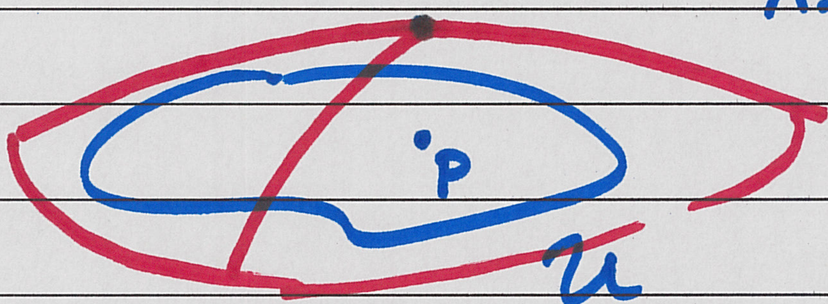
Don't Miss This!

THM
$$\begin{cases} \Delta u = 0 & \text{on a bounded } \underline{\text{domain}}\ \mathcal{U} \\ u\big|_{\partial \mathcal{U}} = 0 \end{cases}$$

Then if $u \in C^2(\bar{\mathcal{U}})$, then $u \equiv 0$.

<u>proof</u>: Let v be a nonzero solution. Assume $v(p) > 0$ for some $p \in \mathcal{U}$.



parabolic (down parabolic) umbrella

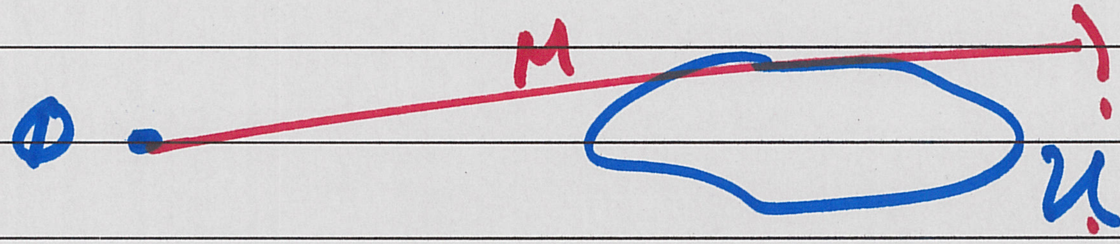
$$w(x) = v(p) - \varepsilon |x-p|^2$$

[redacted]

Claim: If $\varepsilon > 0$ is small enough, then

$$w(x) = v(p) - \varepsilon |x-p|^2 > 0 \quad \text{for } x \in \overline{\mathcal{U}}$$

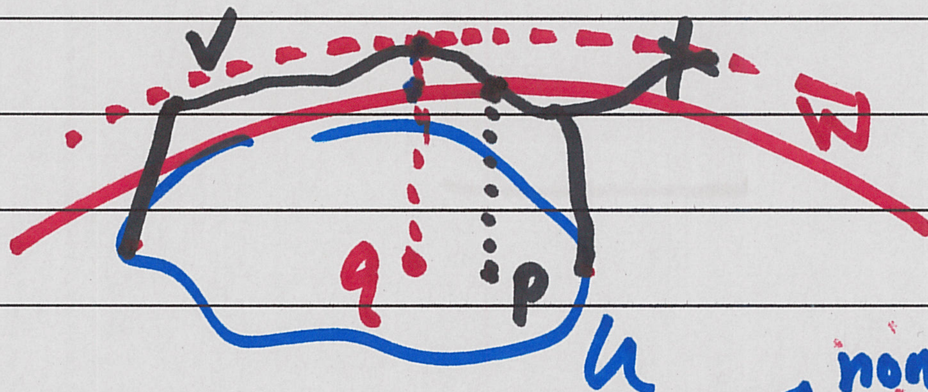
☹ $\mathcal{U}$ is bounded $\Rightarrow$ $|x| \leq M$ for $x \in \overline{\mathcal{U}}$.



