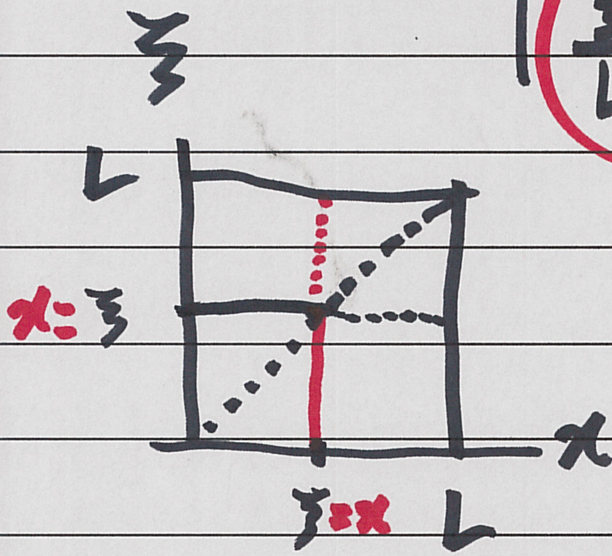
Lecture 10 MATH 6702 Feb. 10, 2020.

$$G(x, \xi) = -\frac{2}{L} \sum_{j=1}^{\infty} \left(\frac{L}{j\pi}\right)^2 \sin\frac{j\pi\xi}{L} \sin\frac{j\pi x}{L}$$

$$= \begin{cases} \frac{\xi - L}{L} x, & \underline{0 \le x \le \xi} \\ \frac{\xi}{L}(x - L), & \xi \le x \le L \end{cases}$$



$$\left\{ \begin{array}{l} \underline{0 \le \xi \le x} \\ x \le \xi \le L \end{array} \right.$$

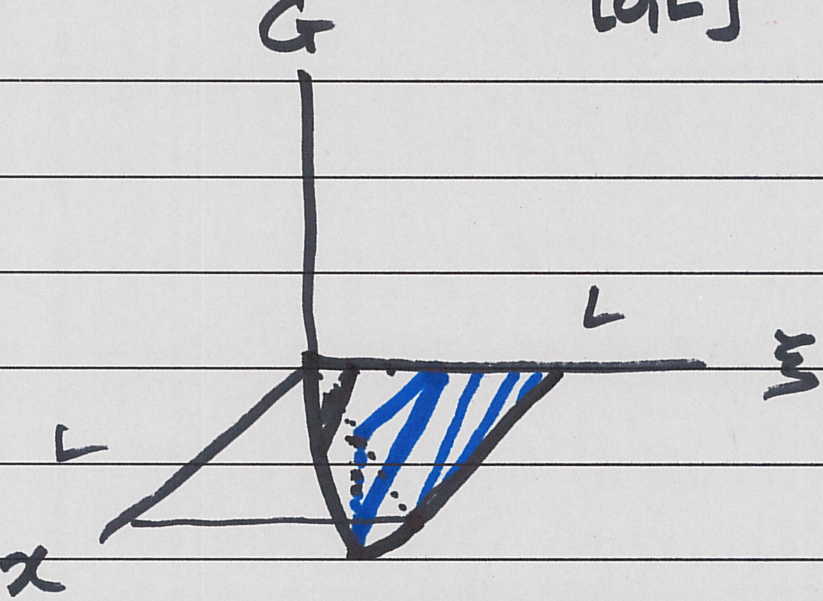
[redacted]

$$G(x,\xi) = \begin{cases} \frac{\xi - L}{L}\, x , & 0 \le x \le \xi \\ \frac{\xi}{L}(x-L) , & \xi \le x \le L . \end{cases}$$

$\uparrow$

Green's function for x'' on $[0,L]$.

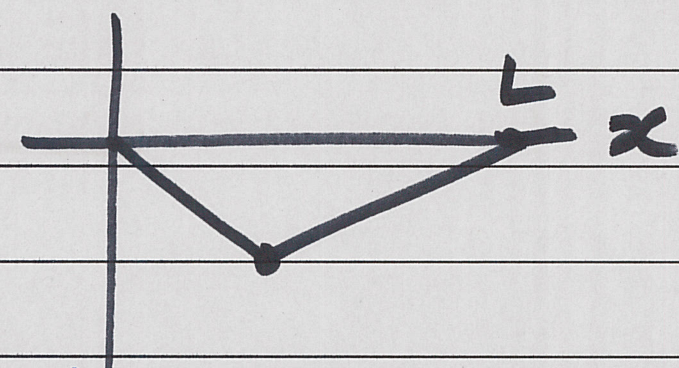
$$f \longmapsto u = \int_{[0,L]} f(\xi) G(x,\xi)$$



[redacted]

