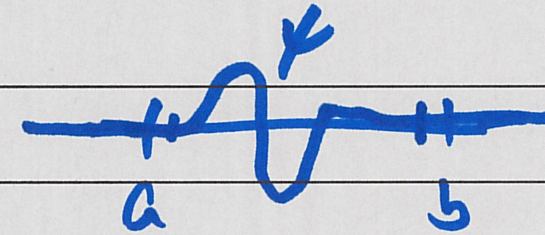
$\int \psi = \int \psi \cdot 1 = \langle \psi, 1 \rangle_{L^2}$

A is the orthogonal complement of span$\{1\}$ in L^2.

[redacted]

$A = \{\psi \in C_c^\infty : \int_{[a,b]} \psi = 0\}$ Why?

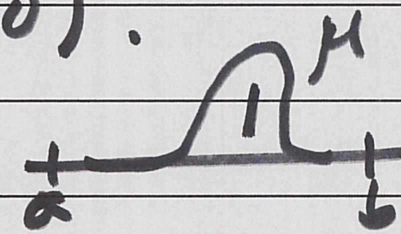
ANS. $\phi(x) = \int_{-\infty}^{x} \psi(t)\,dt$



$\underline{\phi'(x) = \psi(x).}$

Now: Take an arbitrary $\eta \in C_c^\infty$ and "build" a function in A: (probably $\int \eta \neq 0$).

Take also $\mu \in C_c^\infty$ with $\mu \geq 0$, $\int \mu = 1$



↖ fixed.

$\psi = \eta - c\mu$ where $c = \int \eta$

Note: $\psi \in C_c^\infty$.

[redacted]

$\psi = \eta - c\mu$ $\quad \begin{cases} \eta \in C_c^\infty \text{ arbitrary} \\ \mu \in C_c^\infty \text{ fixed } \int \mu = 1 \\ c = \int \eta \end{cases}$

$$\int \psi = \int \eta - c \int \mu = \int \eta - c = 0.$$

$$\psi = \eta - c\mu = \phi'$$

$$\underbrace{\int u\phi' = 0 \quad \forall \phi \in C_c^\infty}_{\text{weak soln } u} \Rightarrow \int u[\eta - c\mu] = 0 \quad \forall \eta \in C_c^\infty$$

$$\Rightarrow \underline{\int u\eta} - c\int u\mu = 0 \quad \forall \eta \in C_c^\infty$$

↑ fundamental lemma?

[redacted]

$$\int u\eta - c\int \underbrace{u\mu}_{} = 0 \quad \forall \eta \in C_c^\infty$$

↑ independent of η : Call it α (constant).

$c = \int \eta$

$$\int u\eta - \left(\int \eta\right)\alpha = 0 \quad \text{for all } \eta \in C_c^\infty$$

$$\int (u-\alpha)\eta = 0 \quad \text{for all } \eta \in C_c^\infty.$$

Fundamental Lemma $\Rightarrow$ $u - \alpha \equiv 0$.

$\Rightarrow$ u is a constant α.

This shows: A weak solution of $u' = 0$ is a classical solution.

[redacted]

Still on $[a,b]$. $u'' = f$ Poisson's ODE

classical $\begin{cases} u \in C^2[a,b] \\ f \in C^0[a,b]. \end{cases}$

Weak Adjoint operator of $\frac{d^2}{dx^2}$

$$\int u''\phi = u'\phi\Big|_\partial - \int u'\phi'$$

$$= -\int u'\phi'$$

$$= -\left[u\phi'\Big|_\partial - \int u\phi'' \right]$$

$$= \int u\phi''$$

$L[u] : C_c^\infty[a,b] \to \mathbb{R}$ by $\phi \mapsto \int u\phi''$

<u>Weak ODE</u> : $\int u\phi'' = \int f\phi \quad \forall \phi \in C_c^\infty$.

L^1_{loc}. $\quad f \in L^1_{loc}$

[redacted]

How about Poisson's Eqn. on $\mathcal{U} \subseteq \mathbb{R}^n$.

$$\begin{cases} \Delta u = f & \text{on } \mathcal{U} \\ u\big|_{\partial \mathcal{U}} = 0 \end{cases}$$

Two approaches to existence:

① Green's function (requires ① Fundamental Soln. on $\mathbb{R}^n$
② Integration. (Green's formulas))

② Weak solutions ← really easy

(requires: ① functional analysis
② Integration.)

[redacted]

Weak formulation for $\begin{cases} \Delta u = f & \text{on } U \subseteq \mathbb{R}^n. \\ u\big|_{\partial U} = 0 \end{cases}$

<u>Key Ideas</u> : 1 Look for a weak solution in $W^{1,2}(U)$.

The Sobolev space

<u>remember</u>

$W^{1,2}$ = set of L^2 functions with ~~x~~ weak derivatives in L^2.

<u>weak derivatives</u>

$$-\int u \frac{\partial \phi}{\partial x_j} = \int g_j \phi \quad \forall \phi \in C_c^\infty$$

$$\boxed{g_j = \frac{\partial u}{\partial x_j} \ (\text{weak})} \quad j = 1, 2, \ldots, n$$

[redacted]

② Weak formulation:

$$-\langle Du, D\phi \rangle_{\cancel{W^{1,2}}} = \langle f, \phi \rangle_{L^2} \quad \forall \phi \in C_c^\infty$$

$$\underline{B(u,\phi)} = \underbrace{-\int \sum_{j=1}^{n} \overset{\overset{g_j}{\shortparallel}}{D_j u}\, D_j \phi}_{\text{adjoint operator}}$$

weak derivatives

$u \in W^{1,2}$

$$\int \Delta u \, \phi$$

$$\shortparallel$$

$$\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

[redacted]