$$w(x) = v(p) - \varepsilon |x-p|^2 \geq v(p) - \varepsilon (|x|+|p|)^2$$
$$\geq v(p) - \varepsilon \cdot 4M^2.$$

If $\varepsilon < \dfrac{v(p)}{8M^2}$, then $w(x) \geq \dfrac{v(p)}{2} > 0$. [redacted]

for $x \in \overline{\mathcal{U}}$. ✓



$$\begin{cases} \Delta v = 0 & \text{on } U \\ v\big|_{\partial U} \equiv 0 \end{cases}$$

non-negative

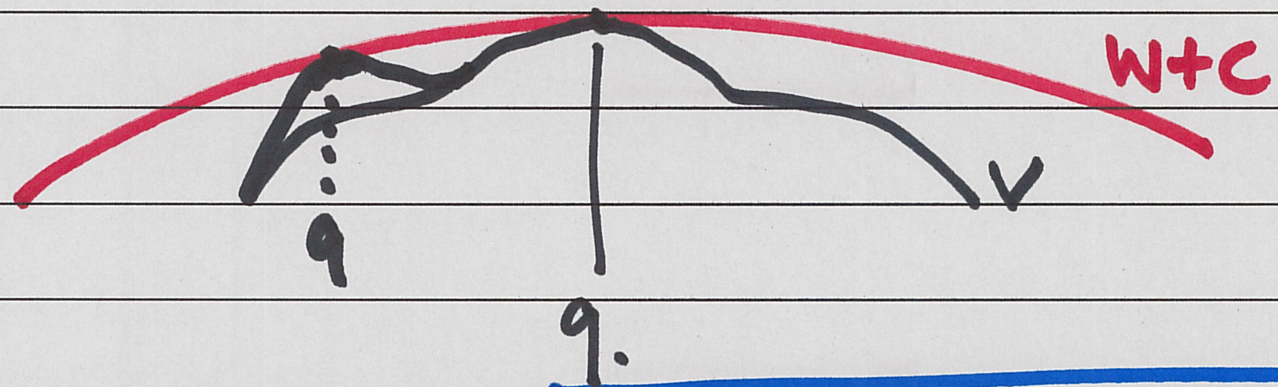
<u>IDEA</u>: Find a ~~positive~~ constant c so that
$v(x) \le w(x) + c$ for all $x \in \bar{U}$ and
$v(q) = w(q) + c$ for some point $q \in \bar{U}$.
If so... then $q \in U$ why?

<u>What is c?</u> $\max_x \left[v(x) - w(x) \right] \geq 0$

$\leftarrow$ c.

$\uparrow$
< 0 on ∂U
$= 0$ at p.

[redacted]



$v \in C^2(\bar{U})$

$$\begin{cases} \Delta v = 0 \text{ on } U \\ v|_{\partial U} = 0 \end{cases}$$

$q \in U$.

Contradiction : $\boxed{\Delta v(q) \leq -2n\varepsilon < 0.}$

$$\Delta w = \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \cdots + \frac{\partial^2 w}{\partial x_n^2} = -2n\varepsilon < 0.$$

$$w = v(p) - \varepsilon \sum_{j=1}^{n} (p_j - x_j)^2$$

$$\frac{\partial^2 w}{\partial x_j^2} = -2\varepsilon < 0.$$

[redacted]

4a.

$$W(x) = v(p) - \varepsilon \sum_{j=1}^{n} (p_j - x_j)^2$$

$$\frac{\partial W}{\partial x_j} = -2\varepsilon(p_j - x_j)(-1)$$

$$\frac{\partial^2 W}{\partial x_j^2} = -2\varepsilon(-\cancel{x}_j)(-1) = -2\varepsilon.$$