Slice $\xi = \text{const}.$

$G(x,\xi) = f(x)$

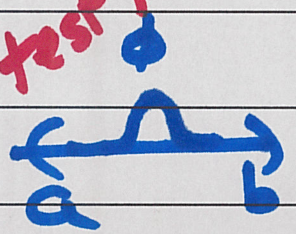


$f \in L^1_{loc}(a,b)$ is said have a **weak derivative**

$g \in L^1_{loc}(a,b)$ if

$$\int_{(a,b)} f\,\phi' = -\int_{(a,b)} g\,\phi$$

test function



for all $\phi \in C_c^\infty(a,b)$

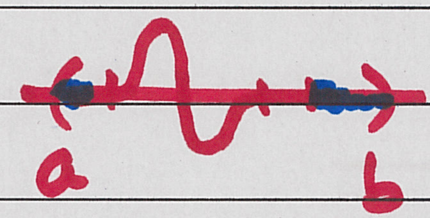
C^∞ functions ϕ with $\operatorname{supp}\phi \subset\subset (a,b)$

[redacted]

$$\int_a^b f(x)\,\phi'(x)\,dx = -\int_a^b f'(x)\,\phi(x)\,dx$$

$f \in C^1[a,b]$

integrate by parts.

ϕ 

$\operatorname{supp}\phi \subset\subset (a,b)$

$$\left.\cancel{f(x)\,\phi(x)}\right|_{x=a}^{b} - \int_a^b f'(x)\,\phi(x)\,dx$$

0

[redacted]

Weak Derivatives:

① If f has a <u>classical derivative</u>, f' then f' is a weak derivative.

② Weak derivatives are unique.

$$\int_{(a,b)} f\phi' = -\int_{(a,b)} g\phi \quad \forall \phi \in C_c^\infty(a,b)$$

"Proof": $\displaystyle\int_{(a,b)} (g_1 - g_2)\phi = 0 \quad \forall \phi \in C_c^\infty(a,b)$

Fundamental Lemma of The Calculus of Variations

(0) If $f \in C^0(a,b)$, and $\int f\phi = 0 \ \forall \phi \in C_c^\infty(a,b)$, Then $f(x) = 0$ for all x.

(1) Also, okay for $f \in L^1_{loc}(a,b)$.

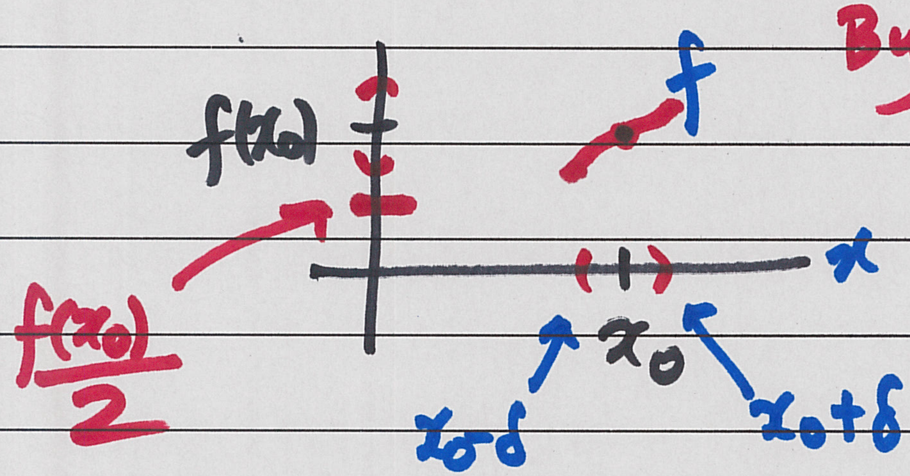
[redacted]

Fundamental Lemma

$$\int f\phi = 0 \quad \forall \phi \implies f = 0.$$

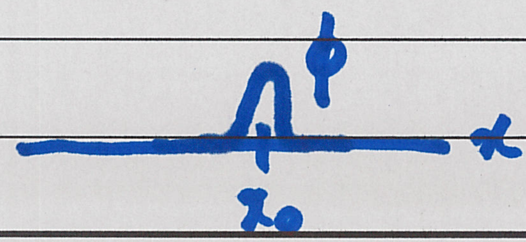
continuous f. Assume (by way of contradiction)

$f(x_0) > 0$ for some x_0.



By continuity, there is some $\delta > 0$ so that $f(x) \geq \frac{f(x_0)}{2}$ if $|x - x_0| < \delta$

$\frac{f(x_0)}{2} > 0$



[redacted]

Take ϕ with

(a) $\phi \geq 0$

(b) $\operatorname{supp}\phi \subseteq (x_0-\delta, x_0+\delta)$

(c) $\phi(x_0) > 0$.