✓

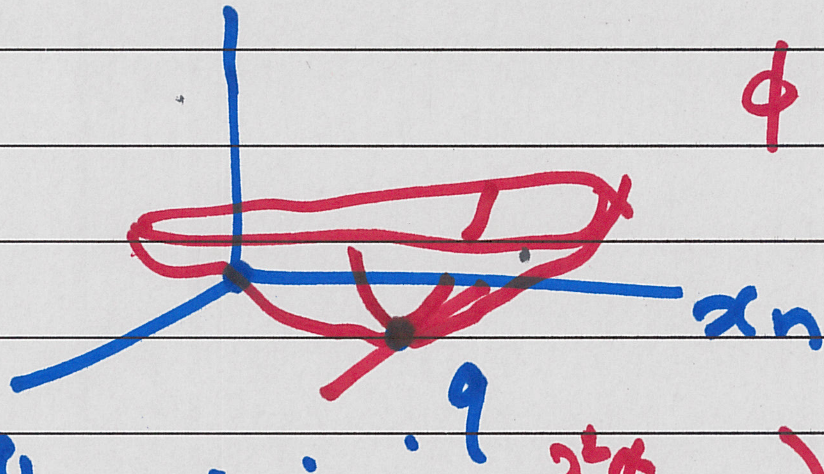
[redacted]

5.

Explanation: ① $\frac{\partial^2 V}{\partial x_j^2}(q) \leq \frac{\partial^2 W}{\partial x_j^2}(q) = -2\varepsilon$. ✓

② $\Delta V(q) = \sum_{j=1}^{n} \frac{\partial^2 V}{\partial x_j^2}(q) \leq -2\varepsilon n < 0$.

$\phi(x) = W(x) + c - V(x) \geq 0$ AND $\phi(q) = 0$.



ϕ has a min at 0.

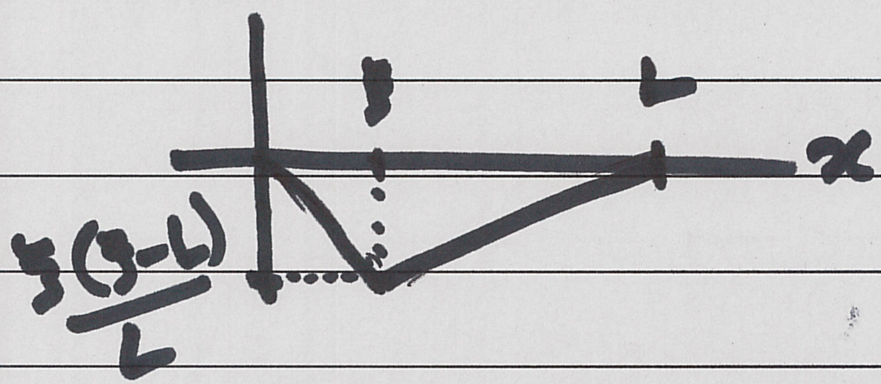
$D\phi_{(q)} = 0$. AND $D^2\phi(q) \geq 0$.

Hessian $D^2\phi = \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)$

[redacted]

Weak Derivatives :

$G(x,\xi)$



A weak derivative g of f has

$$\int f\phi' = -\int g\phi$$

Think about this: If f' is a classical derivative, then

$$\int f\phi' = \cancel{f\phi\Big|}_{\to 0} - \int f'\phi$$

$$\int_0^L G(x,\xi)\,\phi'(x)\,dx$$

$$= \int_0^\xi \frac{\xi-L}{L}\, x\, \phi'(x)\,dx + \int_\xi^L \frac{\xi}{L}(x-L)\,\phi'(x)\,dx$$

$$= \frac{\xi-L}{L}\left(x\phi\Big|_0^\xi - \int_0^\xi \phi(x)\,dx \right) + \frac{\xi}{L}\left((x-L)\phi\Big|_\xi^L - \int_\xi^L \phi(x)\,dx \right)$$

[redacted]

7.

$$\int_0^L G(x,\xi)\,\phi'(x)\,dx$$

$$= \frac{\xi-L}{L}\left(\xi\phi(\xi) - \int_0^{\xi}\phi(x)\,dx\right)$$

$$+ \frac{\xi}{L}\left(-(\xi-L)\phi(\xi) - \int_{\xi}^{L}\phi(x)\,dx\right)$$

$$= \frac{L-\xi}{L}\int_0^{\xi}\phi(x)\,dx - \frac{\xi}{L}\int_{\xi}^{L}\phi(x)\,dx$$

$$= -\int_0^L\left(\frac{\xi-L}{L}\,\chi_{[0,\xi]}(x) + \frac{\xi}{L}\,\chi_{[\xi,L]}(x)\right)\phi(x)\,dx$$

Weak Derivative

$$G_x(x,\xi) = \begin{cases} \dfrac{\xi-L}{L}, & 0 \le x < \xi \\ \dfrac{\xi}{L}, & \xi \le x \le L \end{cases}$$

[redacted]

Characteristic Function:

$$\chi_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A. \end{cases}$$

$\uparrow$ set

$$\int_0^{\xi} \phi(x)\,dx = \int_0^L \phi(x)\,\chi_{[0,\xi)}(x)\,dx.$$

[redacted]

The Point of View of Weak Derivatives:

$$\int f \phi' = -\int f' \phi \quad \leftarrow \text{integral functional}$$

$\uparrow$ weak derivatives

ϕ input or "test function"

$$\mathcal{F}[\phi'] = \mathcal{G}[\phi]$$

$\mathcal{F} : C_c^\infty(a,b) \to \mathbb{R}$ by $\mathcal{F}[\phi] = \int f \phi$

$\mathcal{G} : C_c^\infty(a,b) \to \mathbb{R}$ by

$$\mathcal{G}[\phi] = -\int \underset{\text{"}f'\text{"}}{g} \phi = -\langle g, \phi \rangle_{L^2}$$

[redacted]

Remember Normed Spaces.

$(V, \|\cdot\|)$ $(W, \|\cdot\|)$

normed vector space. $\mathcal{F}: V \to W$ operator.

<u>More Structure</u> : <u>Inner product</u>

$V \times V \longrightarrow \mathbb{R}$

(i) $\langle v, w\rangle = \langle w, v\rangle$ (symmetric)

(ii) bilinear $\langle av+bw, z\rangle = a\langle v, z\rangle + b\langle w, z\rangle$

(iii) positive definite (for inner prod.)
$\langle v, v\rangle \geq 0$ with "$=0$" only for $v = 0$.

$\|v\| = \sqrt{\langle v, v\rangle}$.

[redacted]

Example(s)

$$\mathbb{X}\cdot\mathbb{Y} = \sum_{j=1}^{n} x_j y_j$$

$$\mathbb{X}\cdot\mathbb{X} = \sum_{j=1}^{n} x_j^2 = |\mathbb{X}|^2.$$

L^2 square integrable function $\int |u|^2 < \infty$.

$$\langle u, v\rangle = \int u\cdot v$$

L^1 is normed $\|u\|_{L^1} = \int |u|$.

L^2 is normed $\|u\|_{L^2} = \left(\int |u|^2\right)^{1/2}$

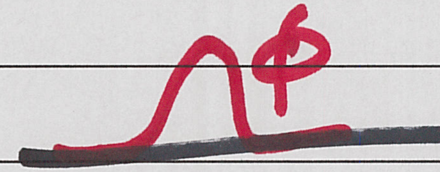
[redacted]

Is L^4 an inner product space?

$$\langle u, v \rangle = \int u^2 v^2$$

symmetric? ✓
bilinear? ✗

$$\langle \cdot , \cdot \rangle : L^4 \times L^4 \to \mathbb{R}$$



$$\begin{cases} \mathcal{F}[\phi] = \int f\phi = \langle f, \phi \rangle_{L^2} \\ \mathcal{F} : C_c^\infty(a,b) \to \mathbb{R} \end{cases}$$

[redacted]