Then $\displaystyle\int_{(a,b)} f\phi = \int_{(x_0-\delta, x_0+\delta)} f\phi$

$\displaystyle \geq \int_{(x_0-\delta, x_0+\delta)} \frac{f(x_0)}{2}\phi$

$\displaystyle = \frac{f(x_0)}{2} \int_{(x_0-\delta, x_0+\delta)} \phi$

> 0. (contradiction)

[redacted]

How about

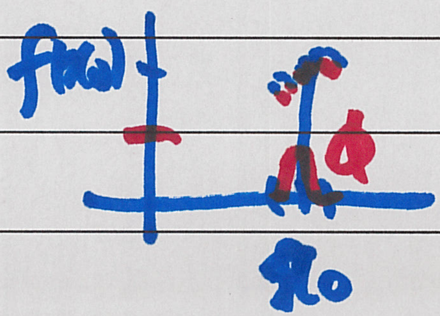
$$\int f\phi = 0 \ \forall \phi \Rightarrow f = 0$$

$$\text{for } f \in L^1 \ ?$$

Theorem: If $f \in L^1_{loc}(a,b)$, then almost every point in (a,b) is a Lebesgue point.

$$\lim_{r \to 0} \frac{1}{2r} \int_{x \in (x_0 - r, x_0 + r)} |f(x_0) - f(x)| = 0 .$$

Lebesgue



[redacted]

Weak Solutions

$x'' = f$ (1-D Poisson Eqn on (a,b))

$\downarrow$

$x''\phi = f\phi$

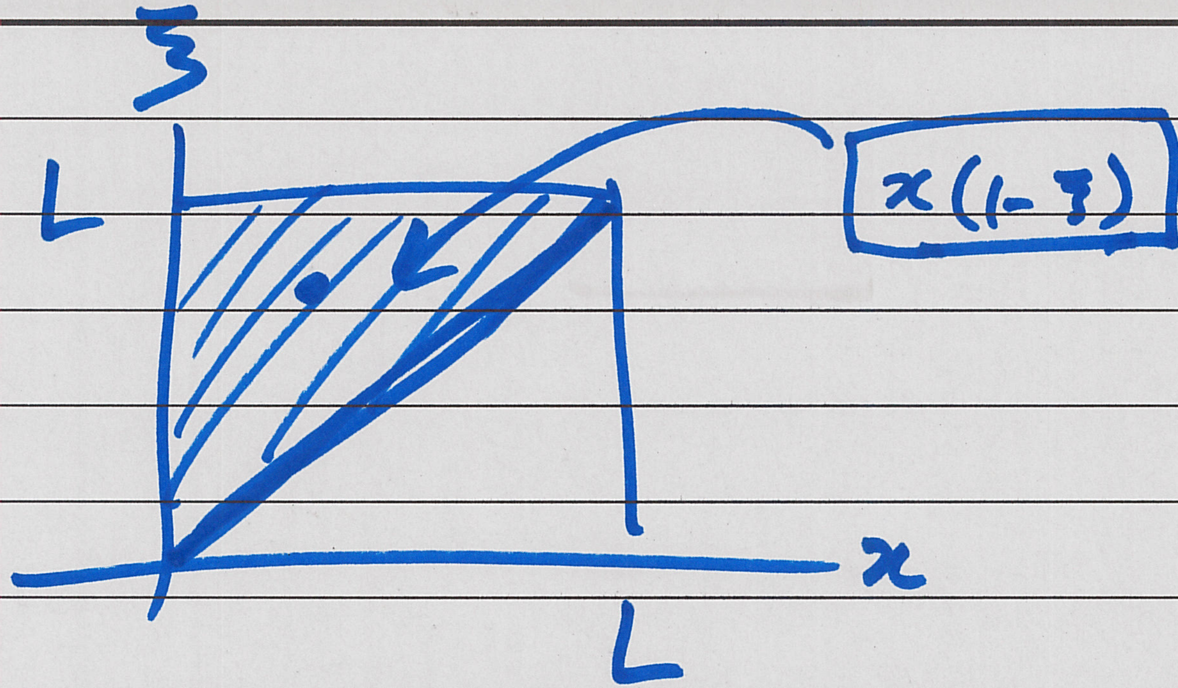
$$\int x''\phi = -\int x'\phi' = \int x\phi''$$

$\int x''\phi = \int f\phi$

$\boxed{\int x\phi'' = \int f\phi}$ $\forall \phi$

$x \in L^1_{loc}$

[redacted]

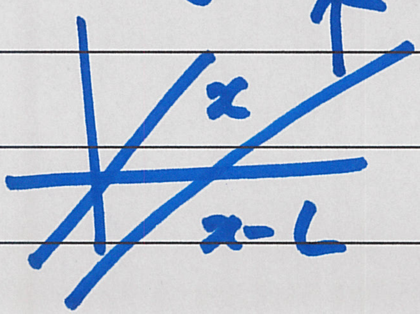


$$\frac{\partial u}{\partial x_j} = D_j u$$

$$\big([0,1] \times (0,1)\big) \setminus \{(x,x) : 0 \le x \le 1\}$$

$$\cancel{b}(a-L) < 0$$

$$y = mx + b$$



[redacted]